JURONG PIONEER JUNIOR COLLEGE
9478 H2 PHYSICS

Gravitational Fields

Introduction

- Isaac Newton (1642 –1726), published *Philosophiæ Naturalis Principia Mathematica ("Mathematical Principles of Natural Philosophy")* in 1687. In it, he formulated the Newton's laws of motion (Dynamics), and the Newton's law of gravitation.

- While the popular story involves Newton observing a falling apple, his actual discovery was more complex. Newton built upon earlier work, including Johannes Kepler's mathematical descriptions of planetary motions from the early 17th century. The law of gravity is celebrated for its elegant simplicity and broad applicability. The applications range from understanding planetary orbits, satellite motion, tidal flow, galaxy formation and stellar dynamics. This law, along with Newton's laws of motion, provided a foundation for understanding the mechanics of the universe.

(a) Recall and use Newton's law of gravitation in the form $F = G\dfrac{m_1 m_2}{r^2}$ **.**

1 GRAVITATIONAL FORCE

1.1 Newton's law of gravitation

Newton's law of gravitation states that the **force of attraction** between two **point masses** is directly proportional to the product of the masses and inversely proportional to the square of the distance between them.

- **Point Mass:** a point mass is a theoretical concept representing an object with a finite mass that occupies zero space, meaning it has no physical dimensions.

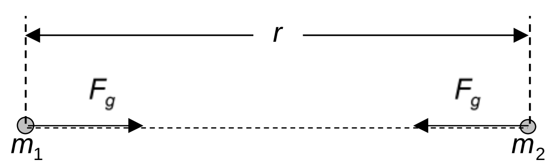


Fig. 1 Gravitational force between **<u>two point masses</u>**

- Mathematically, it is written as $\boxed{F_g = G\dfrac{m_1 m_2}{r^2}}$

  where G is the **<u>gravitational constant</u>** with a value of 6.67×10^{-11} N m^2 kg^{-2},
  m_1 and m_2 are the masses of the two bodies respectively,
  r is the distance between their centres of masses of these 2 point masses, and
  F_g is the magnitude of the gravitational force of **<u>attraction</u>** between them.

- This is known as an **inverse square law**, as the magnitude of the force varies with the inverse square of the separation of the particles, $\boxed{F_g \; \alpha \; \frac{1}{r^2}}$

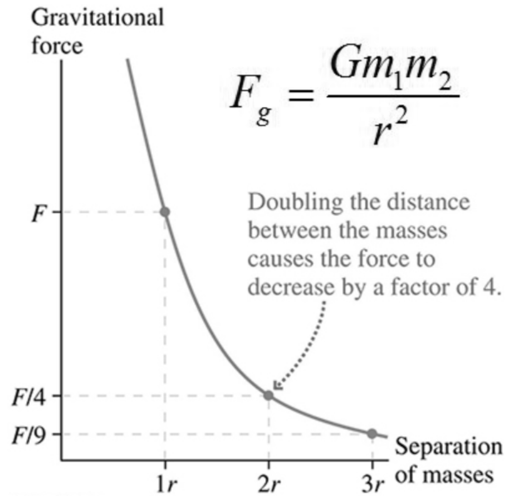


Fig. 2 Magnitude of gravitational force variation with separation

- The gravitational forces of attraction between two bodies form an action-reaction pair, consistent with Newton's third law. Mass m_1 attracts m_2 with the same magnitude of force as m_2 attracts m_1, but in opposite directions on each mass.

- For spherical masses, the distance r is measured between their respective centres of mass (assume the mass has uniform density and concentrated at its centre).

- Sometimes the equation is written with a '–' sign as $\vec{F} = -G\frac{m_1m_2}{r^2}$ to indicate the attractive nature of the force, where r is positive when pointing way from the mass. The negative sign is normally ignored when only magnitude is required. If direction is to be considered, the *F-r* graph is plotted on the negative side.

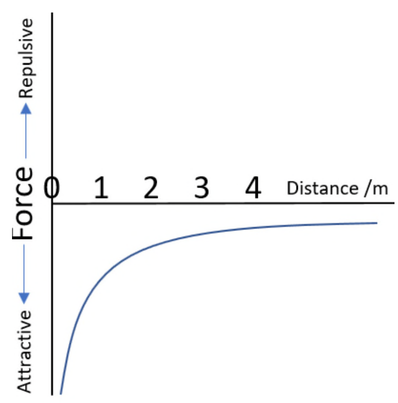


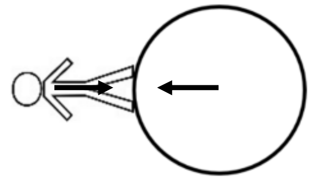
Fig. 3 Gravitational force (including direction) with separation

- Because G is very small in value (6.67×10^{-11}), the gravitational force is often ignored unless one of the masses is sufficiently large like the Moon (7.3×10^{22} kg), the Earth (6.0×10^{24} kg) or the Sun (2.0×10^{30} kg). E.g. we can ignore the negligible gravitational force of attraction between 2 human beings.

Example 1

A man of mass 85.0 kg is standing on the surface of the Earth. Given that the mass of the Earth is 5.97×10^{24} kg with a radius of 6.37×10^{6} m, calculate the gravitational force the Earth exerts on the man. Hence, state the gravitational force the man exerts on the Earth.

Solution: (Diagram not to scale)



$$F_g = G\frac{Mm}{r^2}$$

$$= (6.67\text{x}10^{-11})\frac{(5.97\text{x}10^{24})(85.0)}{(6.37\text{x}10^{6})^2}$$

$$= 834 \text{ N}$$

This is equal to the weight, mg = 85 x 9.81 = 834 N

The gravitational force that the Earth exerts on the man is 834 N towards the centre of Earth.

The gravitational force that the man exerts on the Earth is 834 N from centre of Earth towards the man.

Example 2

Two stationary objects of masses 5.0×10^{3} kg and 1.0×10^{3} kg are placed at a distance of 10 m apart. A third object of mass 1.0 kg, placed between the two masses, experiences no resultant force.

(a) On Fig. 1.1, draw the forces acting on the 1.0 kg object.

(b) Calculate the distance *x* between the 1.0 kg and the 1000 kg objects.

