MOE H3 Math Mathematical Proofs and Reasoning

Problem Set 2

1. Is each of the following statements true or false? Give a proof if it is true, and give a counter-example if it is false.
   (a) For each pair of real numbers x and y, if $x + y$ is irrational, then x is irrational and y is irrational.
   (b) For each pair of real numbers x and y, if $x + y$ is irrational, then x is irrational or y is irrational.
   (c) For each pair of nonzero real numbers x and y, if x is rational and y is irrational, then xy is irrational.

2. (Y2023 Q5)
   (a) Let p be a prime number greater than 2. Write down the possible remainders of p when divided by 4.
   (b) Fermat's Little Theorem states that if p is prime and a is an integer which is not divisible by p, then $a^{p-1} \equiv 1$ (modulo p).

      Use Fermat's Little Theorem to prove that if p is a prime number greater than 2, and there exists an integer z such that $z^2 \equiv -1$ (modulo p), then p is not congruent to 3 (modulo 4).
   (c) Write down the possible remainders of w^2 when divided by 8 where w is an integer.

3. Unique Factorization Theorem states that:
   Every integer $n > 1$ has a unique standard factored form. i.e. there is exactly one way to express
   $$n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$$
   where $p_1 < p_2 < \cdots < p_t$ are distinct primes and $k_1, k_2, \cdots, k_t$ are some positive integers.

   Use the Unique Factorization Theorem to prove that, if a positive integer n is not a perfect square, then $\sqrt{n}$ is irrational.

4. Prove or disprove the statement
   $$\text{If } a^2 \mid b^2, \text{ then } a \mid b \text{ for all integers } a, b.$$

5. Determine whether each of the following real numbers is rational or irrational. Justify your answers.
   (a) $\sqrt{3} + \sqrt{5}$;
   (b) $\sqrt{2} + \sqrt{8}$;
   (c) $\dfrac{1 + \sqrt{2}}{1 + \sqrt{3}}$.

6. Prove that there does not exists a smallest positive real number.

7. Prove that each finite decimal may be written as an infinite decimal in two distinct ways:

$$a_0.a_1a_2\ldots a_{n-1}a_n \;=\; a_0.a_1a_2\ldots a_{n-1}a_n\widehat{0} \;=\; a_0.a_1a_2\ldots a_{n-1}(a_n-1)\widehat{9},$$

where $a_n > 0$ if $n > 0$. (Here $\widehat{0}$ means an infinite tail of 0's, and $\widehat{9}$ an infinite tail of 9's.)

8. Prove that each real number is represented by a unique infinite decimal unless it is representable by a finite decimal, in which case it is representable by precisely two infinite decimals as described in the previous problem.

9. Prove that if there are no nonzero integer solutions to the equation $x^n + y^n = z^n$, then there are no nonzero rational solutions.

Hints

1. *Hint for Question 2.* Do we have all conditions necessary to apply Fermat's Little Theorem? You may want to consider proof by contradiction or contrapositive.

2. *Hint for Question 3.* You may use the fact that n is a perfect square if and only if its standard factored form is $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ where all the k_i's are even.

3. *Hint for Question 4.* Use prime factorization.

4. *Hint for Question 5.* May use the fact that if a positive integer n is not a perfect square, then $\sqrt{n}$ is irrational.

5. *Hint for Question 8.* Requires induction.