RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 2A: Vectors I
Vector Algebra, Ratio Theorem, Scalar and Vector Products

SYLLABUS INCLUDE

- Addition and subtraction of vectors, multiplication of a vector by a scalar, and their geometrical interpretations
- Position vectors, displacement vectors and direction vectors
- Magnitude of a vector
- Unit vectors
- Distance between two points
- Collinearity
- Use of the ratio theorem in geometrical applications
- Concepts of scalar product and vector product of vectors and their properties
- Angle between two vectors
- Geometrical meaning of $|\mathbf{a}\cdot\hat{\mathbf{n}}|$ and $|\mathbf{a}\times\hat{\mathbf{n}}|$, where $\hat{\mathbf{n}}$ is a unit vector

CONTENT

We will now move on to study vectors in three dimensions. When it comes to operations like addition, subtraction and scalar multiplication , the situation is totally analogous to what you have learn in secondary school. The only difference is the introduction of a third (z) component within the vectors. We shall start with a recap of the types of vectors.

In general, there are two types of vectors – position vectors and free vectors.

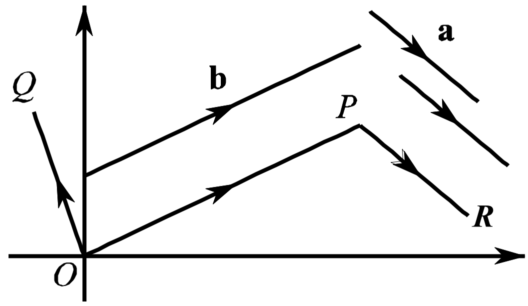
A **position vector** is used to denote the location of a point. It defines the position of one point relative to another point.

A **free vector** (also known as **displacement vector**) is a vector with no associated position.

$\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are the position vectors of points P and Q with respect to the origin O.

On the other hand, the vectors **a** and **b** are free vectors.

In fact, **a** can be represented by $\overrightarrow{PR}$, or any directed line segment of length $|\mathbf{a}|$ and having the same direction as **a**.



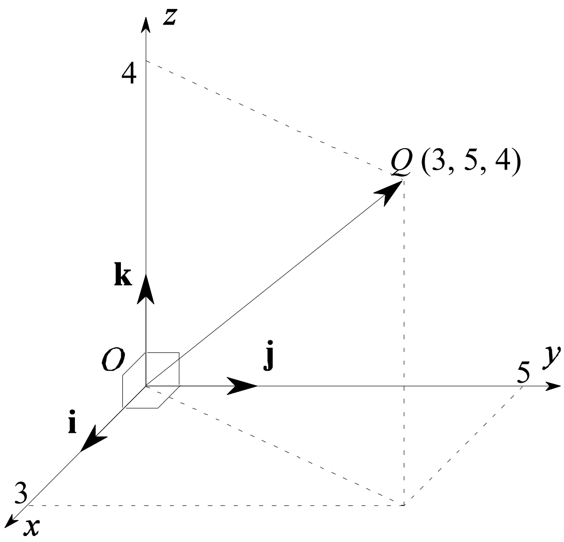
Finally, a **direction vector** is a vector that describes the direction of a line segment (this will be discussed more in the next chapter)

1 Vectors in Three Dimensions (3D Vectors)

To describe a vector in three dimensions, coordinates axes are chosen in the x, y and $z-$directions as shown on the right. Each vector can then be represented by giving its "components" in each of the chosen directions.

In the figure on the right, the vector $\overrightarrow{OQ}$ is 3 units in the x direction, 5 units in the y direction, and 4 units in the z direction.

We write $\overrightarrow{OQ} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$, where **i**, **j** and **k** are vectors of magnitude 1 in the x, y and z directions respectively.



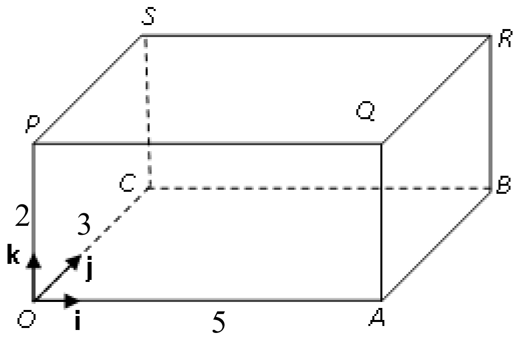
In column vector form, $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$, where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Note: Vectors of magnitude 1 are known as unit vectors which will be discussed in details in section 2.1.2.

Example 1

The diagram shows a cuboid where $OA = 5$ units, $OC = 3$ units and $OP = 2$ units. Using O as the origin, unit vectors **i**, **j** and **k** are taken along OA, OC and OP respectively.

State the corresponding column vectors for $\overrightarrow{OQ}$, $\overrightarrow{SQ}$ and $\overrightarrow{BP}$.



Solution

$\overrightarrow{OQ} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, $\overrightarrow{SQ} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}$, $\overrightarrow{BP} = \begin{pmatrix} -5 \\ -3 \\ 2 \end{pmatrix}$

1.1 Modulus of a Vector

The magnitude of a vector $\mathbf{a}$, denoted by $|\mathbf{a}|$, is also known as its modulus.

If $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $\boxed{|\mathbf{a}| = \sqrt{x^2 + y^2 + z^2}}$ (by Pythagoras' Theorem applied two times).

Example 2

With reference to the origin O, the position vectors of A, B and C are given by

$$\overrightarrow{OA}=\begin{pmatrix}2\\3\\-4\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}5\\-1\\2\end{pmatrix}\text{ and }\overrightarrow{OC}=\begin{pmatrix}11\\\lambda\\14\end{pmatrix}.$$

(i) Find the exact distance between A and B.

(ii) Find the position vector of D in terms of λ if $ABCD$ is a parallelogram.

Solution

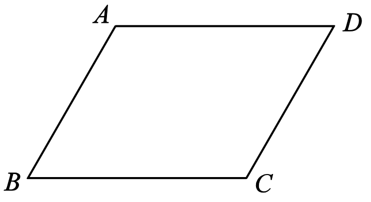
(i) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}5\\-1\\2\end{pmatrix}-\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}3\\-4\\6\end{pmatrix}$

Required distance between A and B

$=\left|\overrightarrow{AB}\right|$

$=\sqrt{3^2+(-4)^2+6^2}=\sqrt{61}$.

