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H2 Mathematics (9758)
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Chapter 7B: Arithmetic Progression and Geometric Progression

SYLLABUS INCLUDES

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- formula for the nth term and the sum of a finite geometric series
- condition for convergence of an infinite geometric series
- formula for the sum to infinity of a convergent geometric series

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INTRODUCTION

One common type of sequence is the **arithmetic progression (AP)**—a sequence in which each term differs from the previous one by a constant amount. For example, saving a fixed amount of money every month or increasing your jogging distance by the same amount each week are everyday situations that involve an AP.

Another important type is the **geometric progression (GP)**, where each term is obtained by multiplying the previous one by a fixed number. GPs appear in many real-world settings too—such as when your savings grow with compound interest, or when a population doubles every few years.

By understanding APs and GPs, you will gain useful tools for recognising patterns, predicting future values, and solving problems that pop up in both personal and financial planning. Let's explore how these special sequences work—and why they matter!

3 SPECIAL SEQUENCE 1: ARITHMETIC PROGRESSION (AP)

3.1 Definition

An **arithmetic progression (AP)** is a sequence of numbers for which the difference between every two consecutive terms is a constant, i.e.

$$\boxed{u_n - u_{n-1} = d}$$

The constant d is called the **common difference.**

Note that we can easily see that AP is a sequence generated by a recurrence relation by writing it as $u_n = u_{n-1} + d$.

3.2 n^{th} term of an Arithmetic Progression

An arithmetic progression with first term a and common difference d can be written as

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

The n^{th} term of the arithmetic progression is

$$\boxed{u_n = a + (n-1)d}$$

Examples

Arithmetic Progression	First Term u_1	Common Difference	Formula for u_n
3, 5, 7, 9, 11, …	3	2	$u_n = 2n+1,\ n=1,2,3,...$
20, 15, 10, 5, 0, … , –30	20	–5	$u_n = 20+(n-1)(-5)$ $= 25-5n,\ n=1,2,3,...11$
$\frac{3}{2},\ 2,\ \frac{5}{2},\ 3,\ \frac{7}{2},\ ...$	$\frac{3}{2}$	$\frac{1}{2}$	$u_n = \frac{3}{2}+(n-1)\left(\frac{1}{2}\right) = 1+\frac{n}{2},$ $n=1,2,3,...$

An **arithmetic series** is formed when the terms of an arithmetic progression are added.
For example, $\underline{3+5+7+9+11+...}$

3.3 Sum of first *n* terms of an Arithmetic Series

We illustrate with a simple example the method to find the sum of $1+2+3+...+8+9+10$.

$$S_{10} = 1+2+3+...+8+9+10 --- (1)$$
$$S_{10} = 10+9+8+...+3+2+1 --- (2)$$

(1) + (2) gives

$$2S_{10} = 11+11+...+11$$
$$2S_{10} = 10(11)$$
$$S_{10} = \frac{110}{2} = 55$$

Let S_n denote the **sum of the first *n* terms** of an arithmetic series with first term a and common difference d .Then

$$\begin{aligned} S_n &= u_1+u_2+u_3+...+u_{n-2}+u_{n-1}+u_n \\ &= a+(a+d)+(a+2d)+...+[a+(n-3)d]+[a+(n-2)d]+[a+(n-1)d] \quad --- (1) \end{aligned}$$

Similarly,

$$\begin{aligned} S_n &= u_n+u_{n-1}+u_{n-2}+...+u_3+u_2+u_1 \\ &= [a+(n-1)d]+[a+(n-2)d]+[a+(n-3)d]+...+(a+2d)+(a+d)+a \quad --- (2) \end{aligned}$$

(1) + (2) gives

$$2S_n = n\left[2a+(n-1)d\right] \quad \Rightarrow \quad S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

Since $u_n = a + (n-1)d$, we can also write S_n as

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}\left[a + a + (n-1)d\right] = \frac{n}{2}\left(u_1 + u_n\right)$$

Sum of the first n terms of an arithmetic series is

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}\left(u_1 + u_n\right)}$$

Remarks: Sum of first *n* terms of an arithmetic series with *a* as its first term and *d* as its common difference can be expressed as $\sum_{k=1}^{n}\left[a + (k-1)d\right]$, which involves a linear expression in terms of *k*. We can apply the sum of first *n* terms formula of an arithmetic series on summation problems involving a general linear term.

