1 Even and Odd Functions

If $f(-x) = f(x)$ for every x in the domain, then f is called an even function. The graph of an even function is **symmetric about the *y*-axis.**

The following functions are examples of even functions:
- $|x|$
- x^{2n}, where $n \in \mathbb{Z}$
- $\cos x$

The sum of any two even functions is also even, and the product of any two even functions is even.

Example 1

Given that $f_i(x)$ are even functions of x, show that $f(x) = \sum_{i=1}^{n} f_i(x)$ is even.

Solution:

$$\begin{aligned} f(-x) &= \sum_{i=1}^{n} f_i(-x) \\ &= \sum_{i=1}^{n} f_i(x) \qquad (\because\ f_i(x) \text{ are even functions, i.e. } f_i(-x) = f_i(x)) \\ &= f(x) \ \ \text{(shown)} \end{aligned}$$

If $f(-x) = -f(x)$ for every x in the domain, then f is called an odd function. The graph of an odd function is **symmetric about the origin.** i.e. we can obtain the entire graph of $y = f(x)$ from itself by rotating through 180° about the origin.

The following functions are examples of odd functions:
- x^{2n+1}, where $n \in \mathbb{Z}$
- $\sin x$

The sum of any two odd functions is odd. However, products work slightly differently: the product of two odd functions is even, and the product of an odd function and an even function is odd.

Example 2 [2019/NJC/Prelim/Q5 (modified)]

A function $\mathrm{f}:\mathbb{R}\to\mathbb{R}$ is $\begin{cases}\text{an odd function, if } \mathrm{f}(-x)=-\mathrm{f}(x)\\ \text{an even function, if } \mathrm{f}(-x)=\mathrm{f}(x)\end{cases}$ for any real value of x.

(a) It is given that $\mathrm{f}:\mathbb{R}\to\mathbb{R}$ is an odd function. Determine with justification, whether each of the following functions is even or odd:

(i) the composite function ff, [2]

(ii) the derivative of f, given that f is differentiable everywhere on $\mathbb{R}$. [3]

(b) Prove that for every function $\mathrm{g}:\mathbb{R}\to\mathbb{R}$, there exists a *unique* decomposition into the sum of an odd function and an even function. (Hint: consider $\dfrac{\mathrm{g}(x)+\mathrm{g}(-x)}{2}$.) [4]

(c) Let the decomposition of e^x be $\sinh(x)$ and $\cosh(x)$ where $\sinh$ and $\cosh$ are odd and even functions respectively.

Show that $y=A\sinh x+B\cosh x$ is a solution to the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=y$. [3]

2 Convex and Concave Functions

Before we dive into convex and concave functions, a brief note on increasing and decreasing functions.

The definition of increasing (and decreasing) functions is **not** based on their derivatives. Non-differentiable functions can also be increasing. We can determine if a function is increasing by comparing its values, but there are two differing conventions:

A function f is **increasing** on an interval *I* if:

- (Convention A): $\mathrm{f}(x_1) \leq \mathrm{f}(x_2)$ for all $x_1 < x_2$ in *I*
- (Convention B): $\mathrm{f}(x_1) < \mathrm{f}(x_2)$ for all $x_1 < x_2$ in *I*

Proponents of Convention A tend to call Convention B "strictly increasing", while proponents of Convention B tend to call Convention A "non-decreasing". Always look at the definition or context to determine which convention is being used.

It is **not** true that $\mathrm{f}(x_1) < \mathrm{f}(x_2)$ for all $x_1 < x_2$ in *I* (i.e. Convention B) corresponds to "the derivative is always positive". Here are two canonical counterexamples:

Example 3

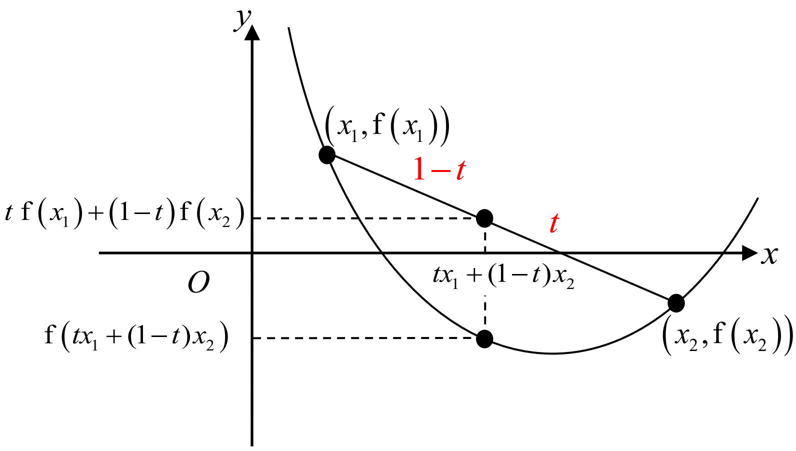
The function $\mathrm{g}(x) = x^{\frac{1}{3}}$ is strictly increasing in $x \in \mathbb{R}$. $\mathrm{g}'(0)$ is undefined.
The function $\mathrm{h}(x) = x^3$ is strictly increasing in $x \in \mathbb{R}$. $\mathrm{h}'(0) = 0$.

A function f is **convex** if for any two points x_1, x_2 in the domain of the function and $0 \leq t \leq 1$, we have

$$\mathrm{f}\left(tx_1 + (1-t)x_2\right) \leq t\,\mathrm{f}\left(x_1\right) + \left(1-t\right)\mathrm{f}\left(x_2\right)$$

Note that equality holds when $x_1 = x_2$.

A function f is strictly convex if the inequality in the previous definition is strict.



Graphically, if f is convex, then the line segment joining two points on the curve lies above or on the curve.
The inequality $\mathrm{f}\left(tx_1 + (1-t)x_2\right) \leq t\,\mathrm{f}\left(x_1\right) + \left(1-t\right)\mathrm{f}\left(x_2\right)$ is sometimes called Jensen's Inequality.

A function f is **concave** if $-\mathrm{f}$ is convex.

Again, convex functions need not be differentiable. However, a sufficient condition for a function to be convex is that its second derivative is non-negative.

Example 4 [2019/NYJC/Prelim/Q4]

The tangent at the point (x, y) on the curve with equation $y = \mathrm{f}(x)$ passes through the point (a, x), where a is a positive constant and $x < a$.

(i) Show that

$$(a-x)\frac{\mathrm{d}y}{\mathrm{d}x} = x - y. \qquad [2]$$

(ii) By making the substitution $y = (a-x)z$, obtain a differential equation in x and z. [3]

(iii) Solve this differential equation and, given that the curve $y = \mathrm{f}(x)$ passes through the origin, show that

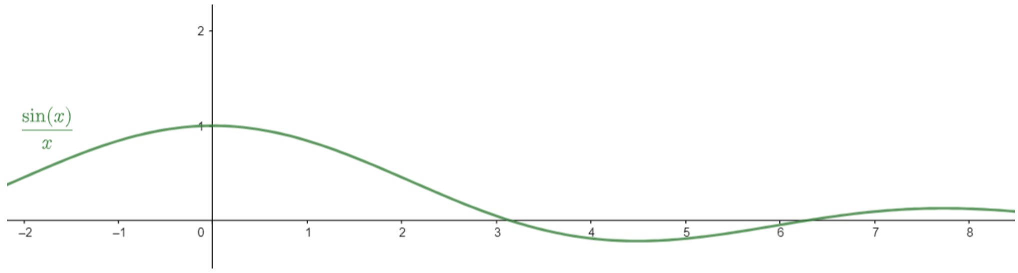
$$\mathrm{f}(x) = (a-x)\ln\left(\frac{a-x}{a}\right) + x. \qquad [6]$$

(iv) What can you say about the concavity of the curve $y = \mathrm{f}(x)$? Justify your answer. [3]

(v) Given that $\mathrm{f}(x) \to a$ as $x \to a$, explain why $\int_0^a \mathrm{f}(x)\,\mathrm{d}x < \frac{1}{2}a^2$. [2]

3 L'Hopital's Rule

How would we find the value of $\lim_{x\to 0} \dfrac{\sin x}{x}$?



