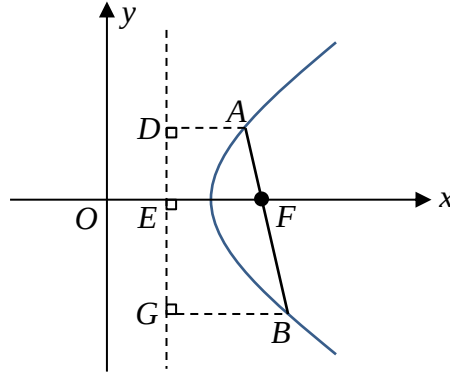


1



In the diagram, a part of a hyperbola with eccentricity e is shown with a chord AB passing through the focal point F . The distance from the nearest directrix to F , denoted by EF , is p .

- (i) Let D, G be the points on the directrix nearest to A and B respectively. Show that

$$\frac{AF}{AB} = \frac{ep - AF}{BF - AF}. \quad [3]$$

- (ii) Hence, or otherwise, find $\frac{1}{AF} + \frac{1}{BF}$ in terms of e and p . [2]

2 Consider the following system of linear equations, where $a, b \in \mathbb{R}$.

$$x + y + az = 0$$

$$x - 2y + az = 0$$

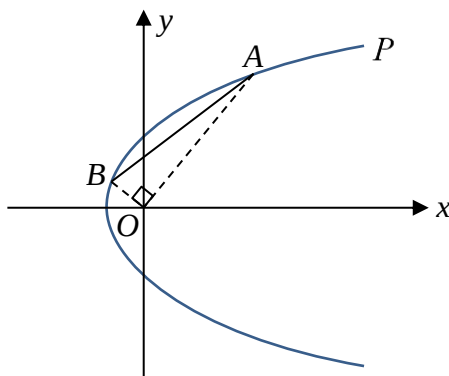
$$x + y + z = b$$

Let S be the solution set of this system of linear equations. By performing suitable row operations, state the possible values of a and b such that

- (i) $S = \{\mathbf{v}\}$, where $\mathbf{v}$ is a nonzero vector, [2]

- (ii) S is not a subspace of $\mathbb{R}^3$. Find the possible sets of S in this case. [3]

3



In the diagram, the origin O is the focus of a parabola P with equation $y^2 = 4h(x+h)$, where h is a positive constant.

- (i) By taking the pole to be at O , find a polar equation of P . [2]
- (ii) A and B are points on P such that OA is perpendicular to OB . By using differentiation, find the exact minimum area of the triangle OAB in terms of h . [You are not required to prove the area found is a minimum.] [5]

- 4 (a) A fixed point of a function $g(x)$ is a real number α such that $\alpha = g(\alpha)$. An approximate value for a fixed point α is to be found by using the fixed point iteration method of the form $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \dots$, with an initial estimate of $x_0 = 1.5$. For each of the following expressions for $g(x)$, explain with reason(s) or with the aid of suitable diagrams, whether it is suitable for fixed point iteration. If the $g(x)$ is found suitable for fixed point iteration, obtain an estimate for α correct to 4 decimal places.

(i) $g(x) = x - x^3 - 4x^2 + 6$ [2]

(ii) $g(x) = \left(\frac{6}{x} - 4x\right)^{\frac{1}{2}}$ [1]

(iii) $g(x) = \left(\frac{6}{4+x}\right)^{\frac{1}{2}}$ [3]

- (b) The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$ is the normal density function with a mean value of zero and standard deviation σ . The probability of a randomly chosen value x as described by this function lying in the interval $[a, b]$, is given by $\int_a^b f(x) dx$.

Use Simpson's rule with 5 ordinates to find an approximation to the probability that a randomly chosen value x will lie in the interval $[-\sigma, \sigma]$. Give your answer correct to 3 decimal places. [4]

[Turn over

5 Do not use a calculator in answering this question.

(i) Show algebraically that

$$\cos\left(\frac{\pi}{10}\right) + \cos\left(\frac{3\pi}{10}\right) + \cos\left(\frac{5\pi}{10}\right) + \cos\left(\frac{7\pi}{10}\right) + \cos\left(\frac{9\pi}{10}\right) = 0. \quad [2]$$

(ii) The equation $z^{10} + (13z - 1)^{10} = 0$ has 10 complex roots denoted by z_i and z_i^* , where z_i^* are the complex conjugates of z_i , $i = 1, 2, 3, 4, 5$, respectively. By expressing these roots as $\frac{1}{z} = 13 - e^{i\theta}$, where $-\pi < \theta \leq \pi$, or otherwise, evaluate

$$\frac{1}{z_1 z_1^*} + \frac{1}{z_2 z_2^*} + \frac{1}{z_3 z_3^*} + \frac{1}{z_4 z_4^*} + \frac{1}{z_5 z_5^*}. \quad [8]$$

6 (i) Let $I_n = \int_0^t x^n (1 + x^2)^{\frac{1}{2}} dx$. Show that

$$(n + 2)I_n = t^{n-1}(1 + t^2)^{\frac{3}{2}} - (n - 1)I_{n-2}, \text{ where } n \geq 2. \quad [3]$$

(ii) By considering the derivative of $\ln(x + \sqrt{1 + x^2})$, find the exact value of $\int_0^{\frac{4}{3}} \sqrt{1 + x^2} dx$ in the form $m + n \ln 3$, where m, n are constants to be determined. [4]

(iii) A curve C has parametric equations

$$x = \tan \theta, \quad y = \frac{1}{1 + \cos 2\theta} \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

A part of C bounded by $x = 0$ and $x = \frac{4}{3}$ is rotated through 2π radians about the x -axis. By converting the equations to cartesian form, or otherwise, find the exact value of the surface area of revolution thus formed. [4]

7 Taking the origin as the pole, the equation of the curve G in polar coordinates $r = f(\theta)$ satisfies the differential equation

$$\frac{d^2 r}{d\theta^2} + 4r = 5 \sin 3\theta \quad \text{for } \frac{\pi}{5} \leq \theta \leq \frac{3\pi}{5}.$$

It is given that when $\theta = \frac{\pi}{2}$, $r = 1$ and $\frac{dr}{d\theta} = -2$.