Solution:

(a)

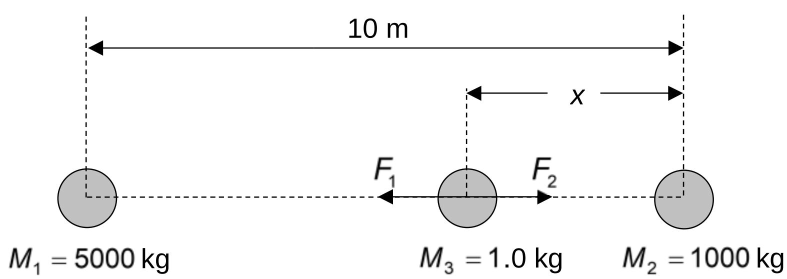


Fig. 1.1

(b) Since the 1.0 kg object experiences zero resultant force,

$$G\frac{M_1M_3}{(10-x)^2} = G\frac{M_2M_3}{x^2}$$

$$\frac{5000}{(10-x)^2} = \frac{1000}{x^2}$$

$$\frac{x^2}{(10-x)^2} = \frac{1000}{5000}$$

Taking square root on both sides of the equation,

$$\frac{x}{(10-x)} = \sqrt{\frac{1}{5}}$$
$$10 - x = \sqrt{5}x$$
$$x \approx 3.1 \text{ m}$$

Therefore, the distance x between the 1.0 kg and 1000 kg objects is 3.1 m.

[from Energy and Fields] Candidates should be able to
(f) show an understanding of the concept of a field as a region of space in which bodies may experience a force associated with the field.
(h) represent gravitational fields by means of field lines
(g) define gravitational field strength at a point as the gravitational force per unit mass on a mass placed at that point
(i) show an understanding that the force on a mass in a gravitational field acts along the field lines

(b) derive, from Newton's law of gravitation and the definition of gravitational field strength, the field strength due to a point mass, $g = G\frac{M}{r^2}$

2 GRAVITATIONAL FIELD

2.1 Concept of a gravitational field

How does the Earth exert an attractive force on the Moon over a vast empty space without making any contact with the Moon?

Physicists use the concept of a field as an intermediary of interaction between masses. The Earth as a mass produces a gravitational field <u>at every point</u> in the space from itself to infinity. Any other mass placed in this field will experience an attractive force. The force acting on the Moon therefore arises from interaction of the gravitational field of the Earth and the mass of the Moon.

- A gravitational field is a region of space in which a mass experiences a force due to the presence of another mass (which generates the gravitational field).

- At every point around a mass, there is a vector-value gravitational field which has both <u>magnitude and direction</u>. This gravitational field can be represented by field lines with <u>vector direction,</u> directed towards the centre of the mass. The relative strength of the field is represented by separation of the field lines. Closer field lines mean stronger field strength. The field direction follows the direction of the gravitational force acting on other masses. The field lines <u>do not cross</u> each other.

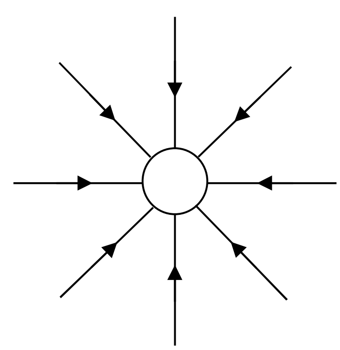


Fig. 4 Gravitational field lines of a mass

2.2 Gravitational field strength

Gravitational field strength at a point is defined as the gravitational force per unit mass acting at that point.

$$\boxed{g = \frac{F_g}{m}}$$

To measure the gravitational field strength produced by *M* at a point X, at distance *r* from *M*, we placed a <u>small test mass *m*</u> at point X. (The small test mass is used to test or measure the force without affecting *M*). The gravitational force acting on *m* by *M*, divided by *m* is the gravitational field strength produced by *M* at X.

M is the source of the field, *m* is used to test the field strength produced by *M*.

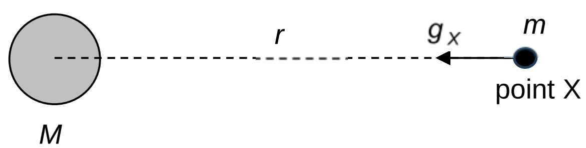


Fig. 5 Gravitational field strength at point X due to an object of mass *M*

- Units for *g* is N kg^{-1}, from the definition.

- If 2 or more fields are present at a point the resultant gravitational field strength is obtained by vector addition. The direction of the resultant gravitational force is along the direction of the resultant field lines with a magnitude of $F = mg_{resultant}$.

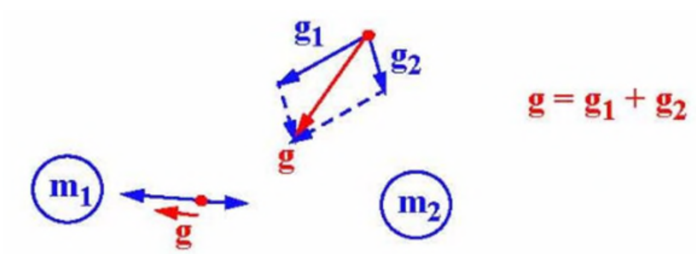
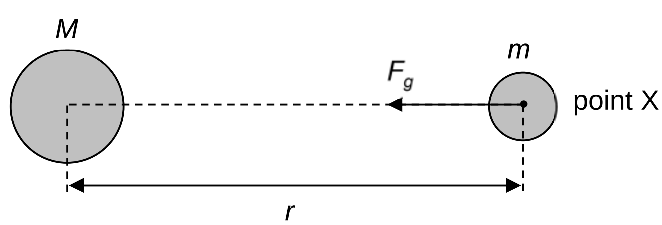


Fig. 6 Vector addition of gravitational field strength

Example 3
Using Newton's law of gravitation and the definition of gravitational field strength, derive the equation $g = \frac{GM}{r^2}$.
Solution:
Consider a mass *M* and another mass *m* placed at point X, which is at a distance *r* away from mass *M*.



By Newton's law of gravitation, the force F_g experienced by mass m is $\dfrac{GMm}{r^2}$.

From the definition of gravitational field strength at point X,

$$g \text{ at point X} = \frac{F_g \text{ experienced by mass } m}{\text{mass } m}$$

$$g = \frac{G\dfrac{Mm}{r^2}}{m}$$

$$g = \frac{GM}{r^2}$$

Hence, the gravitational field strength at point X due to mass *M* is $g = \dfrac{GM}{r^2}$.

Note that the gravitational field strength *g* at *X* is independent of *m*, even though it is defined using *m*. The value of *g* only depends on *r* and *M*.

(c) recall and use $g = G\dfrac{M}{r^2}$ ***for the gravitational field strength due to a point mass to solve problems***

Example 4

The mass of the Earth is approximately 80 times the mass of the Moon and the Earth's radius is 3.7 times that of the Moon. If the gravitational field strength at the Earth's surface is 10 N kg^{-1}, estimate its value on the Moon's surface.

