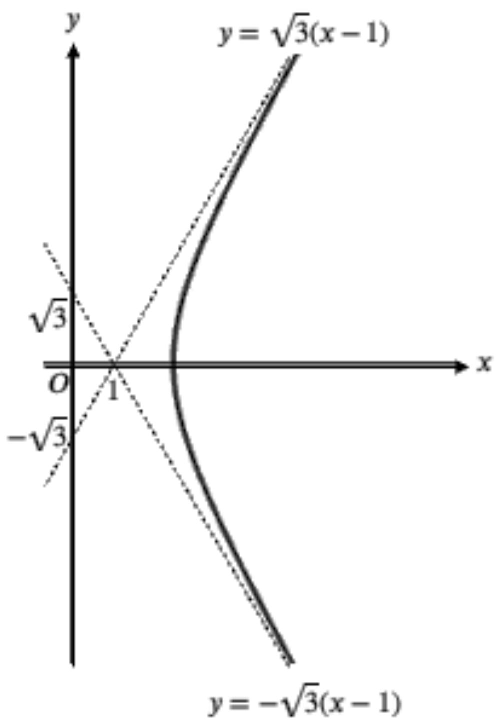
MATHEMATICS

Paper 9758/01

Set I – Paper 1

Topic identification and short answers

Qn	Topic(s)	Part	Answers
1	Complex numbers (cartesian form)		$A = -1$ $B = 1$
2	Sequences and series (arithmetic series)	**(a)**	$a = \dfrac{m^2 + (n-1)(m+n)}{mn}$ $d = -\dfrac{2(m+n)}{mn}$
		(b)	[shown]
3	Complex numbers (modulus, real and imaginary, argument); Graph (sketching, conics section, transformation)	**(a)**	[shown] $y = \pm\sqrt{3}(x-1)$
		(b)	
		(c)	$-\dfrac{\pi}{3} < \arg(z-1) < \dfrac{\pi}{3}$

4	Differentiation (implicit functions, equations of tangents)	**(a)**	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+3y+4}{2y-3x}$
		(b)	[shown]
		(c)	$3x + 5y + 2 = 0$ $11x + 28y + 46 = 0$
5	Complex numbers (polar form, modulus and argument, Argand diagram)	**(a)**	$\frac{u-r}{u+r} = \mathrm{i}\tan\frac{1}{2}\theta$
		(b)	[shown] Right angle at WOZ (or ZOW)
		(c)	$\angle OWZ = \frac{1}{2}\theta$ $\lvert u-r\rvert = 2r\sin\frac{1}{2}\theta$ $\lvert u+r\rvert = 2r\cos\frac{1}{2}\theta$
6	Vectors (ratio theorem, angle between two vectors, scalar products)	**(a)**	$\lambda = \frac{\lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}$
		(b)	[shown]
7	Functions (inverse, composite); System of linear equations	**(a) (i)**	$\mathrm{f}^{-1}(x) = \frac{x+1}{2x-1},\quad x \in \mathbb{R},\quad x \neq \frac{1}{2}$
		(a) (ii)	$\mathrm{g}(x) = \frac{4x-3}{x-1}$
		(b) (i)	$\mathrm{h}(2) = 0.5$ $\mathrm{h}(0.5) = -1$ $\mathrm{h}(-1) = 2.$
		(b) (ii)	$\mathrm{h}(x) = 1 - \frac{1}{x}$
8	Integration techniques (by substitution, by parts); Definite integrals (area of a region)	**(a)**	$I_0 = \frac{\pi a^2}{8}$
		(b)	[shown]
		(c)	$\frac{5\pi}{16}$ units^2
9	Vectors (three dimensions, vector product, magnitude, angle);	**(a)**	[shown] $b = 9.5$
		(b)	$p = \frac{7}{8};\ q = \frac{7}{8};\ r = \frac{9}{32}$ Area ≈ 132.98 units^2 ≈ 133 units^2
		(c)	Distance $\approx 14.60407 \approx 14.6$ units