The issue here is that both numerator and denominator tend to zero as $x \to 0$. Fortunately, we have a practical result that helps us in such circumstances.

Suppose we want to find $\lim_{x\to a} \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$, and we have either:

$$\lim_{x\to a} \mathrm{f}(x) = \lim_{x\to a} \mathrm{g}(x) = 0$$

or:

$$\lim_{x\to a} \mathrm{f}(x) = \pm\infty \text{ and } \lim_{x\to a} \mathrm{g}(x) = \pm\infty .$$

Then the limit can be computed using

$$\lim_{x\to a} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \lim_{x\to a} \frac{\mathrm{f}'(x)}{\mathrm{g}'(x)} \ .$$

Example 5

Using L'Hopital's rule, find (i) $\lim_{x\to 0} \dfrac{\sin x}{x}$, (ii) $\lim_{x\to 0^+} x \ln x$, (iii) $\lim_{x\to -\infty} x\mathrm{e}^x$.

Note: What does this tell you about how <u>polynomial</u>, <u>exponential</u> and <u>logarithmic</u> growth rates compare?

4 Improper Integrals

In H2 Mathematics, definite integrals are usually taken over finite intervals $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$.

In H3 Mathematics, you may have to extend this notion to determine infinite integrals:

If $\mathrm{f}(x)$ is continuous on $[a,\infty)$ for some $a \in \mathbb{R}$, then the improper integral of f over $[a,\infty)$ is:

$$\int_a^\infty \mathrm{f}(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t \mathrm{f}(x)\,\mathrm{d}x$$

If $\mathrm{f}(x)$ is continuous on $(-\infty,b]$ for some $b \in \mathbb{R}$, then the improper integral of f over $(-\infty,b]$ is:

$$\int_{-\infty}^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^b \mathrm{f}(x)\,\mathrm{d}x$$

If $\int_{-\infty}^c \mathrm{f}(x)\,\mathrm{d}x$ and $\int_c^\infty \mathrm{f}(x)\,\mathrm{d}x$ are convergent for some $c \in \mathbb{R}$, then the improper integral of f over $\mathbb{R}$ is:

$$\int_{-\infty}^\infty \mathrm{f}(x)\,\mathrm{d}x = \int_{-\infty}^c \mathrm{f}(x)\,\mathrm{d}x + \int_c^\infty \mathrm{f}(x)\,\mathrm{d}x$$

Limits can similarly be used to determine integrals over discontinuities:

If $\mathrm{f}(x)$ is continuous on $(a,b]$, then the improper integral of f over $(a,b]$ is:

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b \mathrm{f}(x)\,\mathrm{d}x$$

If $\mathrm{f}(x)$ is continuous on $[a,b)$, then the improper integral of f over $[a,b)$ is:

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t \mathrm{f}(x)\,\mathrm{d}x$$

Example 6

Find, if possible, (i) $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$, (ii) $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$, (iii) $\int_0^\infty \frac{1}{x^2}\,\mathrm{d}x$, (iv) $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

4 Integration by Substitution and other Techniques

In H2 Mathematics, we learnt how to integrate by substitution when the substitution is given. How can we recognize the substitution when it is not given?

By the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(\mathrm{g}(x)) = \mathrm{f}'(\mathrm{g}(x))\mathrm{g}'(x) \Rightarrow \int \mathrm{f}'(\mathrm{g}(x))\mathrm{g}'(x)\,\mathrm{d}x = \mathrm{f}(\mathrm{g}(x)) + c$.

Equivalently, when $u = \mathrm{g}(x)$, $\mathrm{g}'(x) = \frac{\mathrm{d}u}{\mathrm{d}x}$ so $\int \mathrm{f}'(\mathrm{g}(x))\mathrm{g}'(x)\,\mathrm{d}x = \int \mathrm{f}'(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \mathrm{f}'(u)\,\mathrm{d}u$.

Example 7 [(ii): 2019/ASRJC/Prelim/Q3a]

Find (i) $\int x\cos x^2 \mathrm{d}x$, (ii) $\int \frac{\cos x}{\sin x - \cos^2 x}\mathrm{d}x$

In more abstract situations, the functions and limits of integration can provide hints on the intended substitution.

Example 8

Prove that $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \int_a^b \mathrm{f}(a+b-t)\,\mathrm{dt}$.

Hence prove that, if $0 < \beta < \frac{\pi}{2}$, then $\int_{\beta}^{\pi-\beta} \left(\frac{x}{\sin x} \right) \mathrm{d}x = \pi \ln\left(\cot \frac{\beta}{2} \right)$.

Example 9 [2019/RI/Prelim/Q4]

(i) It is given that A, B and C are real numbers. Show that the quadratic function $\mathrm{h}(t) = At^2 + 2Bt + C$ is non-negative for all real values of t if and only if $A > 0$ and $AC \geq B^2$. [2]

(ii) Let f and g be continuous functions on the interval $[a, b]$. By considering $\int_a^b \left(t\mathrm{f}(x) + \mathrm{g}(x)\right)^2 \, \mathrm{d}x$, show that

$$\left(\int_a^b \left(\mathrm{f}(x)\right)^2 \mathrm{d}x\right)\left(\int_a^b \left(\mathrm{g}(x)\right)^2 \mathrm{d}x\right) \geq \left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)\mathrm{d}x\right)^2$$

and determine when equality holds. [3]

(iii) Hence show that $\int_0^1 \sqrt{1 + x^3} \, \mathrm{d}x < \frac{\sqrt{5}}{2}$. [3]

(iv) Let k be a continuous and differentiable function on the interval $[0,1]$ such that $\mathrm{k}(1) = 0$. Show that

$$\left(\int_0^1 \mathrm{k}(x) \, \mathrm{d}x\right)^2 \leq \frac{1}{3}\left(\int_0^1 \left(\mathrm{k}'(x)\right)^2 \, \mathrm{d}x\right)$$

and determine when equality holds. [4]

Tutorial Questions

1. The function $\mathrm{f}:[0,\pi]\to\mathbb{R}$ is such that $\mathrm{f}(0)=a$.
It is given that f is twice differentiable and that

$$\int_0^{\pi}\left[\mathrm{f}(x)+\mathrm{f}''(x)\right]\sin x\,\mathrm{d}x=b$$

where b is a real number.
By considering $\int_0^{\pi}\mathrm{f}''(x)\sin x\,\mathrm{d}x$, or otherwise, determine the value of $\mathrm{f}(\pi)$, expressing your answer in terms of a and b.

2. **[2019/DHS-EJC/Prelim/Q2(i)]**
(i) Prove that $\int_a^b \mathrm{f}(x)\,\mathrm{d}x=\int_a^b \mathrm{f}(a+b-t)\,\mathrm{d}t$ for any function f which is continuous on the domain $[a,b]$. [1]

(ii) Using the result in **(i)**, show that $\int_0^{\frac{\pi}{4}}\ln(1+\tan\theta)\ \mathrm{d}\theta=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$. [2]

(iii) Hence, by using an appropriate substitution, find the exact value of $\int_0^1\frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x$. [3]

(iv) Use the result in **(i)** to also find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\cos x}{\cos\left(x-\frac{\pi}{4}\right)}\ \mathrm{d}x$. [3]

3. It is given that n is a positive integer. Using the identity

$$1-x^n\equiv(1-x)\left(1+x+x^2+\cdots+x^{n-1}\right)$$

or otherwise, show that $\frac{1}{x^n(1-x)}\equiv\frac{1}{1-x}+\sum_{r=1}^{n}\frac{1}{x^r}$. Hence find $\int\frac{1}{x^n(1-x)}\,\mathrm{d}x$.
Using a similar method, or otherwise, find

(i) $\int\frac{1}{x^{2n}\left(1-x^2\right)}\,\mathrm{d}x$ (ii) $\int\frac{1}{x\left(1+x^2\right)^n}\,\mathrm{d}x$

