1. Let $\left(\begin{array}{ccc|c} a & -2a & -2b & 4 \\ a & 3a & -3b & 2 \\ 0 & -a & 5 & b \end{array}\right)$ be the augmented matrix of a linear system. Using elementary row operations, simplify the augmented matrix and hence

(a) find the values of a and b for the system to have a unique solution, [2]

(b) give a geometrical interpretation for the system of equations when $a \neq 0$ and $b = -5$. [2]

2. In a four-pole electrical network, the output quantities E_1 and I_1 are given in terms of the input quantities E_2 and I_2 by $\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E_2 \\ I_2 \end{pmatrix}$, where a, b, c and d are non-zero.

Denoting $\mathbf{A}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \frac{1}{c}\begin{pmatrix} a & -|\mathbf{A}| \\ 1 & -d \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}.$$

[2]

Similarly, show that

$$\begin{pmatrix} E_1 \\ I_2 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} \gamma & |\mathbf{A}| \\ 1 & \delta \end{pmatrix}\begin{pmatrix} I_1 \\ E_2 \end{pmatrix},$$

where γ and δ are constants to be determined in terms of a, b, c and d. [3]

Given $\mathbf{A}$ is orthogonal, that is $\mathbf{A}\mathbf{A}^T = \mathbf{I}$, and $|\mathbf{A}| = 1$, b and d are non-zero, find E_1 and E_2 in terms of b, d, I_1 and I_2.
[5]

<table>
<tr><th>Qn</th><th>Solutions</th></tr>
<tr><td>1</td><td>

$$\left(\begin{array}{ccc|c} a & -2a & -2b & 4 \\ a & 3a & -3b & 2 \\ 0 & -a & 5 & b \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} a & 2a & -2b & 4 \\ 0 & a & -b & -2 \\ 0 & -a & 5 & b \end{array}\right)$$

$$\xrightarrow[R_3+R_2]{R_1+2R_2} \left(\begin{array}{ccc|c} a & 0 & 0 & 0 \\ 0 & a & b & -2 \\ 0 & 0 & b+5 & b-2 \end{array}\right)$$

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<tr><td>(a)</td><td>Unique solution if there are 3 pivots. i.e. $a \neq 0$ and $b \neq -5$</td></tr>
<tr><td>(b)</td><td>

If $a \neq 0$ and $b = -5,$ we have $\left(\begin{array}{ccc|c} a & 0 & 0 & 0 \\ 0 & a & -5 & -2 \\ 0 & 0 & 0 & -7 \end{array}\right)$.

System is inconsistent. There is no solution.

The system of equations consists of 3 planes with no common points.

Triangular prism.

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<tr><th>Qn</th><th>Solutions</th></tr>
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$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}$$

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} aE_2 + bI_2 \\ cE_2 + dI_2 \end{pmatrix} \Rightarrow \begin{cases} cE_1 = acE_2 + bcI_2 & ----(1) \\ cE_2 = I_1 - dI_2 & ----(2) \end{cases}$$

Sub (2) into (1):

$$cE_1 = a(I_1 - dI_2) + bcI_2 = aI_1 - (ad - bc)I_2 \quad -----(3)$$

(2) and (3):

$$c\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} a & -(ad-bc) \\ 1 & -d \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

And since $|\mathbf{A}| = ad - bc$

$$\Rightarrow \quad \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \frac{1}{c} \begin{pmatrix} a & -|\mathbf{A}| \\ 1 & -d \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \qquad \text{(shown)}$$

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$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}$$

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} aE_2 + bI_2 \\ cE_2 + dI_2 \end{pmatrix} \Rightarrow \begin{cases} dE_1 = adE_2 + bdI_2 & ----(4) \\ dI_2 = I_1 - cE_2 & ----(5) \end{cases}$$

Sub (5) into (4):

$$\begin{aligned} dE_1 &= adE_2 + b(I_1 - cE_2) \\ &= bI_1 + (ad - bc)E_2 \quad -----(6) \end{aligned}$$

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	(6) and (5): $d\begin{pmatrix}E_1\\I_2\end{pmatrix}=\begin{pmatrix}b & (ad-bc)\\1 & -c\end{pmatrix}\begin{pmatrix}I_1\\E_2\end{pmatrix}$ $\begin{pmatrix}E_1\\I_2\end{pmatrix}=\frac{1}{d}\begin{pmatrix}b & \lvert\mathbf{A}\rvert\\1 & -c\end{pmatrix}\begin{pmatrix}I_1\\E_2\end{pmatrix}$ $\therefore \gamma=b$ and $\delta=-c$
	Method 1: If $\mathbf{A}$ is orthogonal, so $\mathbf{AA}^T=\mathbf{I}$ and $\lvert\mathbf{A}\rvert=1$ $\therefore \mathbf{A}^T=\mathbf{A}^{-1}\Rightarrow\begin{pmatrix}a & c\\b & d\end{pmatrix}=\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$ Thus $a=d,\ c=-b$ Since $\begin{pmatrix}E_1\\E_2\end{pmatrix}=\frac{1}{c}\begin{pmatrix}a & -\lvert\mathbf{A}\rvert\\1 & -d\end{pmatrix}\begin{pmatrix}I_1\\I_2\end{pmatrix}$ $=-\frac{1}{b}\begin{pmatrix}d & -1\\1 & -d\end{pmatrix}\begin{pmatrix}I_1\\I_2\end{pmatrix}$ $=\begin{pmatrix}-\frac{d}{b}I_1+\frac{1}{b}I_2\\-\frac{1}{b}I_1+\frac{d}{b}I_2\end{pmatrix}$
	Method 2: $\mathbf{AA}^T=\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}a & c\\b & d\end{pmatrix}=\begin{pmatrix}a^2+b^2 & ac+bd\\ac+bd & c^2+d^2\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $\Rightarrow\begin{cases}a^2+b^2=1 & ----(1)\\c^2+d^2=1 & ----(2)\\ac+bd=0 & ----(3)\end{cases}$ $\lvert\mathbf{A}\rvert=ad-bc=1 \quad ----(4)$ From (3), $c=-\frac{bd}{a}$. Subst. into (4), $ad-b\left(-\frac{bd}{a}\right)=1$ $d\left(\frac{a^2+b^2}{a}\right)=1$ $d\left(\frac{1}{a}\right)=1$ by (1) $a=d$ Thus, from 3, $c=-b$ $\begin{pmatrix}E_1\\E_2\end{pmatrix}=-\frac{1}{b}\begin{pmatrix}d & -1\\1 & -d\end{pmatrix}\begin{pmatrix}I_1\\I_2\end{pmatrix}=\begin{pmatrix}-\frac{d}{b}I_1+\frac{1}{b}I_2\\-\frac{1}{b}I_1+\frac{d}{b}I_2\end{pmatrix}$