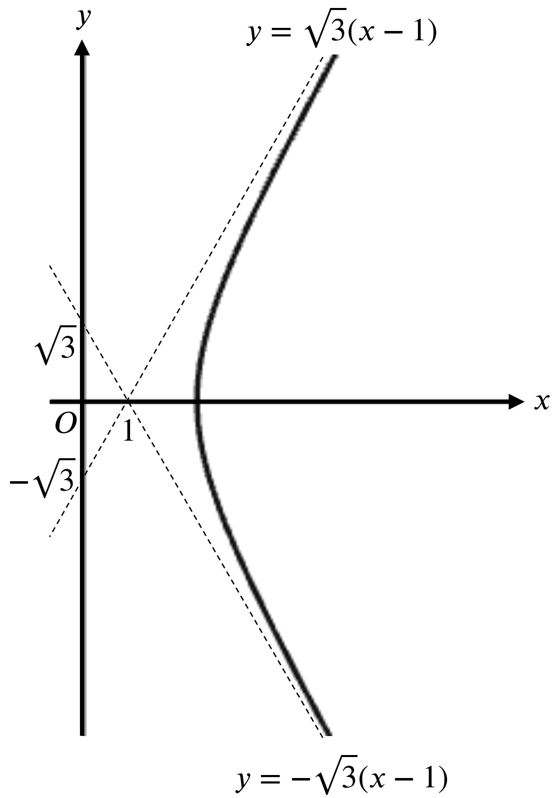
10	Differentiation (parametric functions, stationary points, tangents); Graphs (parametric equation, transformations); Definite integrals (volume of revolution)	**(a)**	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2 - 1}{2t}$
		(b)	$t_1 = \mathrm{e}^{-1}$ $t_2 = \mathrm{e}$
		(c)	Angle $= 49.6°$
		(d)	[shown] $k = \frac{a}{2}\left(\mathrm{e} + \mathrm{e}^{-1}\right)$
		(e)	Volume $= \frac{1}{4}\pi a^3\left(8 - e^2 + 5e^{-2}\right)$ units^3
11	Differential equations; Differentiation (local maxima and minima, connected rates of change)	**(a) (i)**	$\frac{\mathrm{d}R}{\mathrm{d}t} = 0.6R - 0.4RW$ $\frac{\mathrm{d}W}{\mathrm{d}t} = -0.8W + 0.2RW$
		(a) (ii)	[shown]
		(b) (i)	Rabbit: smallest = 900, largest = 10,700. Wolf: smallest = 300, largest = 4,500.
		(b) (ii)	B_2

Suggested solutions and post-mortem

Qn	Suggested Solutions	Post-mortem
1 **[3]**	By remainder theorem, $P(z) = \left(z^2+1\right)g(z) + Az + B$ Divided by $(z+\mathrm{i})$, remainder is $1+\mathrm{i} \rightarrow P(-\mathrm{i}) = -A\mathrm{i} + B = 1+\mathrm{i}$ Divided by $(z-\mathrm{i})$, remainder is $1-\mathrm{i} \rightarrow P(\mathrm{i}) = A\mathrm{i} + B = 1-\mathrm{i}$ Adding the two results to eliminate $A\mathrm{i}$, $-A\mathrm{i} + B + A\mathrm{i} + B = 1+\mathrm{i}+1-\mathrm{i}$ $2B = 2 \rightarrow B = 1$ Substituting $B = 1$ into previous result, $-A\mathrm{i} + 1 = 1 + \mathrm{i} \rightarrow A = -1$	This question mainly assesses on complex numbers expressed in cartesian form. Candidates are expected to recall the factor-remainder theorem from O–Level Additional Mathematics, which will lead to a system of two complex equations relating A and B. The remaining steps follow accordingly with algebraic elimination and substitution.

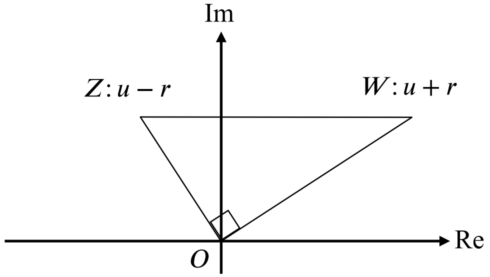
2 (a) [4]	$S_n = m \rightarrow \frac{n}{2}(2a+(n-1)d) = m \rightarrow 2a+(n-1)d = \frac{2m}{n}$ $S_m = n \rightarrow \frac{m}{2}(2a+(m-1)d) = n \rightarrow 2a+(m-1)d = \frac{2n}{m}$ Eliminating $2a$, $(n-1)d-(m-1)d = \frac{2m}{n}-\frac{2n}{m}$ $(n-m)d = 2\left(\frac{m^2-n^2}{mn}\right) = \frac{2(m+n)(m-n)}{mn}$ $\therefore d = -\frac{2(m+n)}{mn}$ Substituting d into S_n, $2a = \frac{2m}{n}-(n-1)\left(-\frac{2(m+n)}{mn}\right) = \frac{2m^2+2(n-1)(m+n)}{mn}$ $\therefore a = \frac{m^2+(n-1)(m+n)}{mn}$	This question mainly deals with arithmetic series. It should be relatively straightforward to find an expression for S_m and S_n. Afterwards, elimination and substitution can be done to yield a and d.
2 (b) [2]	S_{m+n} $= \frac{m+n}{2}\left[2\left(\frac{m^2+(n-1)(m+n)}{mn}\right)+(m+n-1)\left(-\frac{2(m+n)}{mn}\right)\right]$ $= (m+n)\left[\frac{m^2}{mn}+\frac{(n-1)(m+n)}{mn}-\frac{(m+n-1)(m+n)}{mn}\right]$ $= (m+n)\left[\frac{m^2}{mn}+\frac{(m+n)(n-1-m-n+1)}{mn}\right]$ $= (m+n)\left[\frac{m^2}{mn}+\frac{(m+n)(-m)}{mn}\right]$ $= (m+n)\left[\frac{m^2-m^2-mn}{mn}\right]$ $= -(m+n)$	Like the first part, the second part entails making use of the formula for arithmetic series. As a side, a subsequential "show" question such as this one may sometimes prove to be a gainful hindsight, as it helps verify whether previously found results are correct (thereby securing marks in previous parts) so that no errors are carried forward to latter parts (thereby securing marks in latter parts).