4. Evaluate $\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.

5. **[2017/TJC/Prelim/Q7(i)]**
Let n be a positive integer. A sequence of functions f_n is defined as

$$\mathrm{f}_n(x)=\frac{\int_0^x\left(1-t^4\right)^n\,\mathrm{d}t}{\int_0^1\left(1-t^4\right)^n\,\mathrm{d}t}\text{ for }-1\le x\le 1.$$

(a) Show that $\mathrm{f}_n(x)$ is an odd function. [2]

(b) Show that $\mathrm{f}_n(x)$ is an increasing function of x for $0<x<1$. [2]

(c) Show that the gradient of $\mathrm{f}_n(x)$ is a decreasing function of x for $0<x<1$. [1]

(d) Sketch the graph of $\mathrm{f}_n(x)$. [2]

6. [2020/HCI/Prelim/Q3]

(i) Show that, for $m>0$,

$$\int_{1/m}^{m}\frac{x^2}{x+1}\,\mathrm{d}x=\frac{(m-1)^3(m+1)}{2m^2}+\ln m. \qquad [3]$$

(ii) Show by means of substitution $x=\frac{1}{u}$, that

$$\int_b^a\frac{1}{x^n(x+1)}\,\mathrm{d}x=\int_{1/a}^{1/b}\frac{u^{n-1}}{u+1}\,\mathrm{d}u. \qquad [2]$$

(iii) Hence or otherwise, without using calculator, evaluate

$$\int_1^2\frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x. \qquad [3]$$

7. [2017/NYJC/Prelim/Q4]

Throughout this question, n denotes a non-negative integer.

Let $I_n=\displaystyle\int_0^{\pi}\frac{\cos nx}{5-4\cos x}\,\mathrm{d}x$.

Prove that

$$I_n=\begin{cases}\displaystyle\int_0^{\pi}\frac{\cos nx}{5+4\cos x}\,\mathrm{d}x, & n\text{ is even},\\[2ex] \displaystyle-\int_0^{\pi}\frac{\cos nx}{5+4\cos x}\,\mathrm{d}x, & n\text{ is odd}.\end{cases} \qquad [3]$$

Hence or otherwise,

(i) find, in the case when n is even, the exact value of

$$\int_0^{\pi}\frac{\cos x\cos nx}{25-16\cos^2 x}\,\mathrm{d}x\ ,$$

showing your working clearly, [3]

(ii) show that

$$\int_0^{\pi}\frac{\cos^2 x}{25-16\cos^2 x}\,\mathrm{d}x=\frac{1}{4}\int_0^{\pi}\frac{\cos x}{5-4\cos x}\,\mathrm{d}x$$

and use the substitution $t=\tan\frac{x}{2}$ to evaluate the integral

$$\int_0^{\pi} \frac{\cos^2 x}{25-16\cos^2 x}\,\mathrm{d}x$$

Exactly, giving your answer in the form $k\pi$ where k is a rational number to be determined. [7]

8. Given $f:\mathbb{R}\to\mathbb{R}$ such that for all $x, y\in\mathbb{R}\backslash\{0\}$, $f(xy)=f(x)+f(y)$.

Find the value of $f(1)$ and $f(-1)$. Hence show that f is even.

Give an example of a (non-constant) function that satisfies the above properties and sketch its graph.

9. [2021/HCI/Prelim/Q8]

Let $\mathrm{f}:\mathbb{Q}\to\mathbb{Q}$ be a function satisfying the functional equation

$$\mathrm{f}(x+y)=\mathrm{f}(x)+\mathrm{f}(y) \qquad \forall x, y\in\mathbb{Q}$$

(i) Verify that $\mathrm{f}(0)=0$. [1]

A function g is known as an odd function if it satisfies $\mathrm{g}(-x)=-\mathrm{g}(x)$.

(ii) Show that f is an odd function. [2]

(iii) Using Mathematical induction, show that $\mathrm{f}(nx)=n\mathrm{f}(x)$ for all $n\in\mathbb{Z}^+$. [3]

(iv) Using the result in part (iii) or otherwise, show that $\mathrm{f}\left(\frac{x}{m}\right)=\frac{1}{m}\mathrm{f}(x)$ for all $m\in\mathbb{Z}^+$. [2]

(v) Given that the graph $y=\mathrm{f}(x)$ passes through (1, 3), show that $\mathrm{f}(x)=3x, x\in\mathbb{Q}$ is the solution to the function equation. [2]

10. [2021/CJC/Prelim/Q2(a)]

For any real number x, the largest integer less than or equal to x is denoted by $\lfloor x\rfloor$. For example, $\lfloor 3.7\rfloor=3$, $\lfloor 5\rfloor=5$, and $\lfloor -\pi\rfloor=-4$.

(i) Use a sketch graph of $y=\lfloor x\rfloor$ to evaluate $\int_0^5 \lfloor x\rfloor\,\mathrm{d}x$. [2]

(ii) Use a sketch graph of $y=\lfloor \mathrm{e}^x\rfloor$ to show that $\int_0^{\ln n}\lfloor \mathrm{e}^x\rfloor\,\mathrm{d}x=n\ln n-\ln(n!)$, where n is an integer [3]

(iii) Hence, show that $n!>n^n\mathrm{e}^{1-n}$. [3]

11. [2017/Q4]

Let $I_n=\int_0^{\frac{1}{4}\pi}\tan^n x\,\mathrm{d}x$.

(i) For $n>1$, prove that $I_n+I_{n-2}=\dfrac{1}{n-1}$ [4]

(ii) Justify the statement that $\tan x\le\dfrac{4}{\pi}x$ on $\left[0,\frac{1}{4}\pi\right]$. [2]

(iii) Hence prove that I_n tends to zero as n tends to infinity. [3]

(iv) Find the sum of the infinite series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$. [4]

12. [2018/Q7]

The differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x}=x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1,\quad \text{for } x>0 \qquad (1)$$

has a solution curve S such that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is non-zero at all points of S.

(i) By substituting $t=\frac{\mathrm{d}y}{\mathrm{d}x}$ into equation (1) and then differentiating with respect to x, show that S has equation $y^2=4x$. [6]

(ii) Show that a straight line is a tangent to the curve S if and only if it is itself a solution of equation (1). [6]

13. [2019/DHS-EJC/Prelim/Q1]

(i) Find the general solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{(c+t)x}{1-t^2}, \qquad \text{(A)}$$

where c is a constant. [5]

(ii) Make the substitution $y=tx$ in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{ax+by}{bx+ay}, \qquad \text{(B)}$$

where a and b are constants and $a\neq 0,$ and show that the resulting equation is equivalent to equation (A). [3]

(iii) Hence show that equation (B) has solutions of the form

$$|x-y|^{b+a}=D|x+y|^{b-a},$$

where D is an arbitrary constant. [3]

H3 Functions Tutorial Suggested Solutions

1.	The function $\mathrm{f}:[0,\pi]\to\mathbb{R}$ is such that $\mathrm{f}(0)=a$. It is given that f is twice differentiable and that $$\int_0^{\pi}\left[\mathrm{f}(x)+\mathrm{f}''(x)\right]\sin x\,\mathrm{d}x=b$$ where b is a real number. By considering $\int_0^{\pi}\mathrm{f}''(x)\sin x\,\mathrm{d}x$, or otherwise, determine the value of $\mathrm{f}(\pi)$, expressing your answer in terms of a and b.
	$\int_0^{\pi}\mathrm{f}''(x)\sin x\,\mathrm{d}x=\left[\mathrm{f}'(x)\sin x\right]_0^{\pi}-\int_0^{\pi}\mathrm{f}'(x)\cos x\,\mathrm{d}x$ $=0-\left[\left[\mathrm{f}(x)\cos x\right]_0^{\pi}-\int_0^{\pi}-\mathrm{f}(x)\sin x\,\mathrm{d}x\right]$ $=-\left[-\mathrm{f}(\pi)-a+\int_0^{\pi}\mathrm{f}(x)\sin x\,\mathrm{d}x\right]=\mathrm{f}(\pi)+a-\int_0^{\pi}\mathrm{f}(x)\sin x\,\mathrm{d}x$ Rearranging, $\int_0^{\pi}\mathrm{f}''(x)\sin x\,\mathrm{d}x+\int_0^{\pi}\mathrm{f}(x)\sin x\,\mathrm{d}x=\mathrm{f}(\pi)+a$ $b=\mathrm{f}(\pi)+a$ $\mathrm{f}(\pi)=b-a$