(ii)



$\overrightarrow{AB}=\overrightarrow{DC}=\overrightarrow{OC}-\overrightarrow{OD}$

$\overrightarrow{OD}=\overrightarrow{OC}-\overrightarrow{AB}=\begin{pmatrix}11\\\lambda\\14\end{pmatrix}-\begin{pmatrix}3\\-4\\6\end{pmatrix}=\begin{pmatrix}8\\\lambda+4\\8\end{pmatrix}$

2 Some Fundamental Results

2.1 Parallel Vectors and its Applications

2.1.1 Parallel Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors.

We say that $\mathbf{a}$ is parallel to $\mathbf{b}$, denoted by $\mathbf{a} \,//\, \mathbf{b}$, if and only if $\mathbf{b}=\lambda\mathbf{a}$ for some $\lambda\in\mathbb{R}\setminus\{0\}$.

In other words, two non-zero vectors are parallel if one is a scalar multiple of another.

2.1.2 Unit Vectors

A unit vector $\mathbf{a}$ is a vector whose magnitude is unity, i.e. $|\mathbf{a}|=1$.

Recall $\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ are unit vectors.

Other vectors such as $\mathbf{a}=\begin{pmatrix}0.6\\-0.8\\0\end{pmatrix}$, $\mathbf{b}=\dfrac{1}{13}\begin{pmatrix}0\\-5\\12\end{pmatrix}$ are also unit vectors.

For any non-zero vector $\mathbf{a}$, the unit vector in the direction of $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$.

If $\mathbf{b}$ is a vector parallel to $\mathbf{a}$, then $\mathbf{b} = \lambda\hat{\mathbf{a}}$, where $|\lambda|$ is the magnitude of $\mathbf{b}$.

In particular, $\boxed{\mathbf{a} = |\mathbf{a}|\hat{\mathbf{a}} \text{ or } \hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}.}$

Remark: In short, to obtain a unit vector from a given vector, it suffices to divide the vector by its magnitude (length).

Example 3
If $\mathbf{c} = 3\mathbf{a} - \mathbf{b}$, where $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 6\mathbf{j}$, find a unit vector parallel to $\mathbf{c}$.
Hence write down the vector of length 10 and in the same direction as $\mathbf{c}$.

Solution

$$\underset{\sim}{c} = 3\underset{\sim}{a} - \underset{\sim}{b} = 3\begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}-1\\6\\0\end{pmatrix} = \begin{pmatrix}4\\0\\-3\end{pmatrix}$$

A unit vector parallel to $\underset{\sim}{c}$ is given by $\hat{\underset{\sim}{c}} = \frac{1}{|\underset{\sim}{c}|}\underset{\sim}{c} = \frac{1}{\sqrt{4^2 + 0^2 + (-3)^2}}\begin{pmatrix}4\\0\\-3\end{pmatrix} = \frac{1}{5}\begin{pmatrix}4\\0\\-3\end{pmatrix}$

Another unit vector parallel to $\underset{\sim}{c}$ is $-\frac{1}{5}\begin{pmatrix}4\\0\\-3\end{pmatrix}$

The vector of length 10 and in the same direction as $\underset{\sim}{c}$ is $10 \times \frac{1}{5}\begin{pmatrix}4\\0\\-3\end{pmatrix} = 2\begin{pmatrix}4\\0\\-3\end{pmatrix}$

2.1.3 Collinear Points

Three points A, B and C are collinear (i.e. lie on a straight line) if and only if

$\overrightarrow{AB} \,//\, \overrightarrow{AC}$, with A as the common point.

In other words, $\overrightarrow{AB} = \lambda\overrightarrow{AC}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$.

We can also take B or C as the common point instead of A by showing $\overrightarrow{BA} \,//\, \overrightarrow{BC}$ or $\overrightarrow{CA} \,//\, \overrightarrow{CB}$ respectively.

Example 4

With reference to the origin O, the position vectors of A, B and C are given by $\overrightarrow{OA} = 2\mathbf{i} - 3\mathbf{j}$, $\overrightarrow{OB} = 4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\overrightarrow{OC} = 10\mathbf{i} + 21\mathbf{j} - 16\mathbf{k}$. Show that A, B and C are collinear.

Solution

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 10 \\ 21 \\ -16 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \\ -12 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 6 \\ -4 \end{pmatrix} = 3\overrightarrow{AB}.$$

Since $\overrightarrow{BC} \mathbin{/\!/} \overrightarrow{AB}$ with B as the common point, A, B and C are collinear. (shown)

Example 5

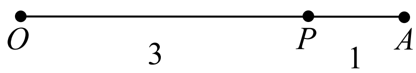
Two points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to the origin O.
The points P on OA and Q on AB are such that $OP = 3PA$ and $5AQ = 7QB$ respectively.
The line segment PQ is produced to R so that $5PQ = 4QR$.
Express $\overrightarrow{AQ}$, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Prove by a vector method that O, B and R are collinear.

Solution

Q on AB such that $5AQ = 7QB$ or $\dfrac{AQ}{QB} = \dfrac{7}{5}$

$$\therefore \ \overrightarrow{AQ} = \frac{7}{7+5}\overrightarrow{AB} = \frac{7}{12}(\mathbf{b} - \mathbf{a})$$

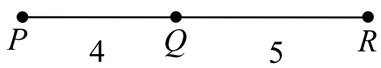
P on OA such that $OP = 3PA$ or $\dfrac{OP}{PA} = \dfrac{3}{1}$



$$\therefore \ \overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \frac{1}{3+1}\mathbf{a} + \frac{7}{12}(\mathbf{b} - \mathbf{a}) = -\frac{1}{3}\mathbf{a} + \frac{7}{12}\mathbf{b}$$

PQ produced to R so that $5PQ = 4QR$ or $\dfrac{PQ}{QR} = \dfrac{4}{5}$

$$\therefore \ \overrightarrow{PR} = \frac{9}{4}\overrightarrow{PQ} = \frac{9}{4}\left(-\frac{1}{3}\mathbf{a} + \frac{7}{12}\mathbf{b}\right) = -\frac{3}{4}\mathbf{a} + \frac{21}{16}\mathbf{b}$$



To show O, B and R are collinear:

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \frac{3}{4}\mathbf{a} + \left(-\frac{3}{4}\mathbf{a} + \frac{21}{16}\mathbf{b}\right) = \frac{21}{16}\overrightarrow{OB}$$

Since $OR \mathbin{/\!/} OB$ with O as the common point, O, B and R are collinear. (proven)

2.2 Linear Combinations of Non-Parallel Vectors

2.2.1 Non-Parallel Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero and non-parallel vectors. If $\lambda\mathbf{a} = \mu\mathbf{b}$ for some $\lambda,\ \mu \in \mathbb{R}$, then $\lambda = \mu = 0$.