3 (a) [3]	Writing z as $x+\mathrm{i}y$, where $x=\mathrm{Re}(z)$ and $y=\mathrm{Im}(z)$, $\lvert x+\mathrm{i}y+3\rvert = \lvert (x+3)+\mathrm{i}y\rvert = 2x$ $\therefore \sqrt{(x+3)^2+y^2}=2x$ Squaring both sides, $x^2+6x+9+y^2=4x^2$ $3x^2-6x-9-y^2=0$ $3x^2-6x+3-y^2=12$ $3\left(x^2-2x+1\right)-y^2=12$ $\dfrac{(x-1)^2}{4}-\dfrac{y^2}{12}=1$ Finding asymptotes, $\dfrac{(x-1)^2}{4}=\dfrac{y^2}{12}$ $y=\pm\sqrt{3}(x-1)$ $\therefore$ x and y are related by the equation of the hyperbola with asymptotes $y=\pm\sqrt{3}(x-1)$.	This question assesses on complex operations, and conic equations, particularly on complex moduli and the equation of a hyperbolic curve respectively. It is perhaps most straightforward to begin substituting $z=x+\mathrm{i}y$ into the given complex equation. The absence of an imaginary number, coupled with the presence of a squared term in the cartesian equation of a hyperbola, should sufficiently hint the use of modulus operation. The remaining part on finding asymptotes of a hyperbola is routine in A–Levels – nothing difficult for well-prepared candidates.

3 (b) [2]		As hinted by the key phrase "part of the hyperbola", not all points on in **(a)** will satisfy the given relationship in z. The left-hand side of the equation $\lvert z+3 \rvert = 2\mathrm{Re}(z)$ contains a modulus, which is always non-negative, and therefore it follows that $\mathrm{Re}(z) \geq 0$. Hence, only the part where $x \geq 0$ on the curve is relevant. A handy technique when sketching a graph with asymptotes is to **start with the asymptotes, followed by the graph itself, and finally the two axes**. When done in this order, the asymptotical behaviour of the graph is clearly shown, uncompromised by existing axes.
3 (c) [2]	From the sketch in **(ii)**, it can be seen that $\arg(z-1)$ has an asymptotic behaviour. $\therefore -\tan^{-1}\left(\sqrt{3}\right) < \arg(z-1) < \tan^{-1}\left(\sqrt{3}\right)$ $\therefore -\frac{\pi}{3} < \arg(z-1) < \frac{\pi}{3}$	It may be useful to consider graph transformation in this question. Note that the value of $\arg z$ is measured with respect to the origin O, which implies that $\arg(z-1)$ takes reference from the point $(1, 0)$. The range of values can then be deduced through the appropriate translation. The relevant angles can be obtained from the gradient of the asymptotes. Candidates may find it noteworthy that angle calculations using gradient values are becoming more common in recent A–Levels.

