FM Recurrence Relations with Complex Roots

1. 2023/VJC/FM Promo/Q5 (modified) [8marks, 14mins]

The sequence $\{v_n\}$ is given by $v_1 = 2$, $v_2 = 0$ and $v_{n+2} - 2v_{n+1} + 2v_n = 4$ for $n \geq 1$. The sequence $\{w_n\}$ satisfies a second order homogeneous recurrence relation and $v_n + w_n = k$, where k is a constant.

(a) Write down a second order homogeneous recurrence relation for $\{w_n\}$ and hence find the value of k. [2]

(b) Obtain an expression for v_n in a form with trigonometric function. [4]

(c) Hence show that v_{4n+2} can be expressed as $k[1+\mathrm{f}(n)]$, where the two possible forms of $\mathrm{f}(n)$ are to be determined. [2]

Solution:

(a)

$v_{n+2} - 2v_{n+1} + 2v_n = 4, \quad v_1 = 2, v_2 = 0$

$v_n + w_n = k$

$\Rightarrow v_n = k - w_n$

Substitute into $v_{n+2} - 2v_{n+1} + 2v_n = 4$

$(k - w_{n+2}) - 2(k - w_{n+1}) + 2(k - w_n) = 4$

$\therefore w_{n+2} - 2w_{n+1} + 2w_n = k - 4$

For this equation to be homogeneous, $k = 4$.

(b)

$w_1 = 4 - v_1 = 2; \quad w_2 = 4 - v_2 = 4$

$w_{n+2} - 2w_{n+1} + 2w_n = 0$

Auxiliary equation: $\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm \mathrm{i}$

$\therefore w_n = \left(\sqrt{2}\right)^n \left[A\cos\dfrac{n\pi}{4} + B\sin\dfrac{n\pi}{4} \right]$

$w_1 = 2 \Rightarrow 2 = \sqrt{2}\left(A\dfrac{1}{\sqrt{2}} + B\dfrac{1}{\sqrt{2}} \right) = A + B$

$w_2 = 4 \Rightarrow 4 = 2(0 + B)$

$\Rightarrow B = 2$

$\Rightarrow A = 0$

Trigo Form: Note that *A* and *B* must be real constants since the sequence generates real numbers.

$\therefore w_n = \left(\sqrt{2}\right)^n \left[2\sin\dfrac{n\pi}{4} \right]$

$\Rightarrow v_n = 4 - 2^{\frac{n}{2}+1}\sin\dfrac{n\pi}{4}$

(b)

$v_{4n+2} = 4 - 2^{\frac{4n+2}{2}+1}\sin\dfrac{(4n+2)\pi}{4} = 4 - 2^{2n+2}\sin(2n+1)\dfrac{\pi}{2}$

$= \begin{cases} 4 - 2^{2n+2}(-1), & n \text{ is odd} \\ 4 - 2^{2n+2}(1), & n \text{ is even} \end{cases}$

$= \begin{cases} 4(1+4^n), & n \text{ is odd} \\ 4(1-4^n), & n \text{ is even} \end{cases}$

Cambridge tested this idea in H2 Math before under Integration Techniques. Line A stuff!

H2 Math Sequences and Series (Objectives – AQ and Interpretation Skills)

2. 2022/VJC/H2 Math Promo/Q11 [12marks, 22mins]

In a laboratory experiment, a tank is transported to and fro between station A and station B.

On the 1st visit to station A, 200 ml of water is added into the tank and on subsequent visits, 200 ml more water than the previous visit to station A will be added. That is, 400 ml and 600 ml of water will be added into the tank on the 2nd and 3rd visit to station A respectively. On each visit to station B, b ml of water is removed from the tank.

It is given that the tank is empty at the start of the experiment and it starts from station A.