To illustrate this, let us revisit Example 5a.

$\sum_{r=0}^{10}(2r+3) = 3 + 5 + 7 + ... + 23$ (Arithmetic Series with 1st term 3 and common difference 3)

$$= \frac{10-0+1}{2}(3+23) = 143$$

This method is more efficient than splitting the series as $\sum_{r=0}^{10}(2r+3) = 2\sum_{r=0}^{10} r + \sum_{r=0}^{10} 3$ which involves application of 2 formulae from Section 1.6 Summation using Standard Results:

(1) $\sum_{r=1}^{n} a = na$.   **(2)** $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Note that $\sum_{r=1}^{n} r$ is also an arithmetic series with *n* terms, first term 1 and common difference 1.

We can easily apply the sum of first *n* terms formula to derive that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Example 9

Find the number of terms in the arithmetic progression 20, 15, 10, … , -135 and the sum of all the terms.

Solution

AP : first term $a = 20$, common difference $d = -5$, number of terms $= n$ $u_n = a + (n-1)d$ $-135 = 20 + (n-1)(-5)$ $n - 1 = \dfrac{155}{5}$ $n = 32$	$S_{32} = \dfrac{n}{2}\left[2a + (n-1)d\right]$ $= \dfrac{32}{2}[2(20) + (32-1)(-5)]$ $= -1840$ OR $S_{32} = \dfrac{n}{2}\left[u_1 + u_n\right]$ $= \dfrac{32}{2}[20 + (-135)]$ $= -1840$

Example 10

In an arithmetic progression, the 8th term is twice the 4th term, and the 20th term is 40.

Find the common difference and the sum of the terms from the 8th to the 20th inclusive.

Solution

AP : first term $= a$, common difference $= d$

$u_8 = 2u_4 \quad \Rightarrow \quad a + 7d = 2(a + 3d) \quad \Rightarrow \quad a = d$

$u_{20} = 40 \quad \Rightarrow \quad a + 19d = 40 \quad \Rightarrow \quad 20d = 40 \quad \Rightarrow \quad a = d = 2$

Sum of the terms from the 8th to the 20th inclusive

$= S_{20} - S_7$

$= \dfrac{20}{2}\left[2(2) + 19(2)\right] - \dfrac{7}{2}\left[2(2) + 6(2)\right] = 364$

Alternatively, we can consider the terms from the 8th to the 20th as a new AP with

1st term, $\quad u_8 = 2 + 7(2) \ = 16$

last term, $u_{20} = 2 + 19(2) = 40$

Number of terms $= 20 - 8 + 1 = 13$.

Required sum $= \dfrac{13}{2}\left[u_8 + u_{20}\right] = \dfrac{13}{2}\left[16 + 40\right]$

$= 364$

Example 11

An arithmetic progression whose first term is 2 and whose n^{th} term is 32 has the sum of its first n terms equal to 357. Find n.

If the k^{th} term has a value exceeding 100, find the least value of k.

Solution

AP: first term $a=2$, common difference $=d$ $S_n = 357 \quad \Rightarrow \quad \frac{n}{2}(2+32)=357$ $\Rightarrow \quad n=21$ $u_n = u_{21} = 32 \quad \Rightarrow \quad 2+20d=32$ $\Rightarrow \quad d=\frac{3}{2}$	Find the least value of k such that $u_k = 2+\frac{3}{2}(k-1) > 100$ $\Rightarrow \quad k-1 > \frac{196}{3}$ $\Rightarrow \quad k > \frac{199}{3} = 66\frac{1}{3}$ Least value of k is 67.