2.	Prove that $\int_a^b\mathrm{f}(x)\,\mathrm{d}x=\int_a^b\mathrm{f}(a+b-t)\,\mathrm{d}t$ for any function f which is continuous on the domain $[a,b]$. Using the result in **(i)**, show that $\int_0^{\frac{\pi}{4}}\ln\left(1+\tan\theta\right)\,\mathrm{d}\theta=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan\theta}\right)\,\mathrm{d}\theta.$ Hence, by using an appropriate substitution, find the exact value of $\int_0^1\frac{\ln\left(1+x\right)}{1+x^2}\,\mathrm{d}x.$ Use the result in **(i)** to also find the exact value of $\int_0^{\frac{\pi}{4}}\frac{\cos x}{\cos\left(x-\frac{\pi}{4}\right)}\,\mathrm{d}x.$
	Using the substitution $y=a+b-x$, we have $\int_a^b\mathrm{f}(x)\,\mathrm{d}x=\int_b^a-\mathrm{f}(a+b-y)\,\mathrm{d}y$ $=\int_a^b\mathrm{f}(a+b-y)\,\mathrm{d}y$ $=\int_a^b\mathrm{f}(a+b-x)\,\mathrm{d}x$ (shown)

	$\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\left(\frac{\pi}{4}\right)-\tan\theta}{1+\tan\left(\frac{\pi}{4}\right)\tan\theta}\right)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$ (shown)
	$\int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta$ by using the substitution $x = \tan\theta$. Also, we have $\int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$. Summing up both equalities, we have $2\int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\ \mathrm{d}\theta + \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)+\ln 2-\ln(1+\tan\theta)\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \ln 2\ \mathrm{d}\theta = \frac{\pi}{4}\ln 2$ $\therefore \int_0^1 \frac{\ln(1+x)}{1+x^2}\ \mathrm{d}x = \frac{\pi}{8}\ln 2$

	$$\int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos\left(x-\frac{\pi}{4}\right)}\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}-x\right)}{\cos\left(\left(\frac{\pi}{4}-x\right)-\frac{\pi}{4}\right)}\,\mathrm{d}x$$ $$= \int_0^{\frac{\pi}{4}} \frac{\cos\left(\frac{\pi}{4}\right)\cos x+\sin\left(\frac{\pi}{4}\right)\sin x}{\cos(-x)}\,\mathrm{d}x$$ $$= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x}{\cos x}\,\mathrm{d}x \quad (\cos x \text{ is an even function})$$ $$= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan x\,\mathrm{d}x$$ $$= \left[\frac{x}{\sqrt{2}}+\frac{1}{\sqrt{2}}\ln\lvert\sec x\rvert\right]_0^{\frac{\pi}{4}}$$ $$= \left(\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}\ln\left\lvert\sec\left(\frac{\pi}{4}\right)\right\rvert\right)-0$$ $$= \frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}\ln\left\lvert\sqrt{2}\right\rvert$$ $$= \frac{\pi}{4\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln 2$$

3.	It is given that n is a positive integer. Using the identity $$1-x^n \equiv (1-x)\left(1+x+x^2+\cdots+x^{n-1}\right)$$ or otherwise, show that $\dfrac{1}{x^n(1-x)} \equiv \dfrac{1}{1-x}+\sum_{r=1}^{n}\dfrac{1}{x^r}$. Hence find $\displaystyle\int \frac{1}{x^n(1-x)}\,\mathrm{d}x$. Using a similar method, or otherwise, find (i) $\displaystyle\int \frac{1}{x^{2n}\left(1-x^2\right)}\,\mathrm{d}x$ (ii) $\displaystyle\int \frac{1}{x\left(1+x^2\right)^n}\,\mathrm{d}x$

$$LHS = \frac{1}{x^n(1-x)}$$
$$= \frac{1-x^n+x^n}{x^n(1-x)}$$
$$= \frac{1-x^n}{x^n(1-x)} + \frac{1}{1-x}$$
$$= \frac{1+x+\cdots+x^{n-1}}{x^n} + \frac{1}{1-x}$$
$$= \sum_{r=1}^{n} \frac{1}{x^r} + \frac{1}{1-x}$$
$$= RHS$$

Hence,

$$\int \frac{1}{x^n(1-x)}\,\mathrm{d}x = \int \sum_{r=1}^{n} \frac{1}{x^r} + \frac{1}{1-x}\,\mathrm{d}x$$
$$= \sum_{r=1}^{n} \int \frac{1}{x^r}\,\mathrm{d}x + \int \frac{1}{1-x}\,\mathrm{d}x$$
$$= \begin{cases} \ln|x| - \ln|1-x| + C, & n=1 \\ \ln|x| - \ln|1-x| - \sum_{r=1}^{n-1} \frac{1}{rx^r} + C, & n \geq 2 \end{cases}$$

[Substituting the variable x within a "dx" integral also requires that the final x be substituted. In this question, since the phrasing is "using a similar method", the idea is to do the substitution before taking integrals.]

(i)

Replace x with x^2 in $\frac{1}{x^n(1-x)} = \sum_{r=1}^{n} \frac{1}{x^r} + \frac{1}{1-x}$:

$$\frac{1}{x^{2n}(1-x^2)} = \sum_{r=1}^{n} \frac{1}{x^{2r}} + \frac{1}{1-x^2}$$
$$\int \frac{1}{x^{2n}(1-x^2)}\,\mathrm{d}x = \int \sum_{r=1}^{n} \frac{1}{x^{2r}} + \frac{1}{1-x^2}\,\mathrm{d}x$$
$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| - \sum_{r=1}^{n} \frac{1}{(2r-1)x^{2r-1}} + C$$

(ii)

Replace x with $1+x^2$ in $\frac{1}{x^n(1-x)} = \sum_{r=1}^{n} \frac{1}{x^r} + \frac{1}{1-x}$:

	$\dfrac{1}{(-x^2)(1+x^2)^n}=\sum_{r=1}^{n}\dfrac{1}{\left(1+x^2\right)^r}+\dfrac{1}{-x^2}$ Hence, $\displaystyle\int\frac{1}{x\left(1+x^2\right)^n}\,\mathrm{d}x=\int\frac{-x}{\left(-x^2\right)\left(1+x^2\right)^n}\,\mathrm{d}x$ $\displaystyle=\int\frac{1}{x}-\sum_{r=1}^{n}\frac{x}{\left(1+x^2\right)^r}\,\mathrm{d}x$ $\displaystyle=\begin{cases}\ln\lvert x\rvert-\frac{1}{2}\ln\left(1+x^2\right)+C, & n=1\\ \ln\lvert x\rvert-\frac{1}{2}\ln\left(1+x^2\right)+\sum_{r=1}^{n-1}\frac{1}{2r\left(1+x^2\right)^r}+C, & n\geq 2\end{cases}$

4.	Evaluate $\lim_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$.
	$\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\frac{x\ln x-x+1}{\left(x-1\right)\ln x}$ $\displaystyle=\lim_{x\to 1}\frac{\ln x+1-1}{\ln x+\frac{x-1}{x}}$ using L'Hopital's Rule $\displaystyle=\lim_{x\to 1}\frac{\ln x}{\ln x+1-\frac{1}{x}}$ $\displaystyle=\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}$ using L'Hopital's Rule $\displaystyle=\frac{1}{2}$

5.	Let n be a positive integer. A sequence of functions f_n is defined as $$\mathrm{f}_n(x)=\frac{\int_0^x\left(1-t^4\right)^n\,\mathrm{d}t}{\int_0^1\left(1-t^4\right)^n\,\mathrm{d}t}\ \text{ for } -1\leq x\leq 1.$$ **(a)** Show that $\mathrm{f}_n(x)$ is an odd function. [2] **(b)** Show that $\mathrm{f}_n(x)$ is an increasing function of x for $0<x<1$. [2] **(c)** Show that the gradient of $\mathrm{f}_n(x)$ is a decreasing function of x for $0<x<1$. [1] **(d)** Sketch the graph of $\mathrm{f}_n(x)$. [2]

Let k_n be the constant $\dfrac{1}{\int_0^1 \left(1-t^4\right)^n \, \mathrm{d}t}$ for each n. Then $\mathrm{f}_n(x) = k_n \int_0^x \left(1-t^4\right)^n \, \mathrm{d}t$.