(a) On the m^{th} visit to station A, 3200 ml of water is added into the tank. Find m. [2]

(b) It is given that immediately after the 16th visit to station B, there is more than 25 litres of water in the tank. Find the set of values that b can take. [3]

In another laboratory experiment, a tank is transported to and fro between station A and station C. 10% of the water in the tank is removed on each visit to station C. It is given that the tank is empty at the start of the experiment, and it starts from station A.

(c) Show that the amount of water in the tank immediately after the 3rd visit to station C is 1009.8 ml. [2]

(d) Show that the amount of water in the tank immediately after the n^{th} visit to station C is $1800n - 16200\left(1 - 0.9^n\right)$. [4]

(e) The experiment stops when the amount of water in the tank immediately after it leaves station C exceeds 10 litres. After the k^{th} visit to station C, the experiment stopped. Find k. [1]

Solution:

(a)

$$200 + (m-1)(200) = 3200$$

$$m - 1 = \frac{3200 - 200}{200}$$

$$m = 16$$

(b)

Amount of water immediately after 16th visit to B

= Total amount of water added to the tank in the 16 visits to A

– Total amount of water removed from tank in the 16 visits to B

$$= \frac{16}{2}\left(2(200) + (16-1)200\right) - 16b > 25000$$

Thus $27200 - 16b > 25000$

$$16b < 2200$$

$$0 \le b < 137.5$$

(c)

Visit (n)	After visiting A	After visiting C
1	200	0.9(200)
2	0.9(200) + 400	0.9[0.9(200) + 400] $= 0.9^2(200) + (0.9)(400)$
3	$0.9^2(200)+(0.9)(400)$ +600	$0.9^3(200)+(0.9)^2(400)$ $+(0.9)(600)$

$0.9^3(200)+(0.9)^2(400)+(0.9)(600) = 1009.8$ ml

(d)

Total amount of water in tank after the nth visit to station C

$$= 0.9^n(200) + (0.9)^{n-1}(400) + (0.9)^{n-2}(600) + \ldots + (0.9)^2(200(n-1)) + (0.9)(200n)$$

$$= 0.9^n(200) + 2(0.9)^{n-1}(200) + 3(0.9)^{n-2}(200) + \ldots + (n-1)(0.9)^2(200) + n(0.9)(200)$$

$$= 0.9^n(200) + (0.9)^{n-1}(200) + (0.9)^{n-2}(200) + \ldots + 0.9^2(200) + (0.9)(200) \quad \text{<-- GP with n terms}$$

$$+ (0.9)^{n-1}(200) + (0.9)^{n-2}(200) + \ldots + 0.9^2(200) + (0.9)(200) \quad \text{<-- GP with n-1 terms}$$

$$+ (0.9)^{n-2}(200) + \ldots + 0.9^2(200) + (0.9)(200) \quad \text{<-- GP with n-2 terms}$$

$$+ \ldots$$

$$+ 0.9^2(200) + (0.9)(200)$$

$$+ (0.9)(200)$$

Nice Qn from VJC.
Useful skill to learn.

$$= 0.9(200)\left(\frac{1-0.9^n}{1-0.9}\right) + 0.9(200)\left(\frac{1-0.9^{n-1}}{1-0.9}\right) + \ldots + 0.9(200)\left(\frac{1-0.9^2}{1-0.9}\right) + 0.9(200)\left(\frac{1-0.9^1}{1-0.9}\right)$$

$$= 1800\left(\left(1-0.9^n\right) + \left(1-0.9^{n-1}\right) + \ldots + \left(1-0.9^2\right) + \left(1-0.9^1\right)\right)$$

$$= 1800\left(n - \left(0.9 + \ldots + 0.9^{n-1} + 0.9^n\right)\right)$$

$$= 1800\left(n - 0.9\left(\frac{1-0.9^n}{1-0.9}\right)\right)$$

$$= 1800n - 16200\left(1-0.9^n\right)$$

11(v)

$1800n - 16200\left(1-0.9^n\right) > 10000$

Visit (n)	$1800n - 16200\left(1-0.9^n\right)$
12	9975.36
13	11317.82

Hence, $k = 13$.