2.2.2 Linear Combinations

Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero and non-parallel vectors.
Any vector $\mathbf{c}$ that is <u>coplanar</u> with (i.e. lie in the same plane as) $\mathbf{a}$ and $\mathbf{b}$ can be expressed as a <u>unique</u> linear combination of $\mathbf{a}$ and $\mathbf{b}$.
In other words, there exist unique $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$.

A useful result:

> If $\mathbf{a}$ and $\mathbf{b}$ are **non-zero** and **non-parallel** vectors such that $\alpha\,\mathbf{a} + \beta\,\mathbf{b} = \lambda\,\mathbf{a} + \mu\,\mathbf{b}$, then $\lambda = \alpha$ and $\mu = \beta$.

Example 6
Suppose $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors.
If $(\alpha - 5)\mathbf{a} + 3\alpha\mathbf{b}$ is parallel to $4\mathbf{a} + 2\mathbf{b}$, find α.

Solution
If $(\alpha - 5)\mathbf{a} + 3\alpha\mathbf{b}$ is parallel to $4\mathbf{a} + 2\mathbf{b}$,

$$\Rightarrow\ (\alpha - 5)\mathbf{a} + 3\alpha\mathbf{b} = \lambda(4\mathbf{a} + 2\mathbf{b}) \text{ for some } \lambda \in \mathbb{R}\setminus\{0\}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors,

$$\alpha - 5 = 4\lambda \quad \text{and} \quad 3\alpha = 2\lambda$$
$$\Rightarrow\ \alpha - 5 = 2(3\alpha)$$
$$\Rightarrow\ \alpha = -1$$

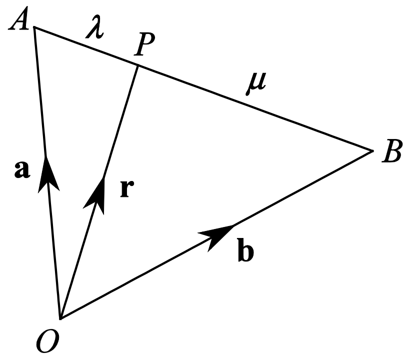
2.3 The Ratio Theorem

Consider a triangle OAB with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

Let P be a point which divides AB in the ratio $\lambda : \mu$,

i.e. $\dfrac{AP}{PB} = \dfrac{\lambda}{\mu}$.

If $\overrightarrow{OP} = \mathbf{r}$, then $\boxed{\mathbf{r} = \dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}}$



Proof: $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$= \mathbf{a} + \frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$$

$$= \mathbf{a} + \frac{\lambda}{\lambda+\mu}(\mathbf{b}-\mathbf{a})$$

$$= \frac{(\lambda+\mu)\mathbf{a} + \lambda\mathbf{b} - \lambda\mathbf{a}}{\lambda+\mu} = \frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$$

If P is the <u>mid-point</u> of AB, then P divides AB in the ratio $1:1$ $\Rightarrow$ $\overrightarrow{OP} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$.

Example 7

Points A, B and R are collinear, with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ respectively, relative to the origin O. Given that $\mathbf{a} = \begin{pmatrix}2\\2\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\6\\-3\end{pmatrix}$, find $\mathbf{r}$ if

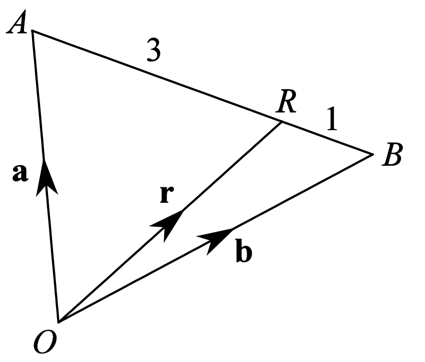
(a) R divides AB in the ratio $3:1$,

(b) R lies on BA produced such that $\frac{AR}{BR} = \frac{2}{5}$.

An equivalent form for (b) is 'R divides AB externally in the ratio $2:5$'.

Solution

(a)

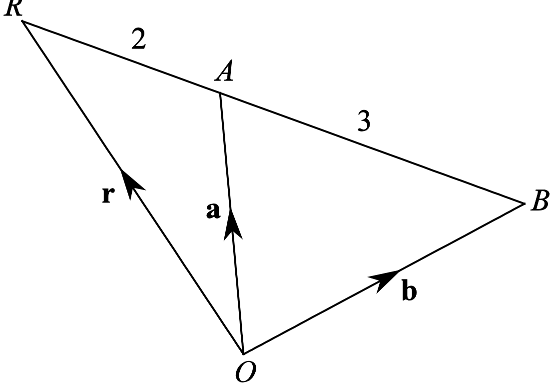


By Ratio Theorem,

$$\mathbf{r} = \frac{1\mathbf{a}+3\mathbf{b}}{1+3}$$

$$= \frac{1}{4}\left[\begin{pmatrix}2\\2\\5\end{pmatrix} + 3\begin{pmatrix}2\\6\\-3\end{pmatrix}\right] = \begin{pmatrix}2\\5\\-1\end{pmatrix}$$

(b)



By Ratio Theorem,

$$\mathbf{a} = \frac{3\mathbf{r}+2\mathbf{b}}{2+3}$$

$$\mathbf{r} = \frac{1}{3}(5\mathbf{a}-2\mathbf{b})$$

$$= \frac{1}{3}\left[\begin{pmatrix}10\\10\\25\end{pmatrix} - \begin{pmatrix}4\\12\\-6\end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix}6\\-2\\31\end{pmatrix}$$

3 The Scalar Product (or Dot Product)

We have seen how to add or subtract vectors. Next we shall look at a product of two vectors and its applications.