Solution:

Using the expression for gravitational field strength at a distance *r*,

$g_E = \dfrac{GM_E}{r_E{}^2}$ at the Earth's surface --------------- (1)

$g_M = \dfrac{GM_M}{r_M{}^2}$ at the Moon's surface --------------- (2)

Dividing the two equations, we have

$$\frac{g_M}{g_E} = \frac{M_M r_E{}^2}{M_E r_M{}^2}$$

$$= \frac{1}{80}(3.7)^2$$

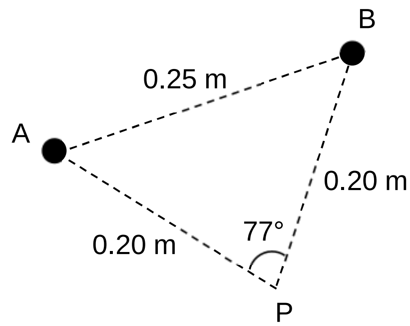
$$g_M = \frac{1}{80}(3.7)^2(10)$$

$$g_M \approx 1.7 \text{ N kg}^{-1}$$

Therefore, the gravitational field strength on the Moon's surface is 1.7 N kg^{-1}.

Example 5

The distance between an object A of mass 8000 kg and another object B of mass 8000 kg is 0.25 m. Calculate the resultant gravitational field strength due to both the objects at a point P 0.20 m away from both A and B.



Solution:

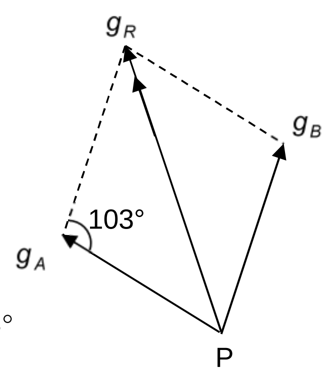
Let the gravitational field strengths due to objects A and B at P be g_A and g_B respectively.

Since masses A and B are the same and P is equidistant from A and B,

$$g_A = g_B = G\frac{M_A}{{r_A}^2}$$

$$= 6.67\times10^{-11}\times\frac{8000}{0.20^2}$$

$$\approx 1.33\times10^{-5}\ \text{N kg}^{-1}$$



Using cosine rule,

$${g_R}^2 = {g_A}^2 + {g_B}^2 - 2g_Ag_B\cos103°$$

$${g_R}^2 = \left(1.33\times10^5\right)^2 + \left(1.33\times10^5\right)^2 - 2\left(1.33\times10^2\right)^2\cos103°$$

$$g_R \approx 2.09\times10^{-5}\ \text{N kg}^{-1}$$

Therefore, the resultant gravitational field strength is $2.09\times10^{-5}\ \text{N kg}^{-1}$ and its direction is pointing from P towards the mid point of AB.

(d) show an understanding that near the surface of the Earth, gravitational field strength is approximately constant and equal to the acceleration of free fall

2.3 Free fall acceleration and field strength

In earlier chapters we used g = 9.81 ms^{-2} the free fall acceleration near the surface of the Earth. The same symbol g is now used as a gravitational field strength. Are they related?

The answer is that they are the same value (ignoring the rotation of the Earth). We can verify this with Newton's second law $F = ma$. Let the mass of the Earth be M, and mass m is allowed to fall freely due to the gravitational force F:

$$F = ma$$

$$G\frac{Mm}{r^2} = ma, \text{ cancelling out m gives}$$

$$a = G\frac{M}{r^2}, \text{ the RHS is the g-field of M.}$$

$$a = g, \text{ substituting M and r for the Earth gives}$$

$$= (6.67x10^{-11})\frac{5.97x10^{24}}{(6.37x10^{6})^2} = 9.81 \text{ ms}^{-2}$$

- Unit for g is Nkg^{-1} , which is the same as the SI based unit of acceleration, m s^{-2}.

- From the surface of the Earth radially outwards, the magnitude of g follows an inverse square law ($\propto \frac{1}{r^2}$).
- Within the Earth (assuming that the Earth is uniform in density), g decreases uniformly ($\propto r$) with the radius r until it is zero at the centre of the Earth. (Derivation and proof not required)

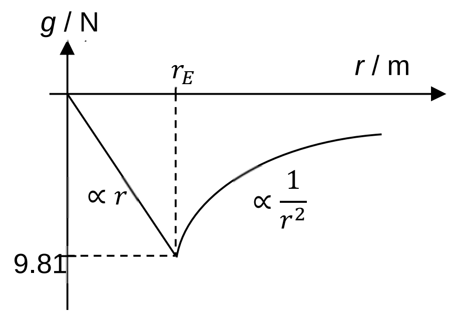


Fig. 7 Variation of Earth's gravitational field

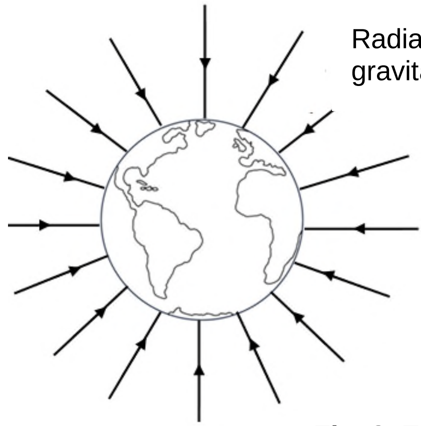
- In reality g varies from the poles to the equator because (i) the Earth is not a perfect sphere (about 21 km smaller at poles, therefore higher g) and (ii) the rotation of the Earth also created differences (smaller g at equator by $r\omega^2 \approx 0.03$ ms^{-2}). These 2 factors explain why g measured at the poles is slightly different from that at the equator. The difference is, however, less than 1 %. [g_{poles}= 9.83 ms^{-2}, $g_{equator}$ = 9.78 ms^{-2}]

2.4 Earth's gravitational field strength at the surface

- The gravitational field strength g near the surface of the Earth is approximately constant at 9.81 N kg^{-1}, also known as acceleration of free fall, with the value of 9.81 m s^{-2}.
- The approximate constancy should be appreciated by considering the value of g at height h above the surface, where h is small compared to the radius R of the Earth. R = 6.4 x 10^6 m, for h = 5 km, the percentage difference in g is less than 0.2 %.

$$g = \frac{GM}{(h+R)^2} \approx \frac{GM}{R^2} \quad \text{since } h << R$$

- Near the surface of the Earth, the gravitational field lines are nearly uniform.



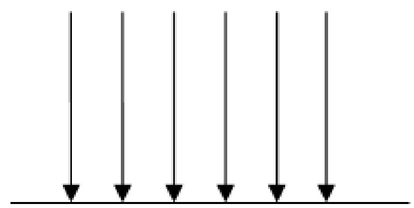


Fig. 8. Earth's gravitational field

(e) define gravitational potential at a point as the work done per unit mass by an external force in bringing a small test mass from infinity to that point.