(i) Find the polar equation of G . [7]

(ii) Sketch G , indicating clearly the point with polar coordinates $(1, \frac{\pi}{2})$ on G . [1]

(iii) Show that the area enclosed by G is given by $a\pi + b(\sin \frac{\pi}{5} - \sin \frac{2\pi}{5})$ square units, where a, b are exact rational numbers to be determined. [5]

- 8** In a habitat, the number of rabbits R_n (in thousands) and wildcats W_n (in thousands) in n years can be modelled by

$$R_n = \frac{3}{2}R_{n-1} - \frac{1}{2}W_{n-1} ,$$

$$W_n = \frac{3}{4}R_{n-1} + \frac{1}{4}W_{n-1} ,$$

where $n \geq 1$, and R_0 and W_0 are the initial number of rabbits and wildcats respectively.

- (i) Find an expression for R_n in terms of R_{n-1} and R_{n-2} . [2]
- (ii) Show that the 2×1 matrix $\begin{pmatrix} R_n \\ W_n \end{pmatrix}$ can be expressed as $\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$, where $\mathbf{P}$ is a 2×2 invertible matrix and $\mathbf{D}$ is a 2×2 diagonal matrix. [4]
- (iii) Using the result in part (ii), find R_n and W_n in terms of n , R_0 and W_0 . [3]
- (iv) Given that the population of both rabbits and wildcats decrease and approach a non-zero equilibrium population, find a relationship between R_0 and W_0 . [3]

- 9** The sequence of integers $\{u_1, u_2, u_3, \dots\}$ is defined by the second-order recurrence relation

$$u_1 = a , u_2 = ka \quad \text{and} \quad u_{n+2} = 2u_{n+1} - 2u_n \quad \text{for } n \geq 1,$$

where a , k are positive constants.

- (i) Show that $u_{n+4} = -4u_n$ for all $n \geq 1$. [2]
- (ii) Prove by mathematical induction that u_{4r-2} is a multiple of ka for all positive integers r . [4]
- (iii) Hence find $\sum_{r=1}^{4N} u_r$ in terms of a , k and N . [3]
- (iv) Find u_n in the form $p^n(A\cos qn + B\sin qn)$, where A , B are constants in terms of a , k and p , q are constants to be determined.
- Hence find $\sum_{r=1}^{20} p^r(A\cos qr + B\sin qr)$ in terms of a and k . [5]

- 10 A *predator-prey model* is a pair of first-order non-linear differential equations used to describe the dynamic interactions between two species in ecological systems, one as predator and the other as prey. A financial analyst adopted the *predator-prey model* to predict the stock market prices of two companies X and Y . The stock prices $\$x$ and $\$y$ of companies X and Y respectively in t months are modelled as follows.

$$\begin{aligned}\frac{dx}{dt} &= 0.2x - 0.01xy, \\ \frac{dy}{dt} &= -0.05y + 0.004xy.\end{aligned}$$

It is given that the initial stock prices of companies X and Y are $\$10.00$ and $\$15.00$ respectively.

- (i) A ‘modified’ Euler method with step size h is given by

$$\begin{aligned}x_2 &= x_1 + h \left. \frac{dx}{dt} \right|_{x=x_1, y=y_1}, \\ y_2 &= y_1 + h \left. \frac{dy}{dt} \right|_{x=x_1, y=y_1}.\end{aligned}$$

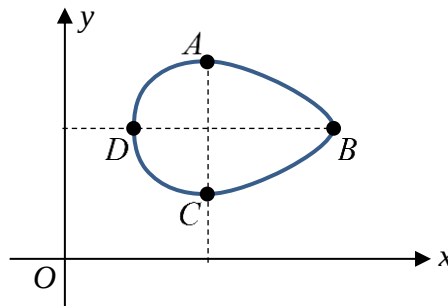
Using this ‘modified’ Euler method with step size 0.5 , estimate the stock prices of companies X and Y after 1 month. Give your answers correct to 2 decimal places. [4]

- (ii) Find an expression for $\frac{dy}{dx}$. Hence, using integration, show that

$$y^g e^{hy} = Kx^m e^{nx},$$

where g, h, K, m, n are constants to be determined. [5]

- (iii) The graph below shows the relationship between x and y .



It is given that the tangents of the curve at the points A and C are parallel to the x -axis, while the tangents of the curve at the points B and D are parallel to the y -axis. By finding the x -coordinate of points A and C , and the y -coordinates of points B and D , determine the coordinates of the points A, B, C and D . [3]

- (iv) Using the graph and answers found in part (iii), comment on what the *predator-prey model* indicates about the stock prices of companies X and Y . [1]

End of Paper