NANYANG JUNIOR COLLEGE

JC2 COMMON TEST 2

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **5th July 2022**

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

NANYANG JUNIOR COLLEGE
Internal Examinations

1 Determine the maximum number of iterations, using the *Euler Method*, with step size 0.1 on the differential equation

$$4y\frac{\mathrm{d}y}{\mathrm{d}x} - xy^2 = x, \quad (y \geq 0)$$

where $y = 1$ when $x = 0$, such that the error between the actual value of y and the approximated value of y is less than 0.1. [5]

2 Let $\mathbf{M}_{2\times 2}(\mathbb{R})$ be a vector space consisting of all 2×2 matrices with real entries. Let V be a subset of $\mathbf{M}_{2\times 2}(\mathbb{R})$ consisting of all symmetric traceless matrices such that

$$V = \left\{\mathbf{A} \in \mathbf{M}_{2\times 2}(\mathbb{R}) : \mathbf{A} = \mathbf{A}^T \text{ and } \operatorname{trace}(\mathbf{A}) = 0\right\},$$

where $\operatorname{trace}(\mathbf{A})$ is the sum of the diagonal entries of $\mathbf{A}$.

(i) Show that V is a subspace of $\mathbf{M}_{2\times 2}(\mathbb{R})$. [2]

Let W be a subspace of $\mathbf{M}_{2\times 2}(\mathbb{R})$ consisting of matrices whose first column has both entries equal to zero.

(ii) By finding a basis for V, show that $V + W = \mathbf{M}_{2\times 2}(\mathbb{R})$. [3]

3 The curve C has polar equation

$$r = 2\cot\theta,$$

where $r \geq 0$ and $0 < \theta < 2\pi$ such that $\theta \neq \pi$.

(i) Show that the lines with cartesian equation $y = 2$ and $y = -2$ are asymptotes of C. [1]

(ii) Sketch C, showing clearly the behaviour of C near the origin. State the cartesian equation of the tangent to C at the pole. [2]

(iii) The region R is bounded by C, the lines with cartesian equation $y = \sqrt{3}$ and $x = 0$. Find the exact area of R. [4]

4 In ecology, a population response model describes the interaction between the availability of food resources and the population of a particular species. The model assumes that the food availability decreases linearly with the population in that same month, while the population growth is proportional to food availability in the previous month.

Let F_n and P_n denote the food availability and population in month n, respectively. The model is defined by:

$$F_n = a - bP_n \quad \text{and} \quad P_n = cF_{n-1},$$

where a, b, c are positive constants. The ecosystem adjusts population each month so that food availability equals population size, such that $F_n = P_n$ for all n.

(a) In terms of a, b, c, write down the first order linear recurrence relation for P_n. Hence solve for P_n given that the initial population is P_0. [5]

It is known that P_n converges to P_e, as $n \to \infty$, and that $P_0 \neq P_e$.

(b) State the condition for convergence of P_n, and write down the value of P_e. [2]

(c) Describe, with reasons, the behaviour of P_n as it approaches P_e. [2]

5 It is required to solve the equation $\ln(x-1) - x + 3 = 0$.

It is given that there are two roots, α and β, where $1.1 < \alpha < 1.2$ and $4.1 < \beta < 4.2$.

(i) The root β can be found using the iterative formula

$$x_{n+1} = \ln(x_n - 1) + 3.$$

(a) Using this iterative formula with $x_1 = 4.15$, find β correct to 3 decimal places. You are to show all your working. [2]

(b) Explain with the aid of a sketch why this iterative formula will not converge to α for any initial value taken. [2]

(ii) Show that the Newton-Raphson iterative formula for this equation can be written in the form

$$x_{n+1} = \frac{3 - 2x_n - (x_n - 1)\ln(x_n - 1)}{2 - x_n}.$$ [2]

(a) Explain with the aid of a sketch why this iterative formula will not converge to α if the initial value $x_1 = 2.2$ is taken. [1]

(b) Use this formula with $x_1 = 1.2$ to find α correct to 3 decimal places. [1]

6 **(a)** Define $I_n = \int \frac{\sin nx}{\sin x}\,\mathrm{d}x$. Prove that, for $n > 2$,

$$I_n - I_{n-2} = \frac{2\sin(n-1)x}{n-1} + C,$$

where C is an arbitrary constant. [2]

Hence find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y\cot x = \sin 5x.$$ [5]

(b) Given that $y = \frac{u}{x}$, show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{x}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{2}{x^2}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{2u}{x^3}.$$ [2]

Hence find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + 25y = 0,\ x > 0.$$ [3]

7 Most bacteria can grow to some extent in the presence of oxygen, known as aerobic culture. A scientist in a laboratory tried to adjust the conditions to suit the target bacterium. Rich nutrient or complete media can be helpful when trying to bulk up a pure culture and get the bacterial cells in good condition. As such, the scientist tried to manage the nutrient for the bacteria he intended to culture and postulated that the amount of nutrients, u_n, measured in suitable units is such that $u_0 = 1$ and $u_1 = \sqrt{2}$, satisfies the following recurrence relation

$$\begin{vmatrix} 2 & 1 & 2 \\ 0 & 2u_{n-1} & 1 \\ 2 & u_n & 2 \end{vmatrix} = u_{n-2} + 2 \quad \text{for} \quad n \geq 2.$$

(i) Solve the recurrence relation. [5]

Subsequently, the scientist found that this recurrence relation is connected to the amount of bacteria, x_n (in billions), cultured in a bottle of fermented milk drink in a laboratory in n days. He postulated that

$$x_n = 20\left(\sqrt{2}\right)^n u_n + \sum_{r=1}^{n}\left(u_r - u_{r-1}\right).$$

(ii) Show that, in long run, the amount of bacteria in the bottle of fermented milk cannot exceed a certain value which is to be determined. [4]

A second scientist disagreed with the first scientist's hypothesis. He felt that that a certain proportion of the bacteria is lost due to temperature fluctuation in the process of culturing. Instead, he postulated that the number of bacteria, x_n is more than 80% of x_{n-1} by 10.