(f) solve problems using the equation $\phi = -G\frac{M}{r}$ ***for the gravitational potential in the field due to a point mass.***

3 GRAVITATIONAL POTENTIAL AND POTENTIAL ENERGY

3.1 Gravitational potential

In mechanics we can solve problems using either vector quantities like force, momentum or acceleration, or using scalar quantities like work or energy.

In a similar way, we can solve gravitational problems using vector quantities like force and field or scalar quantities like energy and work. We now look at the scalar method using a new quantity called gravitational potential.

Like a gravitational field, a mass *M* generates a potential field around itself. This potential has a scalar value at every point around *M*, called the gravitational potential.

To define the magnitude of the gravitational field created by *M* at every point, again we use a small test mass *m*:

Gravitational potential ϕ at a point is the work done per unit mass by an external force in bringing a small test mass from infinity to that point.

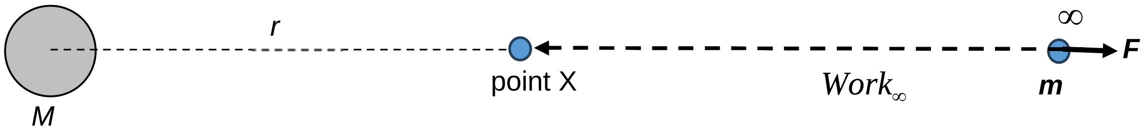


Fig. 9 Gravitational potential definition, ***F*** is the external force.

We use the symbol ϕ (phi) to represent the potential at a point:

$$\boxed{\phi = \frac{Work_{\infty}}{m}}$$

It can be shown (proof not required) using the above definition that for a mass M**,** the gravitational potential ϕ of M at a distance r from M is given by:

$$\boxed{\phi = -\frac{GM}{r}}$$

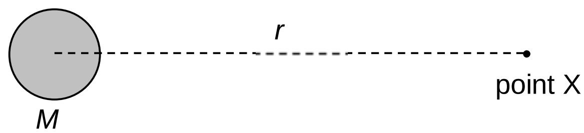


Fig. 10 Gravitational potential at point X due to mass *M* is $\phi_X = -\frac{GM}{r}$

- ϕ is negative in value. The reason is that: (1) The gravitational potential at infinity is zero and (2) because the gravitational force is an attractive force, the external force ***F*** moving mass m must point away from M as it moves m from infinity to X (as shown in Fig 9). The direction of ***F*** is opposite to the displacement, the work done by ***F*** is negative.

- Similar to gravitational field, ϕ does not contain the test mass *m* even though it is defined with *m*. It only depends on *M* and *r*.

- ϕ is a scalar with units of J kg^{-1}.

- The '**–**' sign in $-G\frac{M}{r}$ is part of the value of gravitational potential and not a direction. Therefore, the value of gravitation potential, including the '**–**' sign, increases as *r* increase, unlike *g* whose magnitude decreases with $\frac{1}{r^2}$. The value of gravitational potential becomes less negative or larger in value as *r* gets bigger, eventually reaches the maximum value of 0 at infinity.

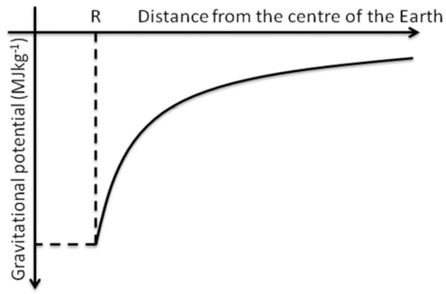
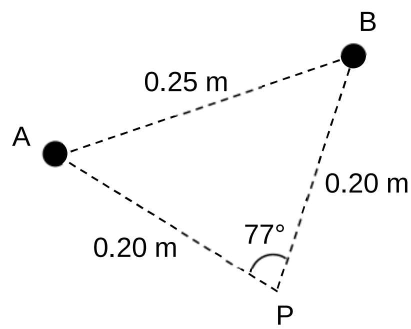


Fig. 11 Variation of gravitational potential with distance

- If there are 2 masses M_1 and M_2, the total gravitational potential of M_1 and M_2 at a point is the simple scalar sum of the potential due to each mass at each point.

Example 6

The distance between an object of mass 8000 kg and another object B of mass 8000 kg is 0.25 m. What is the gravitational potential at P due to the two objects?



Solution:

Let the gravitational potential at point P due to object A be ϕ_A and that due to object B be ϕ_B. Since gravitational potential is a scalar quantity, we can simply add the potentials due to objects A and B.

$$\phi_P = \phi_A + \phi_B$$

$$\phi_P = -\frac{GM_A}{r} + \left(-\frac{GM_B}{r}\right)$$

$$\phi_P = -\frac{6.67 \times 10^{-11} \times 8000}{0.20} + \left(-\frac{6.67 \times 10^{-11} \times 8000}{0.20}\right)$$

$$\phi_P \approx -5.3 \times 10^{-6} \text{ J kg}^{-1}$$

Candidates should be able to ***[Energy and Field] (h) show an understanding of the relationship between equipotential surfaces and field lines*** ***(h) recall that gravitational field strength at a point is equal to the negative potential gradient at that point and use this to solve problems.***

3.2 Equipotential surface

- Like gravitational field, we can also draw lines to represent gravitational potential values. All points at the same distance from the centre of the Earth have the <u>same gravitational potential</u> by the equation $\phi = -G\frac{M}{r}$. The spherical surface where all points on each surface have the same gravitational potential is known as an equipotential surface. Each surface has its own potential value.

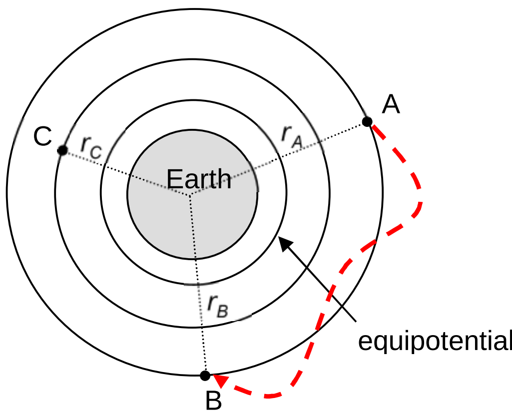


Fig. 12 Equipotential surfaces around Earth

- From Fig. 12, since $r_A = r_B > r_C$, $\phi_A = \phi_B > \phi_C$.

- No work is done if an object is moved along an equipotential surface. Total work done is zero from A to B at the same surface, regardless of path taken.