(a)
Using the substitution $s = -t$,
$\mathrm{f}_n(x) = k_n \int_0^x \left(1-t^4\right)^n \, \mathrm{d}t$
$= k_n \int_0^{-x} \left(1-(-s)^4\right)^n (-1) \, \mathrm{d}s$
$= -k_n \int_0^{-x} \left(1-s^4\right)^n \, \mathrm{d}s$
which has the same numerical value as $-k_n \int_0^{-x} \left(1-t^4\right)^n \, \mathrm{d}t = -\mathrm{f}_n(-x)$
Hence f_n is an odd function.

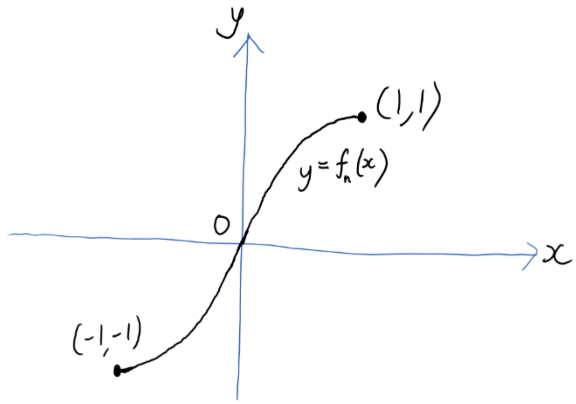
(b) We know that $0 < \left(1-t^4\right)^n < 1$ when $0 < t < 1$, for any n. So, for any x_1, x_2 satisfying $0 < x_1 < x_2 < 1$,

$$\begin{aligned}\mathrm{f}_n(x_2) - \mathrm{f}_n(x_1) &= k_n \left[\int_0^{x_2} \left(1-t^4\right)^n \, \mathrm{d}t - \int_0^{x_1} \left(1-t^4\right)^n \, \mathrm{d}t \right] \\ &= k_n \int_{x_1}^{x_2} \left(1-t^4\right)^n \, \mathrm{d}t \\ &> k_n \int_{x_1}^{x_2} 0 \, \mathrm{d}t \\ &= 0\end{aligned}$$

Thus, f_n is an increasing function for $0 < x < 1$.

(c)
The gradient function is $\mathrm{f}_n'(x) = k_n \left(1-x^4\right)^n$ [by fundamental theorem]
Since
$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[k_n \left(1-x^4\right)^n \right] = -4nk_n \left(1-x^4\right)^{n-1} x^3$
is negative for $0 < x < 1$ $\because n > 0, k_n > 0, \left(1-x^4\right)^{n-1} > 0, x > 0$
we conclude that the gradient function is a decreasing of x for $0 < x < 1$.

(d)



[Drawn graph must look like it has rotational symmetry about origin.]

6.	Show that, for $m > 0$, $$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m.$$ Show by means of substitution $x = \frac{1}{u}$, that $$\int_b^a \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/a}^{1/b} \frac{u^{n-1}}{u+1}\,\mathrm{d}u.$$ Hence or otherwise, without using calculator, evaluate $$\int_1^2 \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x.$$
	$$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \int_{1/m}^{m} (x-1) + \frac{1}{x+1}\,\mathrm{d}x$$ $$= \left[\frac{x^2}{2} - x + \ln(x+1)\right]_{1/\mathrm{m}}^{m}$$ $$= \frac{m^4 - 1 - 2m(m^2-1)}{2m^2} + \ln\left(\frac{m+1}{\frac{1}{m}+1}\right)$$ $$= \frac{(m^2-1)(m^2-2m+1)}{2m^2} + \ln\left(\frac{m^2+m}{m+1}\right)$$ $$= \frac{(m-1)^3(m+1)}{2m^2} + \ln m$$

	$x=\frac{1}{u},\ \frac{\mathrm{d}x}{\mathrm{d}u}=-\frac{1}{u^2}$ $\int_b^a \frac{1}{x^n(x+1)}\,\mathrm{d}x=\int_{1/b}^{1/a}\frac{1}{\left(\frac{1}{u}\right)^n\left(\frac{1}{u}+1\right)}\left(-\frac{1}{u^2}\right)\mathrm{d}u$ $=-\int_{1/b}^{1/a}\frac{u^{n-2}}{\left(\frac{1}{u}+1\right)}\,\mathrm{d}u=\int_{1/a}^{1/b}\frac{u^{n-2}}{\left(\frac{1}{u}+1\right)}\,\mathrm{d}u$ $=\int_{1/a}^{1/b}\frac{u^{n-1}}{(u+1)}\,\mathrm{d}u$
	$\int_1^2 \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x=\int_1^2\frac{x^5}{x^3(x+1)}\,\mathrm{d}x+\int_1^2\frac{x^3}{x^3(x+1)}\,\mathrm{d}x+\int_1^2\frac{1}{x^3(x+1)}\,\mathrm{d}x$ $=\int_1^2\frac{x^2}{(x+1)}\,\mathrm{d}x+\int_1^2\frac{1}{(x+1)}\,\mathrm{d}x+\int_1^2\frac{1}{x^3(x+1)}\,\mathrm{d}x$ $=\int_1^2\frac{x^2}{(x+1)}\,\mathrm{d}x+\left[\ln(1+x)\right]_1^2+\int_1^2\frac{1}{x^3(x+1)}\,\mathrm{d}x$ $=\int_1^2\frac{x^2}{(x+1)}\,\mathrm{d}x+\ln\frac{3}{2}+\int_{1/2}^1\frac{u^2}{(u+1)}\,\mathrm{d}u$ $=\int_{1/2}^2\frac{x^2}{(x+1)}\,\mathrm{d}x+\ln\frac{3}{2}$ $=\frac{(2-1)^3(2+1)}{2\times 2^2}+\ln 2+\ln\frac{3}{2}$ $=\frac{3}{8}+\ln 3$

7.	Throughout this question, n denotes a non-negative integer. Let $I_n=\int_0^{\pi}\frac{\cos nx}{5-4\cos x}\,\mathrm{d}x$. Prove that $I_n=\begin{cases}\int_0^{\pi}\frac{\cos nx}{5+4\cos x}\,\mathrm{d}x, & n \text{ is even,}\\ -\int_0^{\pi}\frac{\cos nx}{5+4\cos x}\,\mathrm{d}x, & n \text{ is odd.}\end{cases}$ Hence or otherwise, **(i)** find, in the case when n is even, the exact value of	[3]

$$\int_0^{\pi} \frac{\cos x \cos nx}{25-16\cos^2 x}\,\mathrm{d}x \ ,$$

showing your working clearly, [3]

