H

TEMASEK JUNIOR COLLEGE

2025 JC2 PRELIMINARY EXAMINATIONS

Higher 3

MATHEMATICS 9820

19 Sep 2025

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae (MF27)

READ THESE INSTRUCTIONS FIRST

Answer all questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2 - 2xy - 4x + 4y + 3.$$

(a) Find the two solutions of the differential equation that have the form $y = mx + c$. [3]

Let $y = \mathrm{f}(x)$ be another solution of the differential equation.

(b) Show that any stationary point on the curve $y = \mathrm{f}(x)$ must lie on one of two lines whose equations are $y = x - 3$ and $y = x - 1$. [2]

(c) Show that any stationary point on the curve $y = \mathrm{f}(x)$ that lies on the line $y = x - 3$ is a minimum point and that on the line $y = x - 1$ is a maximum point. [3]

The curve $y = \mathrm{f}(x)$ has two stationary points and $\mathrm{f}(0) = -2$. It is given that the solution curves of this differential equation do not intersect each other.

(d) On the same diagram, sketch the curve $y = \mathrm{f}(x)$, the two straight line solutions and the two lines which the stationary points lie on. [3]

2 **In this question, the domains of the functions are the set of integers.**

(a) The functions f, g and h are defined by

$$\mathrm{f}(n) = n^2 + 6n + 11,$$

$$\mathrm{g}(n) = n^2 - 4n + 6$$

$$\text{and} \quad \mathrm{h}(n) = n^2 - 2n + 5.$$

By completing the square, show that f and g have the same range. Show also that there are no common integers in the ranges of f and h. [5]

(b) For any integers a and b,

(i) show that $a^2 + ab + b^2 \geq 0$. [1]

(ii) show that $(a-1)^3 - b^3 = (a-1-b)\left[(a-1)^2 + (a-1)b + b^2\right]$. [1]

(c) The functions p and q are defined by

$$\mathrm{p}(n) = n^3 - 3n^2 + 7n$$

$$\text{and} \quad \mathrm{q}(n) = n^3 + 4n - 6.$$

By writing $\mathrm{p}(n)$ in the form $(n+a)^3 + bn + c$, where a, b and c are real constants, find the integers that are common in the ranges of p and q. (Note that the integers common in the ranges may or may not correspond to the same n value.) [4]

3 Let f be a continuous function on the interval $0 \le x \le a$, where a is a constant.

(a) **(i)** Use a substitution to prove that $\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\,\mathrm{d}x$. [2]

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\mathrm{d}x$, where n is a fixed positive integer. [3]

It is further given that $\mathrm{f}(x) = \mathrm{f}(a-x)$.

(b) Show that $\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x = \frac{a}{2}\int_0^a \mathrm{f}(x)\,\mathrm{d}x$. [2]

(c) For non-negative integers n, let $I_n = \int_0^{\frac{\pi}{2}} \cos^n(2x)\,\mathrm{d}x$.

(i) For each $n \ge 2$, show that $nI_n = (n-1)I_{n-2}$. [3]

(ii) Hence find the exact value of $\int_0^{\pi} x\cos^8(2x)\,\mathrm{d}x$. [4]

4 **(a)** Show that for any complex numbers z_1 and z_2,

$$|z_1 + z_2| \le |z_1| + |z_2|.$$ [2]

Use mathematical induction to prove that for any complex numbers $z_1, z_2, \ldots, z_n$,

$$|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n|.$$ [3]

(b) The real numbers $u_1, u_2, \ldots, u_{2023}$ satisfy $u_r - u_{r+1}^2 > 0$ for $r = 1, 2, 3, \ldots, 2022$.
If $W = (u_1 - u_2^2)(u_2 - u_3^2)\ldots(u_{2022} - u_{2023}^2)(u_{2023} - u_1^2)$ and $u_{2023} - u_1^2 > 0$, find the maximum value of $4^{2025}W$. [6]

5 A team is made up of 18 boys.

(a) Each boy writes down two numbers on a piece of paper, which states the number of other boys having the same surname as him, and the number of other boys being born in the same month as he is. When all the pieces of paper were collected, all the integers from 0 to 7 inclusive were seen. Show that there must be at least two boys who share the same surname and the same birth month. [3]

The team of 18 boys takes part in a competition where in the first stage, there are 4 different challenges.