- ***How are g and Φ related***? It can be shown that (proof not required) the negative of the gradient of ϕ -*r* graph gives the gravitational field strength g at that point:

$$\boxed{g = -\frac{d\phi}{dr}}$$

(Verification: $\phi = -G\frac{M}{r}$, differentiate wrt r gives $\frac{d\phi}{dr} = G\frac{M}{r^2} = -g$)

- The '–' sign in the equation is a vector direction, showing that *g* points in the direction of decreasing potential ϕ (Fig 13 below).

- The gravitational field lines are always <u>normal</u> to equipotential surfaces.

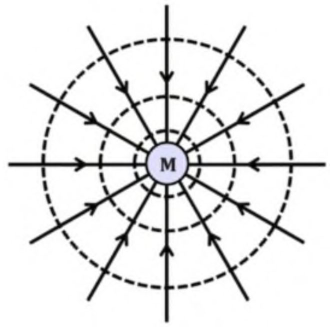
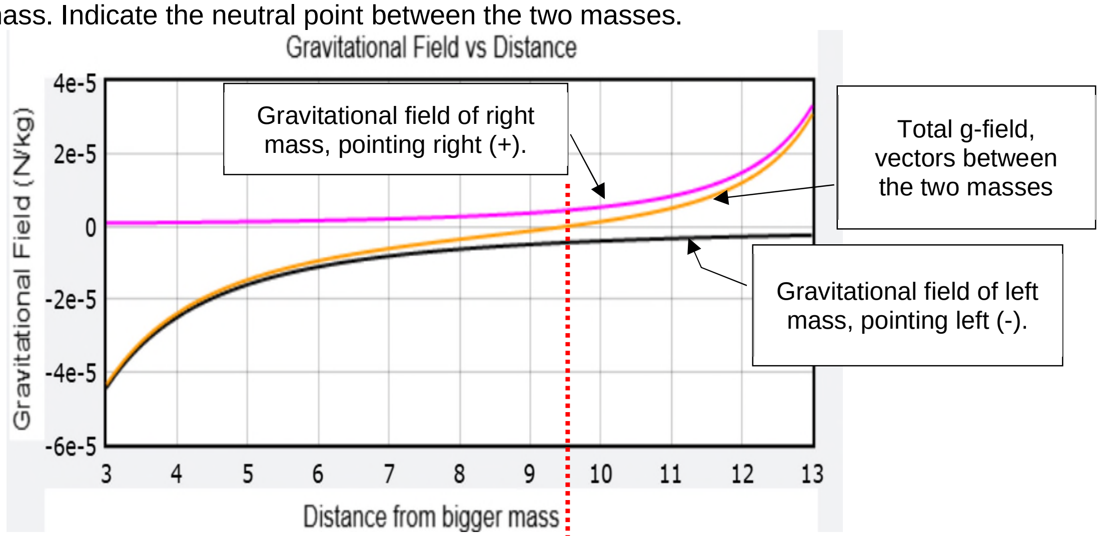
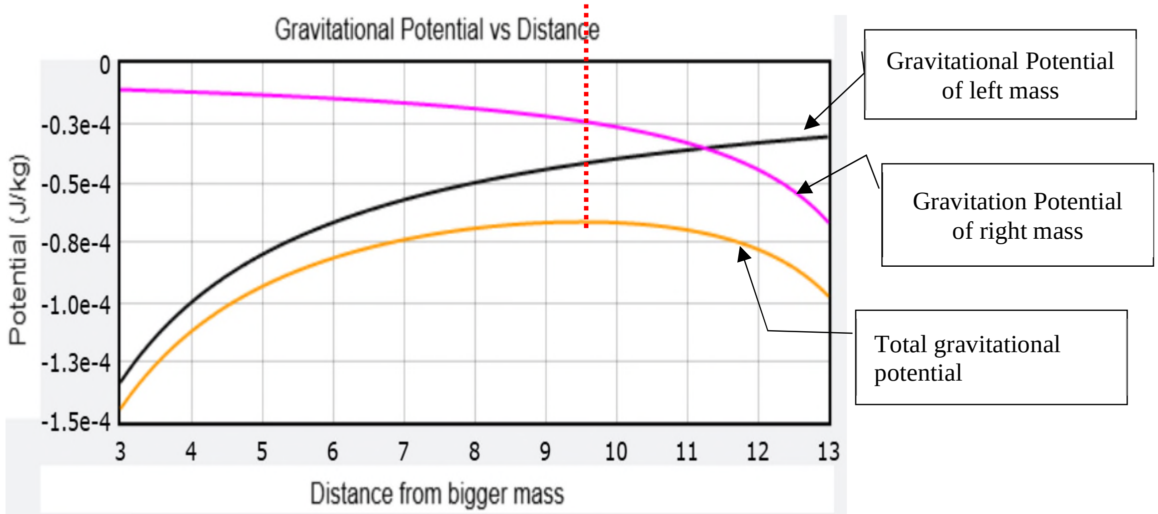


Fig. 13 Equipotential surfaces are perpendicular to field lines.

Example 7

A mass is placed at x =0, another smaller mass is placed at x = 13. The graphs below show the variation of the total gravitational field strength and total potential with distance from the left mass. Indicate the neutral point between the two masses.





The total gravitational potential curve reaches a maximum where the gradient is zero, this corresponds to a field strength value of zero at that point (neutral point), shown by the dotted line in the graphs above.

(g) show an understanding that that the gravitational potential energy of a system of two point masses is $U_g = -G\dfrac{Mm}{r}$ ***.***

[Energy and Field]

(i) show an understanding that the work done by the field in moving the mass is equal to the negative of the change in potential energy

3.3 Gravitational potential energy

When a mass *m* is placed in the gravitational field of another mass *M*, a gravitational potential **energy** is stored in the system of these 2 masses because of the gravitational force of attraction. This stored potential energy can do work or transform to other forms like KE. The value of this gravitational potential energy depends on *m*, *M* and the separation between them.

Gravitational potential energy *U* of the system of mass *m* and *M* separated by a distance *r* is the work done by an external force in moving mass *m* from infinity to a point *r* from *M*.