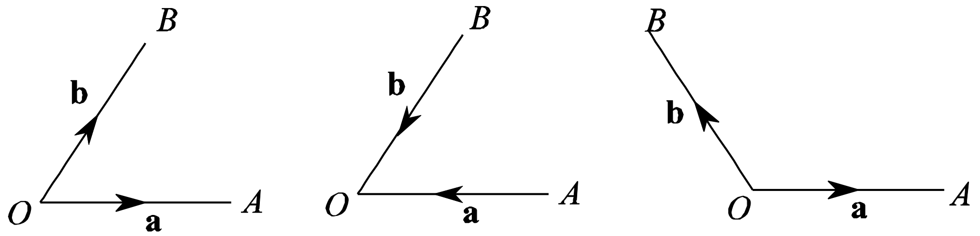
In this section, we shall discuss the product of two vectors whose output is a scalar, known as the scalar product or dot product.

3.1 Definition of Scalar Product

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors.
The scalar product (or dot product) of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\cdot\mathbf{b}$, is defined as

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta}$$

where θ $(= \angle AOB)$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$ and $\mathbf{b}$ are either both leaving from or both meeting at the same point.

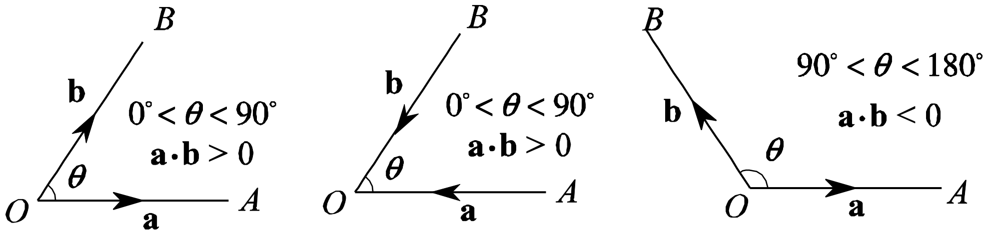


$\mathbf{a}\cdot\mathbf{b}$ is called the scalar product because the product is a scalar, i.e. $\mathbf{a}\cdot\mathbf{b}$ gives a scalar, not a vector. It is also called the dot product because of the notation.

3.2 Basic Properties of Scalar Product

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors, θ be the angle between $\mathbf{a}$ and $\mathbf{b}$, and λ be a non-zero scalar. Then we have the following basic properties of scalar product:

1. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
2. $\mathbf{a}\cdot(\mathbf{b}\pm\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})\pm(\mathbf{a}\cdot\mathbf{c})$
3. $\lambda(\mathbf{a}\cdot\mathbf{b}) = (\lambda\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(\lambda\mathbf{b})$
4. $\mathbf{a}\cdot\mathbf{b} > 0$ if $0° < \theta < 90°$ and $\mathbf{a}\cdot\mathbf{b} < 0$ if $90° < \theta < 180°$



5. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$ or $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$

Remark:
Always check the consistency of vector equations,
i.e. check that both sides of the equation are either both scalars or both vectors.

3.3 Perpendicular Vectors

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, denoted by $\mathbf{a} \perp \mathbf{b}$, if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Here $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 90° = |\mathbf{a}||\mathbf{b}|(0) = 0$.
In particular, $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ since $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular.

Question: Determine whether the following statements are true or false.
If a statement is false, give a reason or provide a counterexample.

1. If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} \cdot \mathbf{b} = 0$.
2. If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$. [Converse of statement **(**1**)**]
3. $[(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}]$ is a scalar.
4. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

Answer:

1. True
2. False
   Counterexample: $\mathbf{i}$ and $\mathbf{j}$ are perpendicular, so $\mathbf{i} \cdot \mathbf{j} = 0$, but $\mathbf{i}$ and $\mathbf{j}$ are non-zero
3. False
   $[(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}]$ is not defined since $\mathbf{a} \cdot \mathbf{b}$ is a scalar while $\mathbf{c}$ is a vector and we cannot talk about the scalar product of a scalar and vector.
4. False in general. Cancellation does not hold for scalar product.
   Counterexample: $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$, $\mathbf{c} = -\mathbf{j}$
   Then $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{0}$, but $\mathbf{b} \neq \mathbf{c}$

Example 8
The angle between two unit vectors $\mathbf{a}$ and $\mathbf{b}$ is $60°$.
If $\mathbf{p} = 2\mathbf{a} - \mathbf{b}$, $\mathbf{q} = -\mathbf{a} + 3\mathbf{b}$ and $\mathbf{r} = 3\mathbf{a} + \mathbf{b}$, prove that **(i)** $\mathbf{p} \cdot (\mathbf{r} + 3\mathbf{q}) = 0$,
(ii) $|\mathbf{p} + 2\mathbf{q} + 2\mathbf{r}| = \sqrt{127}$.

Solution

(i)

$$\begin{aligned}
&\mathbf{p} \cdot (\mathbf{r} + 3\mathbf{q}) \\
&= (2\mathbf{a} - \mathbf{b}) \cdot \left[(3\mathbf{a} + \mathbf{b}) + 3(-\mathbf{a} + 3\mathbf{b})\right] \\
&= (2\mathbf{a} - \mathbf{b}) \cdot (10\mathbf{b}) \\
&= 10\left[(2\mathbf{a} - \mathbf{b}) \cdot \mathbf{b}\right] \\
&= 10\left[2(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{b})\right] \\
&= 10\left[2|\mathbf{a}||\mathbf{b}|\cos 60° - |\mathbf{b}|^2\right] \\
&= 10\left[2(1)(1)\left(\frac{1}{2}\right) - 1^2\right] = 0
\end{aligned}$$

(ii)