3.4 Prove that a Sequence is an Arithmetic Progression

To prove that a sequence $u_1,\ u_2,\ u_3, \ldots,\ u_n, \ldots$ is an arithmetic progression, we need to show that

$$\boxed{u_n - u_{n-1} = \text{constant for all values of } n \geq 2}$$

This constant is the common difference of the AP.

Example 12

(a) The sum of the first n terms of a series is n^2+3n.

Write down the first three terms of the sequence.

Find the n^{th} term and prove that the terms of the sequence form an arithmetic progression.

(b) The sum of the first n terms of a series is given by $S_n=n^2+3n+1$.

Find the first term and show that $S_n-S_{n-1}=2n+2$ for $n\geq 2$.

Show that the terms of the sequence do not form an arithmetic progression.

Solution

(a) $S_n=n^2+3n$ $u_1=S_1=4$ $u_2=$ $S_2-S_1=10-4=6$ $u_3=$ $S_3-S_2=18-10=8$	For $n\geq 2$, $u_n = S_n-S_{n-1}$ $=\left[n^2+3n\right]-\left[(n-1)^2+3(n-1)\right]$ $=\left[n^2-(n-1)^2\right]+\left[3n-3(n-1)\right]$ $=(n+n-1)(1)+3=2n+2$ $u_1=S_1=1^2+3=4$ and it follows the form $u_n=2n+2$ when $n=1$. Thus, $u_n=2n+2$ for $n\geq 1$.

$$u_n-u_{n-1}=\left[2n+2\right]-\left[2(n-1)+2\right]$$
$$=2n+2-2n$$
$$=2 \text{ (constant)}$$

Since the difference between every 2 consecutive terms is a constant, the terms of this sequence form an arithmetic progression. (shown)

(b) $S_n=n^2+3n+1$ $u_1=S_1=5$	$S_n-S_{n-1}=\left[n^2+3n+1\right]-\left[(n-1)^2+3(n-1)+1\right]$ $=\left[n^2-(n-1)^2\right]+\left[3n-3(n-1)\right]+1-1$ $=(n+n-1)(1)+3=2n+2$

So, $u_n=2n+2$ for $n\geq 2$.

$u_2=2(2)+2=6$; $u_3=2(3)+2=8$

$u_3-u_2=2$

$u_2-u_1=1$

Since $u_3-u_2\neq u_2-u_1$, the terms of the sequence do not form an arithmetic progression. (shown)

4 SPECIAL SEQUENCE 2: GEOMETRIC PROGRESSION (GP)

4.1 Definition

A **geometric progression (GP)** is a sequence of numbers for which the ratio of every two consecutive terms is a constant, i.e.

$$\boxed{\frac{u_n}{u_{n-1}} = r}$$

The constant r is called the **common ratio.**

Note that we can easily see that GP is a sequence generated by a recurrence relation by writing it as $u_n = r(u_{n-1})$.

4.2 n^{th} term of a Geometric Progression

A geometric progression with first term a and common ratio r can be written as

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

The n^{th} term of the geometric progression is

$$\boxed{u_n = ar^{n-1}}$$

Examples

Geometric Progression	First Term, u_1	Common Ratio	Formula for u_n
$2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots, \frac{1}{512}$	2	$\frac{1}{2}$	$u_n = 2\left(\frac{1}{2}\right)^{n-1}$, $n = 1, 2, 3, \ldots, 11$
$\frac{1}{4},\ -1,\ 4,\ -16,\ 64, \ldots$	$\frac{1}{4}$	-4	$u_n = \frac{1}{4}(-4)^{n-1}$, $n = 1, 2, 3, \ldots$
$-3,\ 3,\ -3,\ 3, \ldots$	-3	-1	$u_n = -3(-1)^{n-1}$, $n = 1, 2, 3, \ldots$

A **geometric series** is formed when the terms of a geometric progression are added.