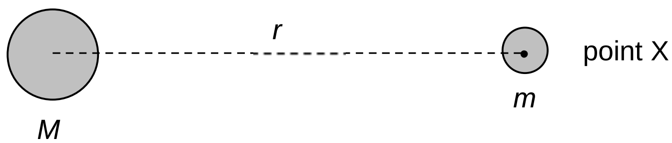


Fig. 14 Gravitational potential energy of mass *m* placed at point X

Recall that the gravitational potential is defined as work done from infinity per unit mass:

$$\phi = \frac{\text{Work}_{\text{from }\infty}}{m}$$

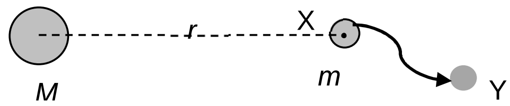
$$\text{Work}_{\text{from }\infty} = m\phi$$

By definition, the work done to bring *m* from infinity is the gravitational potential energy, often designated as U . Therefore,

$$\boxed{\begin{aligned} U &= m\phi \\ &= m(-G\frac{M}{r}) \\ &= -G\frac{Mm}{r} \end{aligned}}$$

- As U is equal to $m\phi$, the shape of the $U - r$ graph is similar to the $\phi - r$ graph.

- If m is moved to a different distance at point Y from M with a different potential, the change in gravitational potential energy is given



$$\boxed{\Delta U = m\Delta\phi}$$

where $\Delta\phi$ is the change in gravitational potential from X to Y.

- Gravitational potential energy are always negative, and increase in value when the separation of the masses increases. The maximum value of gravitational potential energy of 2 masses is zero, when they are separated by infinite distance.

- The gravitation potential energy is a property of the system of 2 masses, it does not belong to either M or m as it requires the presence of both masses.

- The force and potential energy are related. The negative gradient of U-r graph equals the gravitational force or $F_g = -\dfrac{dU}{dr}$. This relation is similar to g and the gradient of $\phi - r$ graph.

  We can derive this relation from $g = -\dfrac{d\phi}{dr}$. Multiply both sides with m gives $mg = -m\dfrac{d\phi}{dr}$ or $F_g = -\dfrac{dU}{dr}$

- The equation $F_g = -\dfrac{dU}{dr}$ shows that the gravitational force points in the direction of decreasing gravitational potential energy.

  Rearranging the equation gives $F_g dr = -dU$. The LHS is a work done term, which means the work done by gravitational force F_g over distance dr is equal to the negative of the change in the gravitational potential energy.

Example 8

Calculate the energy required to move a 3000 kg mass to a height of 200 km above the Earth's surface.

(Given that radius of the Earth $R_E = 6.4 \times 10^6$ m; mass of the Earth $M_E = 6.0 \times 10^{24}$ kg.)

Solution:

Energy required = change in *G.P.E.*

$$= -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right)$$

$$= -GMm\left(\frac{1}{R+h} - \frac{1}{R}\right)$$

$$= -(6.67\times10^{-11})(6.0\times10^{24})\times 3000\times\left[\frac{1}{[(6.4\times10^{6})+(200\times10^{3})]} - \frac{1}{6.4\times10^{6}}\right]$$

$$\approx 5.7\times10^{9}\ \text{J}$$

Can we use ***mgh*** to calculate the change in gravitational potential energy?

The answer is **no** for this case, because g changes over the large distance of 200 km.

However, for small values of h near the surface, the equation *mgh* is good enough:

$$\text{Change in gravitational potential energy} = -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right) = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right)$$
$$= GMm\left(\frac{R+h-R}{R(R+h)}\right) = GMm\frac{h}{R^2},\ \text{for R>>> h}$$
$$= mgh,\ \text{where g} = \frac{GM}{R^2}$$

3.4 Conservation of energy

When the separation between mass *m* and *M* changes, the gravitational potential energy of the system also changes. If the only acting between *M* and *m* is the gravitational force, this change in gravitational potential energy will result in an equal but opposite change in kinetic energy of the bodies, by principle of conservation of energy. Equivalently the sum of Kinetic energies and Gravitational Potential energy is a constant.

Initial GPE + initial KE of (***M+m***) = Final GPE + final KE of (***M+m***)

In most problems, the value of <u>*M* is much bigger than *m*</u>. The effect of the large inertia mass of *M* means that the acceleration on *M* is negligible compared to that of *m*.

Using Newton's second law, $a_M = F/M$ is negligible compared to $a_m = F/m$. We can therefore ignore the change in KE of *M*, and simplify the energy conservation equation as

	Initial GPE + initial KE of ***m*** = Final GPE + final KE of ***m***
Or	change in GPE + change in KE of ***m*** = 0 for ***M>>>m***

Note: We must take into account the KE of both masses <u>if they are comparable in size</u> by including KE of both masses.

(i) analyse problems related to escape velocity by considering energy stores and transfers

3.5 Escape velocity

When a mass is projected vertically up from the surface of the Earth, it will likely fall back. Is there a minimum initial speed of vertical projection such that the object does not fall back at all? Note that this object is only given an **initial vertical velocity without** any rocket propulsion thereafter.

- **Escape velocity (*v_{esc}*)** from a point on the surface of a planet is the minimum velocity required to project a mass *m* to infinity.

- To escape from the Earth's surface, an object needs to have sufficient kinetic energy in order to overcome the gravitational pull of the Earth.

- The minimum kinetic energy required is equal to the gain in gravitational potential energy by the object of mass *m* in leaving Earth to escape to infinity.

  To find the escape speed, use the conservation of energy:

  $(K.E.+G.P.E.)_{\text{surface}} = (K.E.+G.P.E)_{\infty}$ or

  loss in *K.E.* of m = gain in *G.P.E.*

  $$\frac{1}{2}mv_{esc}^{2} - 0 = U_{\infty} - U_{Earth's\ surface}$$ (min *K.E.* at infinity is zero)

  $$\frac{1}{2}mv_{esc}^{2} = \left[0 - \left(-\frac{GM_E m}{R_E}\right)\right]$$ since $U_{\infty} = 0$.

  $$\boxed{v_{esc} = \sqrt{\frac{2GM_E}{R_E}}}$$ (Do not quote this equation, always derive from 1st principle)

  Since $g = \frac{GM_E}{R_E^{\ 2}}$, v_{esc} can also be written as $\boxed{v_{esc} = \sqrt{2gR_E}}$, where g is the gravitational field strength at the Earth's surface.

- Therefore, from the Earth's surface, the escape velocity is

  $$v = \sqrt{2 \times 9.81 \times 6.4 \times 10^{6}} \approx 11200 \text{ m s}^{-1}$$

  This is equivalent to travelling at about 34 times the speed of sound!

- If the launch is not from the surface of the Earth, we cannot use g = 9.81 N kg^{-1}. The escape speed at the top of a mountain is lower than at the bottom.

Example 9
Explain why the Moon does not have an atmosphere.

Answer:
The mass of the Moon is much smaller than the Earth, so the escape velocity is much smaller on the surface of the Moon. Gas molecules move at speeds that are greater than the Moon's escape velocity, causing them to escape into space and preventing the formation of a dense atmosphere.