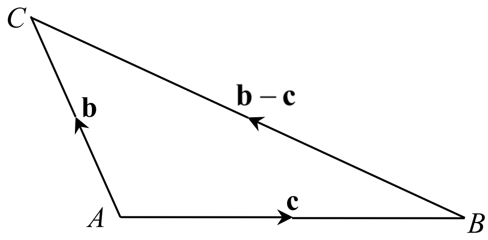
$$\begin{aligned}
&|\mathbf{p} + 2\mathbf{q} + 2\mathbf{r}| \\
&= |(2\mathbf{a} - \mathbf{b}) + 2(-\mathbf{a} + 3\mathbf{b}) + 2(3\mathbf{a} + \mathbf{b})| \\
&= |6\mathbf{a} + 7\mathbf{b}| \\
&= \sqrt{(6\mathbf{a} + 7\mathbf{b}) \cdot (6\mathbf{a} + 7\mathbf{b})} \\
&= \sqrt{36(\mathbf{a} \cdot \mathbf{a}) + 2(6)(7)(\mathbf{a} \cdot \mathbf{b}) + 49(\mathbf{b} \cdot \mathbf{b})} \\
&= \sqrt{36|\mathbf{a}|^2 + 2(42)\left(\frac{1}{2}\right) + 49|\mathbf{b}|^2} \\
&= \sqrt{36(1)^2 + 42 + 49(1)^2} = \sqrt{127}
\end{aligned}$$

Example 9 (9205/2001/02/Q15)

By expanding $(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})$, simplify $|\mathbf{b}|^2+|\mathbf{c}|^2-(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})$.

By taking $\mathbf{b}=\overrightarrow{AC}$ and $\mathbf{c}=\overrightarrow{AB}$, deduce the cosine formula for triangle ABC.

Solution



$$(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})=\mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c}-2\mathbf{b}\cdot\mathbf{c}$$
$$=|\mathbf{b}|^2+|\mathbf{c}|^2-2\mathbf{b}\cdot\mathbf{c}$$
$$|\mathbf{b}|^2+|\mathbf{c}|^2-(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})=2\mathbf{b}\cdot\mathbf{c}$$
$$|\mathbf{b}|^2+|\mathbf{c}|^2-|\mathbf{b}-\mathbf{c}|^2=2|\mathbf{b}||\mathbf{c}|\cos A$$

Since $\mathbf{b}-\mathbf{c}=\overrightarrow{AC}-\overrightarrow{AB}=\overrightarrow{BC}$, the above is the same as

$$AC^2+AB^2-BC^2=2(AC)(AB)\cos A$$ ----- cosine formula

3.4 A Formula in Cartesian Form for Scalar Product

Let $\mathbf{a}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$ be the position vectors of 2 points A and B respectively, relative to the origin O. Then

$$\boxed{\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}\cdot\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=x_1x_2+y_1y_2+z_1z_2}$$

Proof:

$\mathbf{a}\cdot\mathbf{b}$

$=(x_1\mathbf{i}+y_1\mathbf{j}+z_1\mathbf{k})\cdot(x_2\mathbf{i}+y_2\mathbf{j}+z_2\mathbf{k})$

$=(x_1x_2)\mathbf{i}\cdot\mathbf{i}+(x_1y_2)\mathbf{i}\cdot\mathbf{j}+(x_1z_2)\mathbf{i}\cdot\mathbf{k}+(y_1x_2)\mathbf{j}\cdot\mathbf{i}+(y_1y_2)\mathbf{j}\cdot\mathbf{j}+(y_1z_2)\mathbf{j}\cdot\mathbf{k}+(z_1x_2)\mathbf{k}\cdot\mathbf{i}+(z_1y_2)\mathbf{k}\cdot\mathbf{j}+(z_1z_2)\mathbf{k}\cdot\mathbf{k}$

$=x_1x_2+y_1y_2+z_1z_2$ since $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$ and $\mathbf{i}\cdot\mathbf{i}=|\mathbf{i}|^2=1=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}$

Example 10
With reference to the origin O, the points A, B and C have position vectors given by $4\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, $6\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $8\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ respectively. Show that ABC is a right-angled triangle.

Solution

$\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}=\begin{pmatrix}4\\3\\4\end{pmatrix}-\begin{pmatrix}6\\1\\2\end{pmatrix}=\begin{pmatrix}-2\\2\\2\end{pmatrix}$ and $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}8\\2\\3\end{pmatrix}-\begin{pmatrix}6\\1\\2\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix}$

Now $\overrightarrow{BA}\cdot\overrightarrow{BC}=\begin{pmatrix}-2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}=-4+2+2=0$

Since $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are perpendicular, ABC is a right-angled triangle.

3.5 Applications of Scalar Product

Let us look at two applications of the scalar product, namely to find the angle between two non-zero vectors and the length of projection of a vector onto another vector.

3.5.1 Angle between Two Non-Zero Vectors

Let θ be the angle between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$.

Since $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$, we can find θ from $\cos\theta=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

If $\mathbf{a}\cdot\mathbf{b}>0$, θ is acute; If $\mathbf{a}\cdot\mathbf{b}<0$, θ is obtuse.

Example 11
With reference to the origin O, the points A and B have position vectors given by

$$\mathbf{a}=\mathbf{i}+2\mathbf{j}-\mathbf{k} \quad \text{and} \quad \mathbf{b}=-3\mathbf{i}+\mathbf{k}$$

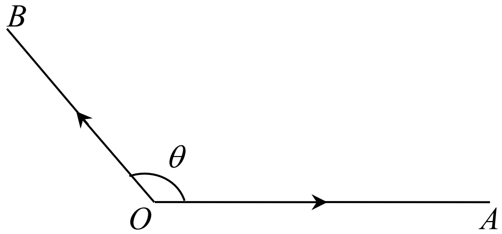
respectively.

Find **(i)** the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$,

(ii) $\angle OAB$,

(iii) The angle between $\overrightarrow{OA}$ and the positive y-direction.

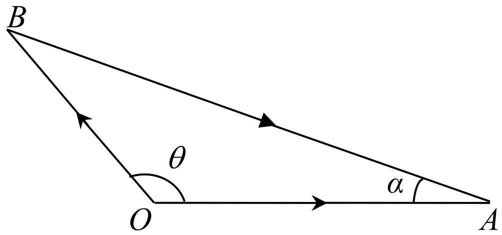
Solution



(i) Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-3\\0\\1\end{pmatrix}}{\sqrt{1^2+2^2+(-1)^2}\sqrt{(-3)^2+0^2+1^2}} = \frac{-3+0-1}{\sqrt{6}\sqrt{10}} = -\frac{2}{\sqrt{15}}$$

$\Rightarrow\ \theta = 121.1°$ (1 d.p.)