(b) **(i)** Find the number of ways in which the team can split themselves to compete in the challenges, if all students take part in exactly one challenge and each challenge must have at least 4 students. [3]

(ii) Six boys take part in one of the challenges where they are tasked to distribute 80 identical dice amongst themselves where they had to then stack the allocated dice in the shortest time possible. Find the number of ways they can distribute 80 dice amongst themselves if each of them must receive at least 5 dice. [2]

The team progresses to the second stage where three of them, Alex, Ben and Charlie, each completes a test of 50 multiple-choice questions. Each of the 50 questions was answered correctly by at least one of the three boys. Alex, Ben and Charlie answered 40, 30 and 20 questions correctly. Questions answered correctly by all three of them were considered easy, while questions answered correctly by only one of the three of them were considered difficult.

(c) Find the difference in the number of easy questions and difficult questions, stating which type has more in number. [3]

This competition is held monthly. Alex has a group of five friends from other teams. At the competition, he met each of them 11 times, every two of them 5 times, every three of them 3 times, every four of them 2 times, and all five of them only once. Alex has participated in the competition 3 times where he did not meet any of these friends.

(d) Find the number of times Alex has taken part in this competition. [2]

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

Continued Fractions

A continued fraction for a real number x is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \quad ,$$

or simply expressed as $x = [a_0; a_1, a_2, \ldots]$, where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{R}$ for each $i \geq 1$.

A continued fraction with $a_i \in \mathbb{Z}^+$ for each $i \geq 1$ is known as a simple continued fraction.

It is known that the simple continued fraction for x is finite if and only if x is rational.
For example, the number

$$\frac{9}{7} = 1 + \cfrac{1}{3 + \cfrac{1}{2}} = [1;3,2] \text{ is rational}$$

while the number

$$\ln 3 = 1 + \cfrac{1}{10 + \cfrac{1}{7 + \cfrac{1}{9 + \ddots}}} = [1;10,7,9,\ldots] \text{ is irrational.}$$

Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with all the terms positive except perhaps a_0. We define two sequences of integers as follows:

$$p_{-1} = 1,\ p_0 = a_0,\ p_n = a_n p_{n-1} + p_{n-2},\ n \geq 1,$$
$$q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2},\ n \geq 1. \qquad (*)$$

For $x = [a_0; a_1, a_2, \ldots]$, and for each $n \in \mathbb{Z}^+$, we define the nth convergent of x as

$$r_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} = [a_0; a_1, a_2, \ldots, a_n] \ .$$

6 **(a)** **(i)** Show that $\sqrt{2} = 1 + \dfrac{1}{1+\sqrt{2}}$ and write $\sqrt{2}$ as a simple continued fraction to show that it is irrational. [2]

(ii) The number $\pi = 3.14159265\ldots$ can be expressed as a simple continued fraction

$$\pi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}}.$$

Find the values of b_0, b_1, b_2, and b_3. [2]

(iii) Let $x = [1; a_1, a_2, a_3, \ldots]$, where for each $n \in \mathbb{Z}^+$, $a_{2n-1} = 3$ and $a_{2n} = 4$. Find the value of x, giving your answer in the form $a + b\sqrt{c}$, where a and b are rational numbers and c is a positive integer. [3]

(b) Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with all the terms positive except perhaps a_0. Let p_n and q_n be defined by the recurrence relations in (*) as stated in the mathematical text.

(i) Prove that for each $n \in \mathbb{Z}^+$ and any positive real number y,

$$[a_0; a_1, a_2, \ldots, a_{n-1}, y] = \frac{yp_{n-1} + p_{n-2}}{yq_{n-1} + q_{n-2}}. \qquad [5]$$

(ii) Hence, show that r_n, the nth convergent of $[a_0; a_1, a_2, \ldots]$, is given by $r_n = \dfrac{p_n}{q_n}$. [1]

(iii) Prove that for each $n \in \mathbb{Z}^+$,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

Hence show that

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n. \qquad [4]$$

(iv) Explain why the nth convergent, $r_n = \dfrac{p_n}{q_n}$, is always in its lowest terms. [1]

(v) Prove that $r_2 < r_4 < r_6 < \ldots < r_5 < r_3 < r_1$. [2]

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