(j) analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes

4. Circular Orbits in gravitational field.

4.1 Circular orbits

- A satellite is kept in orbit by the gravitational attraction of the planet about which it is orbiting. No external rocket propulsion is required once it is placed in orbit.

- Why do satellites not fall back to the surface due to the gravitational force?

  They do fall and travel in curved paths, but they are able to maintain the same height because the surface of the Earth curves as much if they have the right speed. As shown in Fig 15 below, when a satellite speed is too low (A, B), it will fall back to hit the Earth. If it is too fast (D, E), it will not stay in a circular orbit. At the right speed, as shown for speed C, it will stay in a stable circular orbit, never hitting the surface.

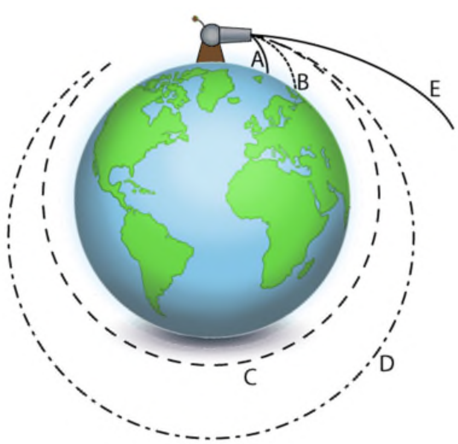


Fig. 15 Satellites with different speeds

- The total energy of the satellite remains constant because the gravitational force does not do any work as it is always perpendicular to the displacement.

- The orbital plane need not be in the equatorial plane. The important point is that the orbital plane contains the centre of the Earth.

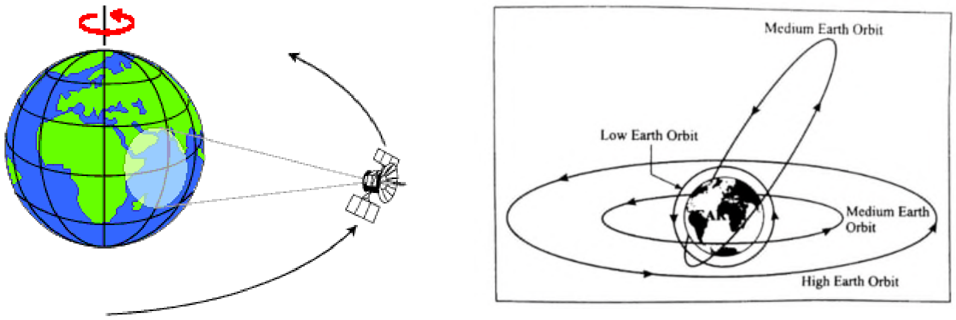


Fig. 16 Satellite orbiting around Earth

- When a satellite of mass *m* orbits around the Earth, **the centripetal force is provided by the Earth's gravitational attraction.** (Important to state this line in explanation of working)

$$F_g = F_c$$

$$\frac{GM_E m}{r^2} = \frac{mv^2}{r}$$

$$\boxed{v = \sqrt{\frac{GM_E}{r}}}$$

- The speed *v* at which the satellite is orbiting around the Earth is $\sqrt{\frac{GM_E}{r}}$, Note that this expression is different from the expression for escape speed! (Compare $v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$.)

- The speed of the satellite only depends on the radius of the orbit. It is independent of the satellite mass. All satellites, regardless of size, orbiting at the same height will be moving at the same speed.

- As the speed of satellite orbit motion decrease when r is larger, the orbital time *T* to complete one round is also larger when the satellite is further away. It can be shown that $T^2 \; \alpha \; r^3$. (See Example 11 next page)

Example 10

Given that a satellite orbits around the Earth at a height of 850 km above its surface, determine

(a) the speed of the satellite,
(b) its period of revolution T.

(Given that radius of the Earth $R_E = 6.4 \times 10^6$ m; mass of the Earth $M_E = 6.0 \times 10^{24}$ kg.)

Solution:

(a)
$$F_g = F_c \Rightarrow \frac{GM_E m}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM_E}{r}}$$
$$\therefore v = \sqrt{\frac{6.67 \times 10^{-11} \times \left(6.0 \times 10^{24}\right)}{\left(6.4 \times 10^6\right) + \left(850 \times 10^3\right)}} \approx 7400 \text{ m s}^{-1}$$

(b) $$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v} = \frac{2\pi \times \left[\left(6.4 \times 10^6\right) + \left(850 \times 10^3\right)\right]}{7430} \approx 6131 \text{ s} \approx 1.7 \text{ hours}$$

(k) show an understanding of satellites in geostationary orbit and their applications

4.2 Geostationary satellite

- The term geostationary implies a satellite is fixed about a position above the Earth.

- This is possible because
  - the satellite has the same period as the Earth's rotation (T = 24 hours),
  - the satellite revolves from West to East, in the same direction as the Earth's rotation,
  - the plane of revolution of the satellite is in the plane of the Equator.

- Their fixed position above a specific location on Earth allows for continuous, stable coverage over large areas, simplifying ground-based equipment and offering benefits for broadcasting, weather monitoring, and telecommunications.

Example 11

For a geostationary satellite, determine the radius of its orbit and its height above the Earth's surface.

Solution:
By Newton's second law,

$$F_g = F_c$$
$$\frac{GM_E m}{r^2} = mr\omega^2$$
$$\frac{GM_E}{r^2} = r\left(\frac{2\pi}{T}\right)^2$$
$$T^2 = \frac{4\pi^2 r^3}{GM_E} \qquad \text{(Kepler's 3}^{\text{rd}}\text{ law, where } T^2 \propto r^3\text{)}$$

For geostationary satellite, T = 24 hours, hence

$$(24 \times 3600)^2 = \frac{4\pi^2 r^3}{6.67 \times 10^{-11} \times (6.0 \times 10^{24})}$$

$$r \approx 4.2 \times 10^7 \text{ m}$$

Hence, the radius of the satellite's orbit is 4.2×10^7 m.

Height of satellite above the Earth's surface

$$= (4.2 \times 10^7) - (6.4 \times 10^6)$$

$$\approx 3.6 \times 10^7 \text{ m}$$

The equation $T^2 = \frac{4\pi^2 r^3}{GM_E}$ allows us to calculate M if r and T are given. Treating the Earth as a satellite of the Sun, T = 1 year, r = 1.52 x10^9 m, the mass of the Sun can be calculated.