For example, $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

4.3 Sum of first *n* terms of a Geometric Series

Let S_n denote the **sum of the first *n* terms** of a geometric series with first term a and common ratio r .Then

$$\begin{aligned} S_n &= u_1 + u_2 + u_3 + ... + u_{n-2} + u_{n-1} + u_n \\ &= a + ar + ar^2 + ... + ar^{n-3} + ar^{n-2} + ar^{n-1} \quad --- \ (1) \end{aligned}$$

$(1) \times r$ gives

$$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-2} + ar^{n-1} + ar^n \quad --- \ (2)$$

$(2)-(1)$ gives $\quad (r-1)S_n = ar^n - a = a(r^n - 1)$

while $(1)-(2)$ gives $\quad (1-r)S_n = a - ar^n = a(1-r^n)$

Sum of the first n terms of a geometric series is

$$\boxed{S_n = \frac{a(r^n - 1)}{r-1} = \frac{a(1-r^n)}{1-r}, \quad r \neq 1}$$

Qn : What if $r = 1$? What will the expression for S_n be?

If $r = 1$, the sequence becomes $a,\ a,\ a,\ ...$, which is both an AP and a GP.

We call such a sequence a <u>constant</u> sequence, and $S_n = a + a + a + ... + a = na$.

Remarks: Sum of first *n* terms of a geometric series with *a* as its first term and *r* as its common ratio can be expressed as $\sum_{k=1}^{n} ar^{k-1}$.

Since $\sum_{k=1}^{n} ar^{k-1}$ can be rewritten as $\sum_{k=1}^{n} \frac{a}{r} r^k$ which is a series involving powers of a constant *r*, we can apply the formula of sum of first *n* terms of a geometric series on summation problems involving a general term consisting of a power of a constant.

An illustration:

$$\begin{aligned} \sum_{r=1}^{n} 2^r &= 2^1 + 2^2 + \cdots + 2^n \quad \text{(Geometric series with 1st term 2 and common ratio 2)} \\ &= \frac{2\left[2^n - 1\right]}{2-1} \\ &= 2\left[2^n - 1\right] \end{aligned}$$

Example 13

In a geometric progression whose terms are positive, the third term is 18 and the fifth term is 162. Find the sum of the first ten terms.

Solution

GP: first term $=a$, common ratio $=r$

$u_3=18 \quad \Rightarrow \quad ar^2=18 \quad ---\ (1) \quad$ and $\quad u_5=162 \quad \Rightarrow \quad ar^4=162 \quad ---\ (2)$

$(2)\div(1):\ \dfrac{ar^4}{ar^2}=\dfrac{162}{18} \quad \Rightarrow \quad r^2=9$

$\Rightarrow \quad r=3$ or $r=-3$ (rejected as terms are positive)

Substitute $r=3$ into (1), $a=2$

Sum of first 10 terms, $S_{10}=\dfrac{2(3^{10}-1)}{3-1}=59048$

4.4 Prove that a Sequence is a Geometric Progression

To prove that a sequence $u_1,\ u_2,\ u_3,...,\ u_n,...$ is a geometric progression, we need to show that

$$\boxed{\frac{u_n}{u_{n-1}} = \text{constant for all values of } n \geq 2}$$

This constant is the common ratio of the GP.

Example 14

The sum of the first n terms of a sequence is given by

$(a-2)^{-n} - 1$, where a is a constant. $a \neq 2$.

Show that the sequence is a geometric progression, and state its common ratio in terms of a.

Solution

$S_n = (a-2)^{-n} - 1$

$S_{n-1} = (a-2)^{-n+1} - 1$

For $n \geq 2$, $u_n = S_n - S_{n-1}$

$$= (a-2)^{-n} - (a-2)^{-n+1}$$
$$= (a-2)^{-n}[1-(a-2)]$$
$$= (a-2)^{-n}(3-a)$$

$$u_1 = S_1 = \frac{1}{a-2} - 1 = \frac{1-a+2}{a-2} = \frac{3-a}{a-2} = (a-2)^{-1}(3-a)$$

u_1 follows the form of $u_n = (a-2)^{-n}(3-a)$ when $n = 1$.