(iii) Write down the new recurrence relation relating x_n and x_{n-1}. Hence, find the amount of bacteria, in billion, in the long run according to this new conjecture if $x_1 = 1$. [4]

8 Sketch on an Argand diagram, the locus of the point P representing the complex number z where $z=1-\mathrm{e}^{\mathrm{i}\theta}\cos\theta$ and $0\le\theta<\pi$. You are to provide full justification for your conclusion.

Find, in terms of θ, the modulus and argument of z.

By considering z^n where n is a non-negative integer, prove that, for $0\le\theta<\pi$,

$$\sum_{r=0}^{n}\left[-1^{\,r}\binom{n}{r}\cos^r\theta\cos r\theta\right]=\sin^n\theta\cos n\left(\theta-\frac{1}{2}\pi\right).$$ [11]

9 The roof of the James S. McDonnell Planetarium in Saint Louis (Figure 1) is designed to resemble a *hyperboloid*, which is created when a hyperbola is rotated about one of its axes of symmetry.

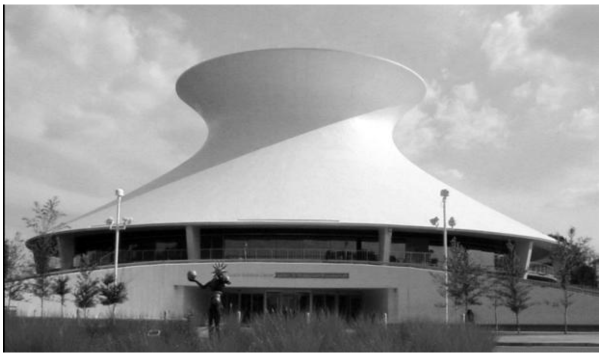


Figure 1

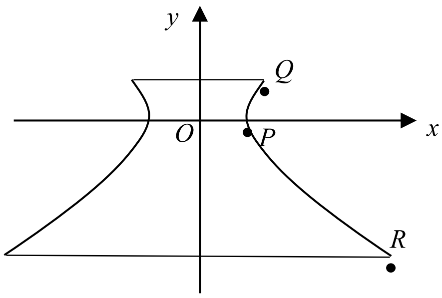


Figure 2

Figure 2 shows the *outline* of the roof. Coordinate axes have been superimposed onto this outline for ease of reference. The defining curve for the outline of the roof is the hyperbola H which contains the points $P(6,0)$, $Q(10,4)$ and $R(2\sqrt{205},-14)$, as shown on Figure 2. P lies on the x-axis, which is one of the axis of symmetry of H. All distances are in metres.

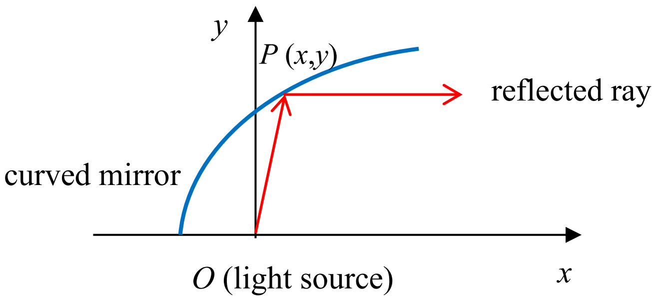
(i) Determine a cartesian equation for H. [2]

(ii) Show that the volume enclosed by the roof is 4392π cubic metres. You may assume that the thickness of the roof is negligible. [3]

(iii) Find the perimeter of the outline of the roof, giving your answer in metres to the nearest integer. [4]

(iv) Find the definite integral which, when evaluated, will give exactly the curved surface area of the roof. Evaluate this integral using Simpson's Rule with four strips, giving your answer to the nearest integer. [5]

10 The diagram shows a curved mirror such that any ray of light emanating from the source located at the origin O of a cartesian coordinate system is always reflected parallel to the x-axis.



The ray emanating from O strikes the curved mirror at P with coordinates (x, y). Let α be the acute angle which OP makes with the tangent to the curved mirror at P and let β be the angle which OP makes with the x-axis. Assume that the law of reflection of light holds.

(i) Express β in terms of α. [2]

(ii) Write down expressions for $\tan\alpha$ and $\tan\beta$ in terms of x, y and $\frac{\mathrm{d}y}{\mathrm{d}x}$, whichever is appropriate. [2]

(iii) Using the results in **(i)** and **(ii)**, prove that

$$\left[1-\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]\left(\frac{y}{x}\right)=2\frac{\mathrm{d}y}{\mathrm{d}x}. \qquad (1)$$ [2]

(iv) Use the substitution $p=\frac{\mathrm{d}y}{\mathrm{d}x}$ to show that the differential equation (1) can be reduced to the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x}+x=\pm\sqrt{x^2+y^2}. \qquad (2)$$ [2]

(v) Use the substitution $z=x^2+y^2$ to find the general solution of the differential equation (2). [4]

(vi) Given that P has coordinates $(-a, 0)$, find the polar equation of the curved mirror, expressing your answer in the form of $r=\mathrm{f}(\theta)$, where $\mathrm{f}(\theta)$ is a function of θ and a. [4]

———END OF PAPER———