(ii) Let α be $\angle OAB$

$$\overrightarrow{BA} = \mathbf{a}-\mathbf{b} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}-3\\0\\1\end{pmatrix} = \begin{pmatrix}4\\2\\-2\end{pmatrix}$$

$$\cos(\angle OAB) = \frac{\overrightarrow{OA}\cdot\overrightarrow{BA}}{|\overrightarrow{OA}||\overrightarrow{BA}|} = \frac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\2\\-2\end{pmatrix}}{\sqrt{1^2+2^2+(-1)^2}\sqrt{4^2+2^2+(-2)^2}} = \frac{4+4+2}{\sqrt{6}\sqrt{24}} = \frac{5}{6}$$

$\Rightarrow\ \angle OAB = 33.6°$ (1 d.p.)

(iii) Let φ be the angle between $\overrightarrow{OA}$ and the positive y-direction. Then φ is the angle between $\overrightarrow{OA}$ and $\mathbf{j}$.

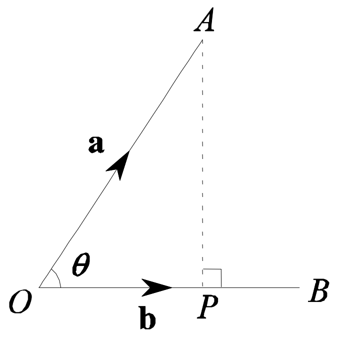
Thus

$$\cos\varphi = \frac{\mathbf{a}\cdot\mathbf{j}}{|\mathbf{a}||\mathbf{j}|} = \frac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{\sqrt{1^2+2^2+(-1)^2}\sqrt{0^2+1^2+0^2}} = \frac{2}{\sqrt{6}}$$

$\Rightarrow\ \varphi = 35.3°$ (1 d.p.)

3.5.2 Length of Projection

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors, represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.
Let P be the foot of perpendicular from A to the line OB, and let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.



Then the **length of projection of vector $\mathbf{a}$ onto vector $\mathbf{b}$** is given by

$$\boxed{\left|\overrightarrow{OP}\right| = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|}$$

where $\hat{\mathbf{b}}$ is a unit vector in the direction of $\mathbf{b}$.

Proof: (For θ acute)

$$\left|\overrightarrow{OP}\right| = |\mathbf{a}|\cos\theta$$
$$= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$
$$= \mathbf{a}\cdot\hat{\mathbf{b}}$$

Since $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$,

$$|\mathbf{a}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$

Note:

When θ is acute, $\mathbf{a}\cdot\mathbf{b} > 0$, so $\mathbf{a}\cdot\hat{\mathbf{b}} = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$.

When θ is obtuse, $\mathbf{a}\cdot\mathbf{b} < 0$, so we have $\left|\overrightarrow{OP}\right| = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$ as length OP is non-negative.

Example 12

With reference to the origin O, the points A, B and C have position vectors given by $\mathbf{i}+2\mathbf{j}-\mathbf{k}$, $3\mathbf{i}+\mathbf{k}$ and $2\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ respectively. Find the length of projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$.

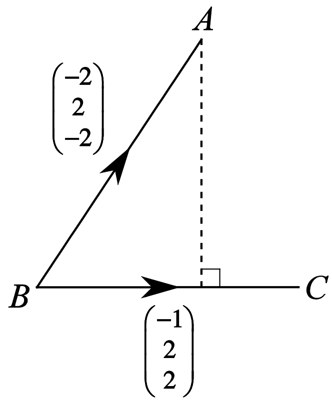
Solution

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}3\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\2\\-2\end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}2\\2\\3\end{pmatrix} - \begin{pmatrix}3\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\2\end{pmatrix}$$

Length of projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$ is given by

$$\frac{\left|\overrightarrow{BA}\cdot\overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|} = \frac{\left|\begin{pmatrix}-2\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\2\end{pmatrix}\right|}{\sqrt{(-1)^2+2^2+2^2}} = \frac{|2+4-4|}{\sqrt{9}} = \frac{2}{3}$$



4 The Vector Product (or Cross Product)

4.1 Definition of Vector Product

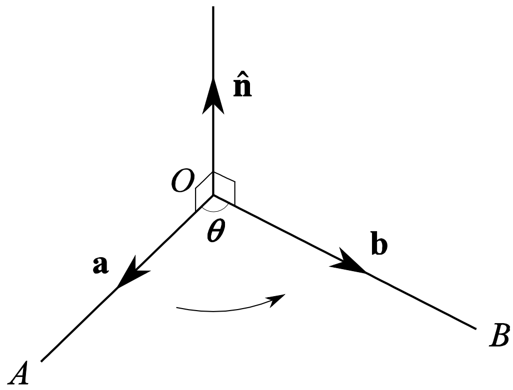


Figure 1

Let **a** and **b** be two non-zero vectors that are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Then the vector product (or cross product) of **a** and **b**, denoted by $\mathbf{a}\times\mathbf{b}$, is defined as

$$\boxed{\mathbf{a}\times\mathbf{b}=\left(|\mathbf{a}||\mathbf{b}|\sin\theta\right)\hat{\mathbf{n}}},$$

where $\theta=\angle AOB$ is the angle between **a** and **b**, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both **a** and **b**, with direction determined by the right hand rule, as shown in Figure 1.
Thus $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both **a** and **b**.

$\mathbf{a}\times\mathbf{b}$ is called the vector product because the product is a vector. It is also called the cross product because of the notation.

Note: $\mathbf{a}\cdot\mathbf{0}=0,$ the real number zero, whereas $\mathbf{a}\times\mathbf{0}=\mathbf{0},$ the zero vector.