$\therefore u_n = (a-2)^{-n}(3-a)$ for $n \geq 1$.

$$\frac{u_n}{u_{n-1}} = \frac{(3-a)(a-2)^{-n}}{(3-a)(a-2)^{-n+1}},\ n \geq 2$$

$$= \frac{1}{a-2} \text{ (constant)}$$

Since the ratio of every 2 consecutive terms is a constant, the terms of this sequence form a geometric progression with common ratio $\frac{1}{a-2}$. (shown)

Example 15 (involving both AP and GP)

The first term of an increasing arithmetic progression is 1. The sum of the first 5 terms, the sum of the first 10 terms and the sum of the first 20 terms are consecutive terms of a geometric progression.

Find the common difference of the arithmetic progression and the common ratio of the geometric progression.

Solution

AP: first term $a=1$, common difference $=d$

Since S_5, S_{10}, S_{20} are consecutive terms of a GP, $\Rightarrow \quad \dfrac{S_{10}}{S_5}=\dfrac{S_{20}}{S_{10}}$

$$\Rightarrow \quad \left(S_{10}\right)^2=S_5S_{20}$$

Now

$$S_5=\frac{5}{2}[2(1)+4d]=5(1+2d)$$

$$S_{10}=\frac{10}{2}[2(1)+9d]=5(2+9d)$$

$$S_{20}=\frac{20}{2}[2(1)+19d]=10(2+19d)$$

So

$$\left[5(2+9d)\right]^2=\left[5(1+2d)\right]\left[10(2+19d)\right]$$
$$(2+9d)^2=2(1+2d)(2+19d)$$
$$4+36d+81d^2=2(2+23d+38d^2)$$
$$5d^2-10d=0$$
$$5d(d-2)=0$$

$d=2$ or $d=0$ (rejected as AP is increasing)

Common ratio of GP $=\dfrac{S_{10}}{S_5}=\dfrac{5\left[2+9(2)\right]}{5[1+2(2)]}=\dfrac{20}{5}=4$

4.5 Sum to Infinity of a Geometric Series

Consider the geometric series with first term $\frac{1}{2}$ and common ratio $\frac{1}{2}$.

We know that

$$S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = 1-\left(\frac{1}{2}\right)^n.$$

What about the sum of the infinite series? Does the geometric series converge?

i.e. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+... = ?$

Notice that as $n \to \infty$, $S_n = 1-\left(\frac{1}{2}\right)^n \to 1$, since $\left(\frac{1}{2}\right)^n \to 0$.

$\therefore \; S_\infty = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+... = 1$

For a geometric series with first term a and common ratio r, $S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$.

If $|r|<1$, then as $n \to \infty$, $r^n \to 0$ and $S_n \to \frac{a}{1-r}$.

In this case, we say the geometric series is **convergent**, and the **sum to infinity** is given by

$$S_\infty = \frac{a}{1-r}, \quad |r|<1.$$

Example 16

Determine if each of the following geometric series is convergent, and find its sum to infinity if it exists.

(a) $\frac{1}{2}-1+2-4+...$ **(b)** $4+1+\frac{1}{4}+\frac{1}{16}+...$

Solution

(a) $\frac{1}{2}-1+2-4+...$ Common ratio $= r_1 = -2$ Since $\lvert r_1\rvert = 2 > 1$, the geometric series is not convergent and the sum to infinity does not exist.	**(b)** $4+1+\frac{1}{4}+\frac{1}{16}+...$ Common ratio $= r_2 = \frac{1}{4}$ Since $\lvert r_2\rvert = \frac{1}{4} < 1$, the geometric series is convergent and $S_\infty = \frac{4}{1-\frac{1}{4}} = \frac{16}{3}$

Example 17

Determine if the following series converges and find the sum to infinity if it exists.