(ii) show that

$$\int_0^{\pi} \frac{\cos^2 x}{25-16\cos^2 x}\,\mathrm{d}x = \frac{1}{4}\int_0^{\pi} \frac{\cos x}{5-4\cos x}\,\mathrm{d}x$$

and use the substitution $t = \tan\frac{x}{2}$ to evaluate the integral

$$\int_0^{\pi} \frac{\cos^2 x}{25-16\cos^2 x}\,\mathrm{d}x$$

Exactly, giving your answer in the form $k\pi$ where k is a rational number to be determined. [7]

$$I_n = \int_0^{\pi} \frac{\cos nx}{5-4\cos x}\,\mathrm{d}x$$

$$= \int_{\pi}^{0} \frac{\cos\left[n(\pi-t)\right]}{5-4\cos(\pi-t)}\cdot(-1)\,\mathrm{d}t$$

$$= \int_0^{\pi} \frac{\cos\left[n(\pi-t)\right]}{5-4\cos(\pi-t)}\,\mathrm{d}t$$

$$= \int_0^{\pi} \frac{\cos n\pi \cos nt + \sin n\pi \sin nt}{5+4\cos t}\,\mathrm{d}t$$

$$= \int_0^{\pi} \frac{\cos n\pi \cos nt}{5+4\cos t}\,\mathrm{d}t$$

$$= \cos n\pi \int_0^{\pi} \frac{\cos nt}{5+4\cos t}\,\mathrm{d}t$$

$$= \cos n\pi \int_0^{\pi} \frac{\cos nx}{5+4\cos x}\,\mathrm{d}x$$

$$= \begin{cases} \displaystyle\int_0^{\pi} \frac{\cos nx}{5+4\cos x}\,\mathrm{d}x, & n \text{ is even} \\ \displaystyle -\int_0^{\pi} \frac{\cos nx}{5+4\cos x}\,\mathrm{d}x, & n \text{ is odd} \end{cases}$$

(i)

[It is tempting to write the integrand as a product, but then what would we do with it? We cannot easily split it that way since $\int uv\,\mathrm{d}x \neq \left(\int u\,\mathrm{d}x\right)\cdot\left(\int v\,\mathrm{d}x\right)$ in general.]

$$\int_0^{\pi} \frac{\cos x \cos nx}{25-16\cos^2 x}\,\mathrm{d}x = \int_0^{\pi} \frac{\cos x \cos nx}{(5-4\cos x)(5+4\cos x)}\,\mathrm{d}x$$

Using an idea similar to partial fractions, we can write

$$\frac{\cos x}{(5-4\cos x)(5+4\cos x)} = \frac{A}{5-4\cos x} + \frac{B}{5+4\cos x}$$

for some constants A, B. Re-writing the RHS as a single fraction and comparing coefficients, we obtain $A = \frac{1}{8}, B = -\frac{1}{8}$, and so

$$\int_0^{\pi} \frac{\cos x \cos nx}{25-16\cos^2 x}\,\mathrm{d}x = \int_0^{\pi} \frac{\cos nx}{8(5-4\cos x)} - \frac{\cos nx}{8(5+4\cos x)}\,\mathrm{d}x$$

$$= \int_0^{\pi} \frac{\cos nx}{8(5+4\cos x)} - \frac{\cos nx}{8(5+4\cos x)}\,\mathrm{d}x \qquad \text{(using the previous result)}$$

$$= 0$$

when n is even.

(ii)

$$\int_0^{\pi} \frac{\cos^2 x}{25-16\cos^2 x}\,\mathrm{d}x = \int_0^{\pi} \frac{\cos x \cos x}{(5-4\cos x)(5+4\cos x)}\,\mathrm{d}x$$

$$= \int_0^{\pi} \frac{\cos x}{8(5-4\cos x)} - \frac{\cos x}{8(5+4\cos x)}\,\mathrm{d}x$$

$$= \int_0^{\pi} \frac{\cos x}{8(5-4\cos x)} + \frac{\cos x}{8(5-4\cos x)}\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^{\pi} \frac{\cos x}{5-4\cos x}\,\mathrm{d}x$$

where the second step uses the same splitting as in (i).

When $t = \tan\frac{x}{2}$,

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2} \quad \text{and}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1+t^2\right) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2}{1+t^2}, \text{ so}$$

$$\int_0^{\pi} \frac{\cos^2 x}{25-16\cos^2 x}\,\mathrm{d}x = \frac{1}{4}\int_0^{\pi} \frac{\cos x}{5-4\cos x}\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^{\infty} \frac{\left(\frac{1-t^2}{1+t^2}\right)}{5-4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$$

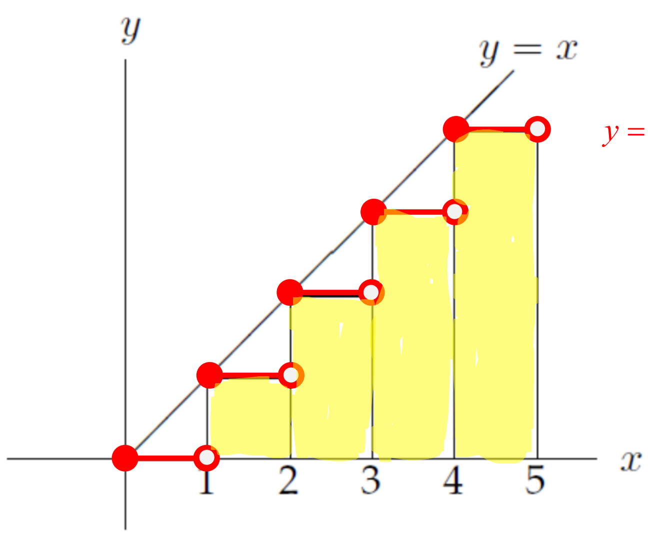
	$= \frac{1}{2}\int_0^\infty \frac{1-t^2}{(1+t^2)(1+9t^2)}\,\mathrm{d}t$ $= \frac{1}{2}\int_0^\infty \frac{-1}{4(1+t^2)} + \frac{5}{4(1+9t^2)}\,\mathrm{d}t$ $= -\frac{1}{8}\left[\tan^{-1} t\right]_0^\infty + \frac{5}{24}\left[\tan^{-1} 3t\right]_0^\infty = \frac{\pi}{24}$

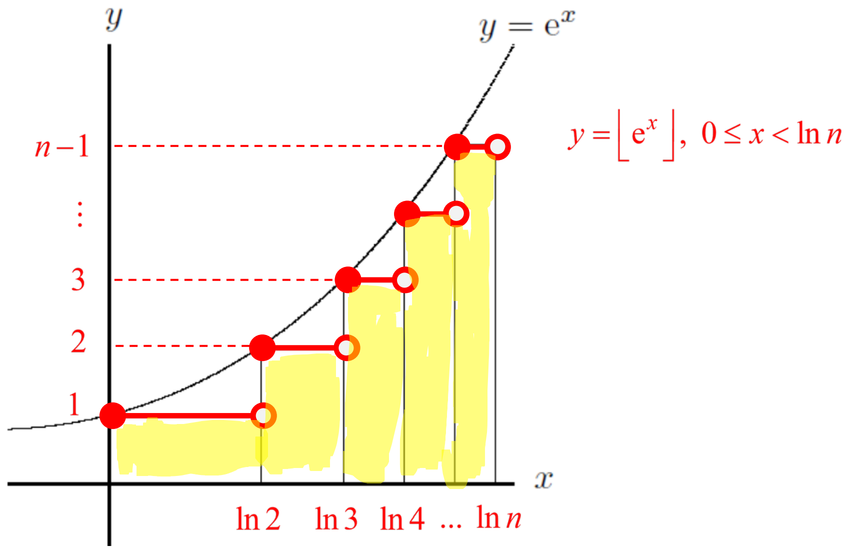
8.	Given $f:\mathbb{R}\to\mathbb{R}$ such that for all $x, y \in \mathbb{R}\backslash\{0\}$, $f(xy) = f(x) + f(y)$. Find the value of $f(1)$ and $f(-1)$. Hence show that *f* is even. Give an example of a (non-constant) function that satisfies the above properties and sketch its graph.
	$f(1) = f(1) + f(1)$ $\Rightarrow f(1) = 0$ $f(1) = f(-1) + f(-1)$ $\Rightarrow 0 = 2f(-1)$ $\Rightarrow f(-1) = 0$ For all $x \neq 0$, $f(-x) = f((-1)(x))$ $= f(-1) + f(x)$ $= 0 + f(x)$ $= f(x)$ and so *f* is even. A possible function is $f(x) = \ln\lvert x\rvert$