4.2 Properties of Vector Product

Let **a**, **b** and **c** be three non-zero vectors, θ be the angle between **a** and **b**, and λ be a non-zero scalar. Then we have the following basic properties of vector product:

1. $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a}),$ so $\mathbf{a}\times\mathbf{b}\neq\mathbf{b}\times\mathbf{a}.$
2. $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c})=(\mathbf{a}\times\mathbf{b})\pm(\mathbf{a}\times\mathbf{c}).$
3. $\lambda(\mathbf{a}\times\mathbf{b})=(\lambda\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\lambda\mathbf{b}).$
4. $|\mathbf{a}\times\mathbf{b}|=|\mathbf{b}\times\mathbf{a}|=|\mathbf{a}||\mathbf{b}||\sin\theta|\leq|\mathbf{a}||\mathbf{b}|.$
5. Two non-zero vectors **a** and **b** are <u>perpendicular</u> if and only if $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|.$
   In particular, $\mathbf{i}\times\mathbf{j}=\mathbf{k},$ $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}.$
6. Two non-zero vectors **a** and **b** are <u>parallel</u> if and only if $\mathbf{a}\times\mathbf{b}=\mathbf{0}.$
   In particular, $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}.$

Example 13
Three non-zero vectors **a**, **b** and **c** satisfy $\mathbf{a}\times(3\mathbf{b})=(2\mathbf{a})\times\mathbf{c}.$
What can be said about the vector $3\mathbf{b}-2\mathbf{c}$?

Solution

$$[\mathbf{a}\times(3\mathbf{b})]-[2\mathbf{a}\times\mathbf{c}]=\mathbf{0}$$

$$[\mathbf{a}\times(3\mathbf{b})]-[\mathbf{a}\times(2\mathbf{c})]=\mathbf{0}$$

$$\mathbf{a}\times(3\mathbf{b}-2\mathbf{c})=\mathbf{0}$$

Hence **a** and $3\mathbf{b}-2\mathbf{c}$ are parallel OR $3\mathbf{b}-2\mathbf{c}=\mathbf{0}$.

Question: Determine whether the following statements are true or false.

If a statement is false, explain why it is false or provide a counterexample.

1. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
2. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.
3. $[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$ is a vector.
4. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.

Answer:

1. True, since by definition, $\mathbf{a} \times \mathbf{a} = \left(|\mathbf{a}||\mathbf{a}|\sin 0^\circ\right)\hat{\mathbf{n}} = \mathbf{0}$. Or we can note that **a** is parallel to **a**.
2. False in general. Counterexample:
   Take $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{j}$.
   $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$, while $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq \mathbf{0}$
3. False.
   [(**a** × **b**) **.** **c**] is a scalar since **a** × **b** is a vector and the scalar product of two vectors (**a** × **b**) and **c** is a scalar.
4. True since by definition, **a** × **b** is a vector perpendicular to both **a** and **b** and hence its scalar product with both **a** and **b** are both zero.

4.3 Vector Product for Vectors in Cartesian Form

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in three-dimensional space and let the position vectors of A and B with respect to the origin O be **a** and **b** respectively. Then

$$\boxed{\mathbf{a} \times \mathbf{b} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}}.$$

Proof:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \times (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) \\
&= (x_1 x_2)\mathbf{i} \times \mathbf{i} + (x_1 y_2)\mathbf{i} \times \mathbf{j} + (x_1 z_2)\mathbf{i} \times \mathbf{k} \\
&\quad + (y_1 x_2)\mathbf{j} \times \mathbf{i} + (y_1 y_2)\mathbf{j} \times \mathbf{j} + (y_1 z_2)\mathbf{j} \times \mathbf{k} \\
&\quad + (z_1 x_2)\mathbf{k} \times \mathbf{i} + (z_1 y_2)\mathbf{k} \times \mathbf{j} + (z_1 z_2)\mathbf{k} \times \mathbf{k} \\
&= (x_1 y_2)\mathbf{k} - (x_1 z_2)\mathbf{j} - (y_1 x_2)\mathbf{k} + (y_1 z_2)\mathbf{i} + (z_1 x_2)\mathbf{j} - (z_1 y_2)\mathbf{i} \\
&= (y_1 z_2 - z_1 y_2)\mathbf{i} + (z_1 x_2 - x_1 z_2)\mathbf{j} + (x_1 y_2 - y_1 x_2)\mathbf{k}
\end{aligned}$$

Example 14

Find a unit vector that is perpendicular to both vectors $(2\mathbf{i}-\mathbf{j}+\mathbf{k})$ and $(3\mathbf{i}+4\mathbf{j}-\mathbf{k})$.
Find also the sine of the angle between the two vectors.

Solution

Let $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}$.

Then $\mathbf{a}\times\mathbf{b}$ is a vector that is perpendicular to both **a** and **b**.

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix}\times\begin{pmatrix}3\\4\\-1\end{pmatrix}=\begin{pmatrix}1-4\\-(-2-3)\\8+3\end{pmatrix}=\begin{pmatrix}-3\\5\\11\end{pmatrix}$$

The required unit vector is

$$\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}=\frac{1}{\sqrt{(-3)^2+5^2+11^2}}\begin{pmatrix}-3\\5\\11\end{pmatrix}=\frac{1}{\sqrt{155}}\begin{pmatrix}-3\\5\\11\end{pmatrix}$$

Let θ, $0^\circ\le\theta\le180^\circ$, be the angle between **a** and **b**.

Now, $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}||\sin\theta|=|\mathbf{a}||\mathbf{b}|\sin\theta$ (since $0^\circ\le\theta\le180^\circ$), so

$$\sin\theta=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}=\frac{\sqrt{155}}{\sqrt{2^2+(-1)^2+1^2}\sqrt{3^2+4^2+(-1)^2}}=\frac{\sqrt{155}}{\sqrt{6}\sqrt{26}}=\sqrt{\frac{155}{156}}$$

4.4 Applications of Vector Product

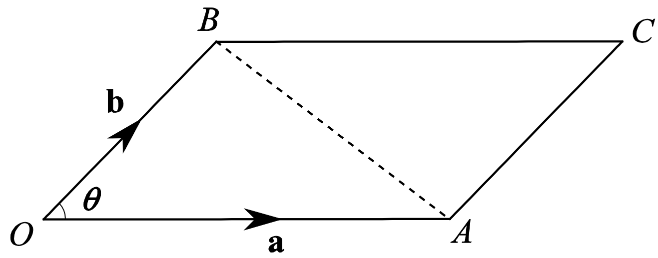
Let us explore two applications of the vector product, firstly to find the areas of triangles and parallelograms, and secondly, to find the perpendicular or shortest distance from a point to a line.