(a) $\sum_{r=1}^{n-1}\left(r^2+2^r\right)$, **(b)** $\sum_{r=0}^{n}\left(-\frac{1}{4}\right)^r$.

Solution	
(a) $\sum_{r=1}^{n-1}\left(r^2+2^r\right)=\sum_{r=1}^{n-1}r^2+\sum_{r=1}^{n-1}2^r$ $=\frac{1}{6}(n-1)(n)(2n-1)+\frac{2\left(2^{n-1}-1\right)}{2-1}$ $=\frac{1}{6}(n-1)(n)(2n-1)+2\left(2^{n-1}-1\right)$ As $n\to\infty$, $\frac{1}{6}(n-1)(n)(2n-1)\to\infty$ and $2\left(2^{n-1}-1\right)\to\infty$ Therefore, this series diverges and sum to infinity does not exist.	**(b)** $\sum_{r=0}^{n}\left(-\frac{1}{4}\right)^r=1+\left(-\frac{1}{4}\right)+\left(-\frac{1}{4}\right)^2+....+\left(-\frac{1}{4}\right)^n$ is a geometric series with first term $a=1$ and common ratio $r=-\frac{1}{4}$. Since $\lvert r\rvert<1$, this series converges and the sum to infinity, $\sum_{r=0}^{\infty}\left(-\frac{1}{4}\right)^r = S_\infty=\frac{a}{1-r}=\frac{1}{1-\left(-\frac{1}{4}\right)}=\frac{4}{5}$

Example 18

A geometric series has first term 1 and common ratio $\frac{7}{8}$. Find the least value of n for which the sum of the first n terms is greater than twice the sum to infinity of the remaining terms.

Solution

GP: first term = 1, common ratio = $\frac{7}{8}$, $(n+1)^{\text{th}}$ term = $\left(\frac{7}{8}\right)^n$

To find the least number of terms, n, such that

$$\frac{1-\left(\frac{7}{8}\right)^n}{1-\frac{7}{8}}>2\left[\frac{\left(\frac{7}{8}\right)^n}{1-\frac{7}{8}}\right]$$

$$1-\left(\tfrac{7}{8}\right)^n>2\left(\tfrac{7}{8}\right)^n$$

$$\left(\tfrac{7}{8}\right)^n<\tfrac{1}{3}$$

$$n>\frac{\lg\left(\frac{1}{3}\right)}{\lg\left(\frac{7}{8}\right)}=8.227$$

$\therefore$ Least value of n is 9.

Alternatively, you may use the GC (table of values) to solve the inequality.

n	$\left(\frac{7}{8}\right)^n-\frac{1}{3}$
8	$0.01028>0$
9	$-0.0327<0$

$\therefore$ Least value of n is 9.

Example 19

By using a geometric series, express the repeating decimal 2.353535… as a rational number.

Solution

$$
\begin{aligned}
2.35353535\ldots &= 2+0.35+0.0035+0.000035+0.00000035+\ldots \\
&= 2+\frac{35}{100}+\frac{35}{100^2}+\frac{35}{100^3}+\frac{35}{100^4}+\ldots \\
&= 2+\frac{35}{100}\left(1+\frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\ldots\right) \\
&= 2+\frac{35}{100}\left(\frac{1}{1-\frac{1}{100}}\right) \\
&= 2\frac{35}{99}
\end{aligned}
$$

5 MISCELLANEOUS

Example 20 (number pattern, involving both AP and GP)

Positive odd integers starting from 1 are grouped into sets containing 1, 2, 4, 8,… odd integers, so that the number of odd integers in each set after the first is twice the number of odd integers in the previous set:

$$\{1\}, \{3, 5\}, \{7, 9, 11, 13\}, \{15, 17, 19, 21, 23, 25, 27, 29\} \ldots$$

Find, in terms of k, **(i)** the number of terms in the k^{th} set,

(ii) the number of terms in the first k sets,

(iii) the last integer in the k^{th} set,

(iv) the sum of all integers in the first k sets.