9.	Let $\mathrm{f}:\mathbb{Q}\to\mathbb{Q}$ be a function satisfying the functional equation $$\mathrm{f}(x+y) = \mathrm{f}(x) + \mathrm{f}(y) \qquad \forall x, y \in \mathbb{Q}$$ **(i)** Verify that $\mathrm{f}(0) = 0$. [1] A function g is known as an odd function if it satisfies $\mathrm{g}(-x) = -\mathrm{g}(x)$. **(ii)** Show that f is an odd function. [2] **(iii)** Using Mathematical induction, show that $\mathrm{f}(nx) = n\mathrm{f}(x)$ for all $n \in \mathbb{Z}^+$. [3] **(iv)** Using the result in part (iii) or otherwise, show that $\mathrm{f}\left(\frac{x}{m}\right) = \frac{1}{m}\mathrm{f}(x)$ for all $m \in \mathbb{Z}^+$. [2]

	(v) Given that the graph $y=\mathrm{f}(x)$ passes through (1, 3), show that $\mathrm{f}(x)=3x,\ x\in\mathbb{Q}$ is the solution to the function equation. [2]
	(i) We have $\mathrm{f}(0)=\mathrm{f}(0)+\mathrm{f}(0)$ so subtracting $\mathrm{f}(0)$ from both sides yields the result. (ii) Substituting $y=-x$ we get $0=\mathrm{f}(0)=\mathrm{f}(x)+\mathrm{f}(-x)$ i.e. $\mathrm{f}(x)=-\mathrm{f}(-x)$ so f is an odd function. (iii) Let P_n be the statement $\mathrm{f}(nx)=n\mathrm{f}(x)$ for positive integers *n*. P_1 is true since both sides are equal to $\mathrm{f}(x)$. Suppose P_k is true, then consider P_{k+1}. $LHS=\mathrm{f}((n+1)x)=\mathrm{f}(nx+x)=\mathrm{f}(nx)+\mathrm{f}(x)=n\mathrm{f}(x)+\mathrm{f}(x)=(n+1)\mathrm{f}(x)=RHS$ so P_{k+1} is true. Since P_1 is true, and $P_k\Rightarrow P_{k+1}$, by Mathematical Induction, P_n is true for all positive integer *n*. (iv) First set $n=m$ in the result of part (iii), obtaining $\mathrm{f}(mx)=m\mathrm{f}(x)$. Since *x* is an arbitrary rational number, we can replace *x* with $\frac{x}{m}$ which yields $\mathrm{f}(x)=m\mathrm{f}\left(\frac{x}{m}\right)$. Dividing both sides of this equation by *m* gives the required result. (v) We divide into 3 cases: (A) *x* is positive (B) *x* is negative (c) *x* is zero. Case (A): Let $x\in\mathbb{Q}^+$. Since *x* is rational, let $x=\frac{p}{q}$ where $p,q\in\mathbb{Z}^+$. Then $\mathrm{f}(x)=\mathrm{f}\left(\frac{p}{q}\right)$ $=\frac{1}{q}\mathrm{f}(p)$ (using part (iv)) $=\frac{1}{q}\cdot p\mathrm{f}(1)$ (using part (iii)) $=3\cdot\frac{p}{q}=3x$.

	Case (B): x is negative From part (ii), since f is odd, if $x \in \mathbb{Q}^-$ then $\mathrm{f}(x) = -\mathrm{f}(-x) = -(-3x) = 3x$ using the result in Case (A). Case (C): if $x = 0$ then $\mathrm{f}(0) = 0 = 3 \cdot 0$. So we have shown $\mathrm{f}(x) = 3x$ for all 3 cases.

10	For any real number x, the largest integer less than or equal to x is denoted by $\lfloor x \rfloor$. For example, $\lfloor 3.7 \rfloor = 3$, $\lfloor 5 \rfloor = 5$, and $\lfloor -\pi \rfloor = -4$. **(i)** Use a sketch graph of $y = \lfloor x \rfloor$ to evaluate $\int_0^5 \lfloor x \rfloor \, \mathrm{d}x$. [2] **(ii)** Use a sketch graph of $y = \lfloor \mathrm{e}^x \rfloor$ to show that $\int_0^{\ln n} \lfloor \mathrm{e}^x \rfloor \, \mathrm{d}x = n \ln n - \ln(n!)$, where n is an integer [3] **(iii)** Hence, show that $n! > n^n \mathrm{e}^{1-n}$. [3]
	(i)  $\int_0^5 \lfloor x \rfloor \mathrm{d}x =$ Sum of areas of the rectangles shown $= 0+1+2+3+4$ $= 10$ (ii)



$\int_0^{\ln n} \lfloor e^x \rfloor dx$

$=$ Sum of areas of the rectangles shown

$= 1(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) + 3(\ln 4 - \ln 3) + ... + (n-1)\left(\ln n - \ln(n-1)\right)$

$= -\ln 1 - \ln 2 - \ln 3 - \ldots - \ln(n-1) + (n-1)\ln n$

$= -\left(\ln 1 + \ln 2 + \ln 3 + \ldots + \ln n\right) + n\ln n$

$= n\ln n - \ln\left(1\cdot 2\cdot 3\cdot \ldots \cdot n\right) = n\ln n - \ln\left(n!\right)$

(iii) Since $\lfloor e^x \rfloor \le e^x$,

$$\int_0^{\ln n} \lfloor e^x \rfloor dx \le \int_0^{\ln n} e^x dx$$

$$\Rightarrow \quad -\ln(n!) + n\ln n \le \left[e^x\right]_{x=0}^{x=\ln n}$$

$$= e^{\ln n} - e^0$$

$$= n - 1$$

$$\Rightarrow \quad \ln(n!) \ge n\ln n + 1 - n$$

$$\ln(n!) \ge \ln n^n + \ln e^{1-n}$$

$$\ln(n!) \ge \ln(n^n e^{1-n})$$

$$n! \ge n^n e^{1-n} \quad \text{(shown)}$$

11	Let $I_n = \int_0^{\frac{1}{4}\pi} \tan^n x \, dx$. (i) For $n > 1$, prove that $I_n + I_{n-2} = \dfrac{1}{n-1}$ [4]

	(ii) Justify the statement that $\tan x \le \frac{4}{\pi} x$ on $\left[0, \frac{1}{4}\pi\right]$. [2] (iii) Hence prove that I_n tends to zero as n tends to infinity. [3] (iv) Find the sum of the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$. [4]
	(i) $I_n + I_{n-2} = \int_0^{\frac{1}{4}\pi} \tan^n x + \tan^{n-2} x \, \mathrm{d}x$ $= \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \left(1 + \tan^2 x\right) \mathrm{d}x$ $= \int_0^{\frac{1}{4}\pi} \tan^{n-2} x \sec^2 x \, \mathrm{d}x$ $= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{1}{4}\pi}$ $= \frac{1}{n-1}$ (ii) The function $\mathrm{f}(x) = \tan x$ is convex in the interval $\left[0, \frac{1}{4}\pi\right]$ because $\mathrm{f}''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^2 x\right) = 2\sec^2 x \tan x \ge 0$ in the interval. Therefore, the line joining the endpoints $(0, 0)$ and $\left(\frac{\pi}{4}, 1\right)$ lies completely above the graph $y = \tan x$. But this line is simply $y = \frac{4}{\pi} x$, thus $\tan x \le \frac{4}{\pi} x$. (iii) Since $0 \le \tan x \le \frac{4}{\pi} x$ in the interval $\left[0, \frac{1}{4}\pi\right]$, integrating over this interval yields $\int_0^{\frac{1}{4}\pi} 0 \, \mathrm{d}x \le \int_0^{\frac{1}{4}\pi} \tan^n x \, \mathrm{d}x \le \int_0^{\frac{1}{4}\pi} \left(\frac{4x}{\pi}\right)^n \mathrm{d}x$ $0 \le I_n \le \left(\frac{4}{\pi}\right)^n \left[\frac{x^{n+1}}{n+1}\right]_0^{\frac{1}{4}\pi}$ $0 \le I_n \le \left(\frac{4}{\pi}\right)^n \left(\frac{\pi}{4}\right)^{n+1} \left(\frac{1}{n+1}\right)$ $0 \le I_n \le \frac{\pi}{4(n+1)}$ As $n \to \infty$, the RHS tends to zero as well, so by squeeze theorem $I_n \to 0$. (iv) This infinite series is $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}$.