4.4.1 Areas of Triangles and Parallelograms

Let the points A and B have position vectors **a** and **b** with respect to the origin O, and let θ be the angle between **a** and **b**. Let C be the point such that $OACB$ is a parallelogram. Then we have the following two results:

(1) Area of triangle OAB is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

(2) Area of parallelogram $OACB$ is $|\mathbf{a}\times\mathbf{b}|$.



Proof: **(1)** Area of the triangle OAB is $\frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

(2) Area of the parallelogram $OACB$ is 2(Area of triangle OAB) $=|\mathbf{a}\times\mathbf{b}|$.

Example 15

Referred to the origin O, the position vectors of the points A, B and C are $2\mathbf{i}-\mathbf{j}$, $3\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ and $-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ respectively. Use a vector product to find the exact area of triangle ABC.

Solution

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}3\\-2\\2\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}-2\\1\\2\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}-4\\2\\2\end{pmatrix}$$

Area of triangle ABC $=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}-4\\2\\2\end{pmatrix}\right|=\frac{1}{2}\left|\begin{pmatrix}-6\\-10\\-2\end{pmatrix}\right|=\frac{1}{2}\sqrt{140}=\sqrt{35}$ units^2

Example 16

Relative to the origin O, the points P and Q have position vectors $\frac{1}{3}(\mathbf{b}-\mathbf{a})$ and $\frac{1}{5}(\mathbf{a}+3\mathbf{b})$ respectively.

(i) Show that the area of triangle OPQ is $k\left|\mathbf{a}\times\mathbf{b}\right|$, where k is a constant to be determined.

(ii) Given that $|\mathbf{a}|=\sqrt{2}$, $|\mathbf{b}|=3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{6}$ radians, find the exact area of triangle OPQ.

Solution

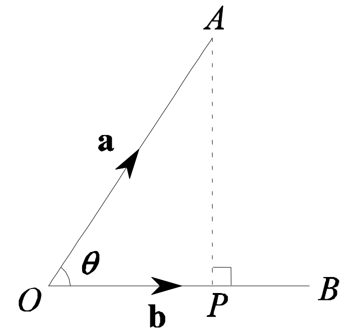
(i) Area of triangle OPQ
$$\begin{aligned}&=\frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right|\\&=\frac{1}{2}\left|\frac{1}{3}(\mathbf{b}-\mathbf{a})\times\frac{1}{5}(\mathbf{a}+3\mathbf{b})\right|\\&=\frac{1}{30}\left|(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times3\mathbf{b})-(\mathbf{a}\times\mathbf{a})-(\mathbf{a}\times3\mathbf{b})\right|\\&=\frac{1}{30}\left|-(\mathbf{a}\times\mathbf{b})-3(\mathbf{a}\times\mathbf{b})\right|\\&=\frac{1}{30}\left|-\left(4\mathbf{a}\times\mathbf{b}\right)\right|\\&=\frac{2}{15}\left|\mathbf{a}\times\mathbf{b}\right|\end{aligned}$$

$\therefore k=\frac{2}{15}$

(ii) Area of triangle OPQ $=\frac{2}{15}\left|\mathbf{a}\times\mathbf{b}\right|=\frac{2}{15}|\mathbf{a}||\mathbf{b}|\sin\frac{\pi}{6}=\frac{2}{15}(\sqrt{2})(3)\left(\frac{1}{2}\right)=\frac{\sqrt{2}}{5}$ units^2

4.4.2 Perpendicular distance from a point to a line

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors, represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.
Let P be the foot of perpendicular from A to the line OB, and θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.



Then the **perpendicular distance** from point A to the line OB is given by

$$\boxed{\left|\overrightarrow{AP}\right| = \left|\mathbf{a}\times\hat{\mathbf{b}}\right|}$$

where $\hat{\mathbf{b}}$ is a unit vector in the direction of $\mathbf{b}$.

Proof:

$$\left|\overrightarrow{AP}\right| = |\mathbf{a}|\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|} = \left|\mathbf{a}\times\hat{\mathbf{b}}\right|$$

Since $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$,

$$|\mathbf{a}|\sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$$

So, in a right-angle triangle OAP,

OP = length of projection of $\mathbf{a}$ onto $\mathbf{b}$ = $\left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$ [Recall in 3.5.2]

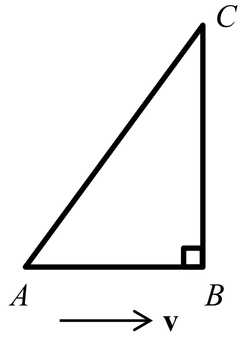
AP = perpendicular distance from point A to line OB = $\left|\mathbf{a}\times\hat{\mathbf{b}}\right|$

Example 17
The triangle ABC has a right angle at B and AB is parallel to the vector $\mathbf{v} = 2\mathbf{i}+2\mathbf{j}-\mathbf{k}$.
Given that $\overrightarrow{AC} = \mathbf{i}-3\mathbf{j}+\mathbf{k}$, find the lengths of AB and BC.

Solution

AB = Length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$

$$= \left|\overrightarrow{AC}\cdot\hat{\mathbf{v}}\right| = \left|\begin{pmatrix}1\\-3\\1\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}\right| = \frac{1}{3}|-5| = \frac{5}{3} \text{ units}$$



BC = Perpendicular distance from point C to line AB

$$= \left|\overrightarrow{AC}\times\hat{\mathbf{v}}\right| = \left|\begin{pmatrix}1\\-3\\1\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}\right| = \frac{1}{3}\left|\begin{pmatrix}1\\3\\8\end{pmatrix}\right|$$

$$= \frac{1}{3}\sqrt{1+9+64}$$

$$= \frac{\sqrt{74}}{3} \text{ units}$$

Note : Check your values using Pythagoras' Theorem, i.e. ensure that $AC^2 = AB^2 + BC^2$

SUMMARY