Solution

Set	Number of terms
1	$1 = 2^0$
2	$2 = 2^1$
3	$4 = 2^2$
4	$8 = 2^3$

The number of terms in each set follows a GP with first term 1 and common ratio 2.

(i) Number of terms in the k^{th} set $= 2^{k-1}$

(ii) Number of terms in the first k sets $= 2^0 + 2^1 + 2^2 + ... + 2^{k-1} = \dfrac{(1)(2^k - 1)}{2-1} = 2^k - 1$

The odd integers in the sets follow an AP with first term 1 and common difference 2.

The last integer in the k^{th} set is the $(2^k - 1)^{\text{th}}$ term of the AP.

(iii) Last integer in the k^{th} set $= 1 + (2^k - 1 - 1)(2) = 2^{k+1} - 3$

(iv) Sum of all integers in the first k sets $= \dfrac{2^k - 1}{2}\left[1 + (2^{k+1} - 3)\right] = (2^k - 1)^2$

Example 21 [9205/1996/01/Q13]

An athlete is considering two plans for a training programme.

Plan I. Run 2000 m on day 1, 2400 m on day 2, and on each successive day increase the distance run by 400m.

Plan II. Run 2000 m on day 1, 2200 m on day 2, and on each successive day multiply the distance run by a constant factor $\frac{11}{10}$.

(i) For Plan I find, in the form $A(n+B)$, where A and B are constants to be determined, the distance, in metres, run on day n.

(ii) For Plan I find, in the form $Cn(n+D)$, where C and D are constants to be determined, the total distance, in metres, run on days 1 to n, inclusive.

(iii) For Plan I find the least value of n for which the total distance run on days 1 to n, inclusive exceeds 200 000 m.

(iv) For Plan II find an expression for the distance, in metres, run on day N.

(v) For Plan II find the least value of N for which the distance run on day N exceeds 16 000 m.

Solution

(i) For Plan I, distance run forms an AP with first term 2000 and common difference 400.
Distance, in metres, run on day n is $2000+(n-1)(400)=400(n+4)$
Hence $A=400,\ B=4$.

(ii) For Plan I, total distance, in metres, run on days 1 to n, inclusive

$$=\frac{n}{2}\left[2(2000)+(n-1)400\right]=200n(n+9)$$

Hence $C=200,\ D=9$.

(iii) For Plan I, $200n(n+9)>200\ 000 \quad\Rightarrow\quad n^2+9n-1000>0$

Consider $n^2+9n-1000=0$

$$\Rightarrow\quad n=\frac{-9\pm\sqrt{9^2-4(-1000)}}{2}=\frac{-9\pm\sqrt{4081}}{2}\approx -36.441 \text{ or } 27.441$$

$n^2+9n-1000>0 \Rightarrow n<-36.441$ or $n>27.441$

Since $n\in\mathbb{Z}^+$, $n>27.441 \Rightarrow$ least n is 28.

OR

For Plan I, $200n(n+9)>200\ 000 \quad\Rightarrow\quad n^2+9n-1000>0$

Let $y=n^2+9n-1000$

From G.C.,

n	y
27	$-28<0$
28	$36>0$

Hence least n is 28.

(iv) For Plan II, distance run forms a GP with first term 2000 and common ratio $\frac{11}{10}$.

Distance, in metres, run on day $N = 2000\left(\frac{11}{10}\right)^{N-1}$

(v) For Plan II, $2000\left(\frac{11}{10}\right)^{N-1} > 16\ 000 \quad \Rightarrow \quad \left(\frac{11}{10}\right)^{N-1} > 8 \quad \Rightarrow \quad N > \frac{\ln 8}{\ln(\frac{11}{10})} + 1 = 22.8$

Hence least value of N is 23.

SUMMARY