	Since each term $\dfrac{(-1)^{k+1}}{2k-1}$ can be written as $(-1)^{k+1}\left(I_{2k-2}+I_{2k}\right)$, the partial sums of the first n terms is given by $\displaystyle\sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k-1}=\sum_{k=1}^{n}\left[(-1)^{k+1}\left(I_{2k-2}+I_{2k}\right)\right]$ $=I_0+(-1)^{n+1}I_{2n}$ since all the intermediate terms cancel using Method of Differences Taking $n\to\infty$, $(-1)^{n+1}I_{2n}\to 0$, and so $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=I_0=\int_0^{\frac{1}{4}\pi}1\,\mathrm{d}x=\dfrac{\pi}{4}$

12.	The differential equation $$y\frac{\mathrm{d}y}{\mathrm{d}x}=x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1,\quad \text{for } x>0 \qquad (1)$$ has a solution curve S such that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is non-zero at all points of S. (i) By substituting $t=\dfrac{\mathrm{d}y}{\mathrm{d}x}$ into equation (1) and then differentiating with respect to x, show that S has equation $y^2=4x$. [6] (ii) Show that a straight line is a tangent to the curve S if and only if it is itself a solution of equation (1). [6]
	(i) $yt=xt^2+1$ Differentiating, $t^2+y\dfrac{\mathrm{d}t}{\mathrm{d}x}=t^2+2xt\dfrac{\mathrm{d}t}{\mathrm{d}x}$ $\left(y-2xt\right)\dfrac{\mathrm{d}t}{\mathrm{d}x}=0$ $\left(y-2xt\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ Since $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is non-zero on S, $y=2xt$ $t=\dfrac{y}{2x}$. Substituting this into $yt=xt^2+1$, we get $\dfrac{y^2}{2x}=\dfrac{xy^2}{4x^2}+1$ $\dfrac{y^2}{2x}=\dfrac{y^2}{4x}+1$

$$\frac{y^2}{4x}=1$$

$y^2=4x$ (shown)

(ii) Differentiating $y^2=4x$ with respect to x produces $2y\frac{\mathrm{d}y}{\mathrm{d}x}=4$.

Let (x_0,y_0), $x_0\neq 0$ be a point on S. The equation of ℓ, the tangent to S at that point is

$$\ell: y-y_0=\left(\frac{2}{y_0}\right)(x-x_0)$$

$$\ell: y=\left(\frac{2}{y_0}\right)x+\left(y_0-\frac{2x_0}{y_0}\right)$$

Since (x_0,y_0) lies on S, we have $x_0=\frac{y_0^2}{4}$, and so

$$\ell: y=\left(\frac{2}{y_0}\right)x+\left(\frac{y_0}{2}\right)$$

The equation of ℓ thus satisfies

$$ym_\ell=y\left(\frac{2}{y_0}\right)$$

$$=\left(\frac{2x}{y_0}\right)\left(\frac{2}{y_0}\right)+\left(\frac{y_0}{2}\right)\left(\frac{2}{y_0}\right)$$

$$=x\left(\frac{2}{y_0}\right)^2+1$$

$=x(m_\ell)^2+1$ where m_ℓ is the gradient of ℓ

i.e. ℓ satisfies the given differential equation.

Conversely, suppose a line L satisfies the given differential equation. Let the line have gradient t. Then, substituting into the given D.E., we have

$$yt=xt^2+1\Rightarrow y=tx+\frac{1}{t}$$

If L is tangent to the S, since S satisfies $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{y}$, L must be tangent to S at

$$t=\frac{2}{y}\Leftrightarrow y=\frac{2}{t}.$$

Indeed, from the above, the equation of the tangent to S at (x_0,y_0) is

$$\ell: y=\left(\frac{2}{y_0}\right)x+\left(\frac{y_0}{2}\right).$$

Substituting $y_0 = \dfrac{2}{t}$, we get $\ell : y = tx + \dfrac{1}{t}$, which is exactly the equation of L.

So we have checked that L is tangent to S at $y = \dfrac{2}{t}$, where t is the gradient of L.

13. Find the general solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(c+t)x}{1-t^2}, \qquad \text{(A)}$$

where c is a constant.

Make the substitution $y = tx$ in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax+by}{bx+ay}, \qquad \text{(B)}$$

where a and b are constants and $a \neq 0$, and show that the resulting equation is equivalent to equation (A).

Hence show that equation (B) has solutions of the form

$$|x-y|^{b+a} = D|x+y|^{b-a},$$

where D is an arbitrary constant.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(c+t)x}{1-t^2}$$

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(c+t)}{1-t^2}$$

$$\int \frac{1}{x}\,\mathrm{d}x = \int \frac{c}{1-t^2} - \frac{1}{2}\frac{-2t}{1-t^2}\,\mathrm{d}t$$

$$\ln|x| - d = \frac{c}{2}\ln\left|\frac{1+t}{1-t}\right| - \frac{1}{2}\ln\left|1-t^2\right|$$

$$\ln\left|\frac{x}{k}\right| = \ln\left|\frac{1+t}{1-t}\right|^{\frac{c}{2}} - \ln\left|(1+t)(1-t)\right|^{\frac{1}{2}}$$

$$\left|\frac{x}{k}\right| = \frac{|1+t|^{\frac{c-1}{2}}}{|1-t|^{\frac{c+1}{2}}}$$

$$x = \frac{p|1+t|^{\frac{c-1}{2}}}{|1+t|^{\frac{c+1}{2}}},$$ where p is an arbitrary constant.

$$y = tx$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = t + x\frac{\mathrm{d}t}{\mathrm{d}x}$$

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$$t+x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{ax+b\left(tx\right)}{bx+a\left(tx\right)}$$

$$t+x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a+b\left(t\right)}{b+at}$$

$$x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a+b\left(t\right)}{b+at}-t$$

$$x\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{a-at^2}{b+at}$$

$$\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{b+at}{a\left(1-t^2\right)}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\left(\frac{b}{a}+t\right)x}{1-t^2}$$

and the resulting equation is equivalent to equation (A) with $c=\frac{b}{a}$.

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$$\left|\frac{x}{k}\right|=\frac{\left|1+\frac{y}{x}\right|^{\frac{\frac{b}{a}-1}{2}}}{\left|1-\frac{y}{x}\right|^{\frac{\frac{b}{a}+1}{2}}}$$

$$|x||x-y|^{\frac{\frac{b}{a}+1}{2}}|x|^{-\frac{\frac{b}{a}+1}{2}}=|k||x+y|^{\frac{\frac{b}{a}-1}{2}}|x|^{-\frac{\frac{b}{a}-1}{2}}$$

$$|x||x-y|^{\frac{\frac{b}{a}+1}{2}}=|k||x+y|^{\frac{\frac{b}{a}-1}{2}}|x|^{\frac{\frac{b}{a}+1}{2}-\frac{\frac{b}{a}-1}{2}}$$

$$|x||x-y|^{\frac{b+a}{2a}}=|k||x+y|^{\frac{b-a}{2a}}|x|$$

$$|x||x-y|^{b+a}=|k|^{2a}|x+y|^{b-a}|x|$$

$$|x-y|^{b+a}=D|x+y|^{b-a}$$

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