4.3 Energy of satellite in orbit

- For a satellite of mass m in a circular orbit of radius r about a large mass M, the gravitational potential energy E_P of the system is given by

$$\boxed{E_P = -\frac{GMm}{r}}$$

- The kinetic energy E_K of an orbiting satellite can be found by applying Newton's second law on the satellite.

$$F_g = F_c$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

$$\boxed{E_K = \frac{GMm}{2r}}$$

- The total energy E_T is the sum of the gravitational potential and kinetic energies. (We assume that M is at rest with K.E. of M = 0.)

$$E_T = E_P + E_K$$

$$E_T = -\frac{GMm}{r} + \frac{GMm}{2r}$$

$$\boxed{E_T = -\frac{GMm}{2r}}$$

- The '–' sign of total energy shows that the satellite is bound by the Earth's gravitational field. It is not able to escape to infinity but only circling around the Earth. The energies vary with radius of orbit as show in the graph below.

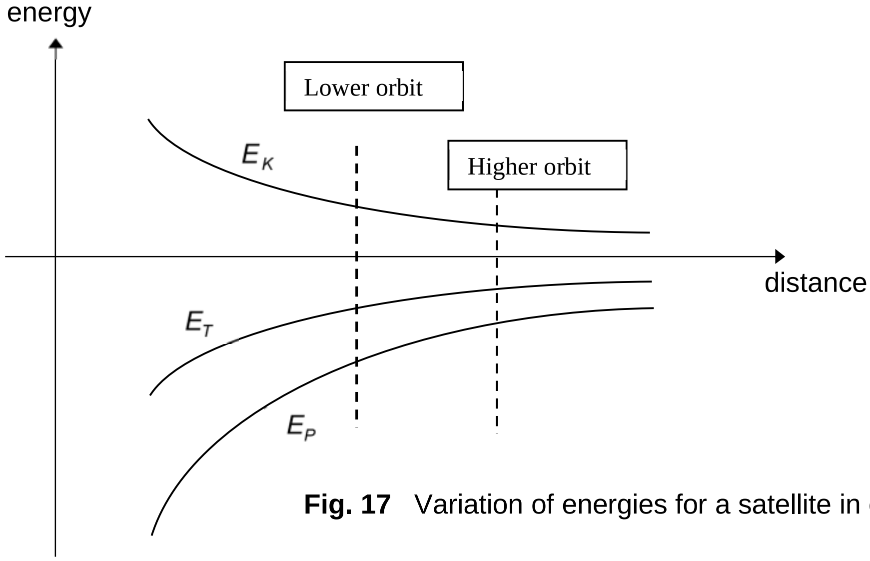


Fig. 17 Variation of energies for a satellite in orbit

- A satellite in a higher orbit or larger r has a higher total energy (less negative), a higher gravitational potential energy but a lower kinetic energy.

- The equations $E_k = G\frac{Mm}{2r}$ and $E_p = -G\frac{Mm}{r}$ also show that the magnitude of E_k is half the magnitude of E_p. When a satellite is moved from lower to higher orbit it slows down, but the decrease in E_k is not sufficient to account for the increase in E_p. External energy from rocket propulsion must be provided to enable this orbital transfer from lower to higher orbit.

  Energy required = $|\text{Gain in } E_p| - |\text{loss in } E_k| = |\text{loss in } E_k| = \frac{1}{2}|\text{gain in } E_p|$

- Conversely, if a satellite falls from higher to lower orbit due to atmospheric friction, it speeds up but the loss in E_p is double the gain in E_k. The difference in these 2 energy changes is accounted for by the work done against atmospheric friction.

  WD against friction = $|\text{loss in } E_p| - |\text{gain in } E_k| = |\text{gain in } E_k| = \frac{1}{2}|\text{loss in } E_P|$

- In principle, once a satellite is placed in a specific orbital radius with the correct speed, it will stay in orbit forever without any propulsion. Gravitational force does not do any work on a satellite in circular orbit because the gravitational force is always perpendicular to the displacement. In practice, however, satellites eventually lose altitude due to factors like atmospheric drag, solar wind or micro-meteoroid impacts, ultimately causing them to re-enter the Earth's atmosphere in a spiral path with increasing speed and burn up due to extremely high temperature produced.

Example 12

A satellite, of mass 120 kg, is orbiting Earth with an orbital radius *R* of 6610 km at a speed of 7780 m s^{-1}. It is boosted into higher orbit of radius 6890 km.

(a) Show that the speed of the satellite in the new orbit is 7620 m s^{-1}.

(b) Calculate the change in
 (i) kinetic energy,
 (ii) gravitational potential energy,
 (iii) total energy.

(c) State whether this change in total energy is an increase or decrease.

Solution:

(a) $F_g = F_c \Rightarrow \frac{GM_E m}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM_E}{r}}$

$7780 = \sqrt{\frac{GM_E}{6610 \times 10^3}}$

$GM_E = 4.001 \times 10^{14}$

Therefore, speed of satellite in new orbit,

$v' = \sqrt{\frac{GM_E}{r'}}$

$= \sqrt{\frac{4.001 \times 10^{14}}{6890 \times 10^3}}$

$\approx 7620 \ \text{m s}^{-1}$

(b) (i) $\Delta K.E. = \frac{1}{2}m(v_2)^2 - \frac{1}{2}m(v_1)^2$

$= \frac{1}{2}(120)(7620^2 - 7780^2)$

$\approx -1.48 \times 10^8$ J

(ii) $\Delta G.P.E. = U_2 - U_1 = \left(\frac{-GM_E m}{R_2}\right) - \left(\frac{-GM_E m}{R_1}\right) = GM_E m\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$

$= (4.001 \times 10^{14}) \times 120 \times \left[\frac{1}{(6610 \times 10^3)} - \frac{1}{(6890 \times 10^3)}\right]$

$\approx 2.95 \times 10^8$ J

(iii) Total change in energy

$= \Delta K.E. + \Delta G.P.E.$

$= (-1.48 \times 10^8) + (2.95 \times 10^8)$

$= 1.47 \times 10^8$ J

(c) The total energy has increased by 1.47×10^8 J.

Summary of key equations

Force	$-G\dfrac{Mm}{r^2}$
Field	$g=\dfrac{F}{m},\ g=-\dfrac{GM}{r^2}$
Potential	$\phi=\dfrac{W_{from\ \infty}}{m},\ \phi=-G\dfrac{M}{r},\ g=-\dfrac{d\phi}{dr}$
Potential Energy	$U=m\phi,\ U=-G\dfrac{Mm}{r},\ F=-\dfrac{dU}{dr}$
Circular motion	$G\dfrac{Mm}{r^2}=\dfrac{mv^2}{r}$, $T^2\alpha\ r^3$, $E_k=G\dfrac{Mm}{2r},\ E_p=-G\dfrac{Mm}{r},\ E_T=-G\dfrac{Mm}{2r}$

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