4 (a) [1]	By implicit differentiation, $2x+3y+3x\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}+4=0$ $2x+3y+4=\frac{\mathrm{d}y}{\mathrm{d}x}(2y-3x)$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+3y+4}{2y-3x}$	Implicit differentiation should pose little challenge to careful candidates.
4 (b) [3]	At the tangent point (x, y), the tangent equation is $y+4=m(x-6)$, where m is the gradient. Now since $\frac{\mathrm{d}y}{\mathrm{d}x}=m$, $\rightarrow \frac{2x+3y+4}{2y-3x}=\frac{y+4}{x-6}$ $\rightarrow (x-6)(2x+3y+4)=(2y-3x)(y+4)$ $\rightarrow 2x^2+3xy+4x-12x-18y-24=2y^2-3xy+8y-12x$ $\rightarrow 2\left(x^2+3xy-y^2\right)+4x-26y-24=0$ Since (x, y) lies on C, we use the equation of C to obtain $x^2+3xy-y^2=1-4x$. $\rightarrow 2(1-4x)+4x-26y-24=0$ $\rightarrow 2-4x-26y-24=0$ $\rightarrow 2x+3y=11$	This question assesses candidates on forming tangent line equations and finding relationships between their gradients and their intersection point. Successful candidates will recognise, after equating the gradient function with the gradient of a general line passing through $(6,-4)$, that **the equation of C itself helps eliminate implicit terms in the intermediate steps**. With some algebraic manipulation, the result follows.
4 (c) [3]	$2x+3y=11 \rightarrow x=-\frac{13y+11}{2}$ Substituting x into the equation of C, $\left(-\frac{13y+11}{2}\right)^2+3\left(-\frac{13y+11}{2}\right)y-y^2=1-4\left(-\frac{13y+11}{2}\right)$ Using calculator, $y=-1$ or $y=-\frac{1}{3}$. Substituting these y values into $2x=3y=11$, we have the following: • When $y=-1$, $x=1$ and $m=-\frac{3}{5} \rightarrow$ equation of tangent: $3x+5y+2=0$ • When $y=-\frac{1}{3}$, $x=-\frac{10}{3}$ and $m=-\frac{11}{28} \rightarrow$ equation of tangent: $11x+28y+46=0$	After substituting into C accordingly, the resulting quadratic equation can be solved with ease using calculator, though proceeding manually is welcomed albeit not required. The remaining steps follow.

5 (a) [3]	$\frac{u-r}{u+r}$ $=\frac{r\cos\theta+\mathrm{i}\,r\sin\theta-r}{r\cos\theta+\mathrm{i}\,r\sin\theta+r}$ $=\frac{r[\cos\theta+\mathrm{i}\sin\theta-1]}{r[\cos\theta+\mathrm{i}\sin\theta+1]}$ $=\frac{1-2\sin^2\frac{1}{2}\theta+\mathrm{i}\left(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right)-1}{2\cos^2\frac{1}{2}\theta-1+\mathrm{i}\left(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right)+1}$ $=\frac{-2\sin^2\frac{1}{2}\theta+\mathrm{i}\left(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right)}{2\cos^2\frac{1}{2}\theta+\mathrm{i}\left(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right)}$ $=\frac{2\sin\frac{1}{2}\theta\left(-\sin\frac{1}{2}\theta+\mathrm{i}\cos\frac{1}{2}\theta\right)}{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+\mathrm{i}\sin\frac{1}{2}\theta\right)}$ $=\frac{2\mathrm{i}\sin\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+\mathrm{i}\sin\frac{1}{2}\theta\right)}{2\cos\frac{1}{2}\theta\left(\cos\frac{1}{2}\theta+\mathrm{i}\sin\frac{1}{2}\theta\right)}$ $=\mathrm{i}\tan\frac{1}{2}\theta$	This question assesses candidates' capacity in manipulating complex number expressions using the trigonometric identities in the List of Formulae (MF27). A small manipulation is required in the last bit, in which an imaginary number is factored out as a common term in the nominator to allow for cancelation.

5 (b) [1]	From **(i)**, it can be deduced that: $\rightarrow \arg\left(\frac{u-r}{u+r}\right) = \arg\left(\mathrm{i}\tan\frac{1}{2}\theta\right) = \arg(\mathrm{i}) = \frac{\pi}{2}$ $\rightarrow \arg(u-r) - \arg(u+r) = \frac{\pi}{2}$ $\therefore$ Triangle OWZ is a right triangle with right angle at WOZ (or ZOW).	This part mainly assesses on complex number operations involving arguments. It should be apparent that triangle OWZ is a right triangle once candidates eventually obtain the value $\frac{\pi}{2}$. Candidates then only need to deduce where the angle is.
5 (c) [4]	Using information in **(i)** to roughly sketch triangle OWZ on an Argand diagram:  Using the sketch and the result in **(i)**, we have: $\rightarrow \tan\angle OWZ = \frac{OZ}{OW} = \frac{\lvert u-r\rvert}{\lvert u+r\rvert} = \left\lvert\frac{u-r}{u+r}\right\rvert = \left\lvert\mathrm{i}\tan\frac{1}{2}\theta\right\rvert = \tan\frac{1}{2}\theta$ $\therefore \angle OWZ = \frac{1}{2}\theta$ Distance between Z and W, $\lvert ZW\rvert$ $= \lvert(u+r)-(u-r)\rvert$ $= \lvert 2r\rvert = 2r$ Using trigonometry to find $\lvert u-r\rvert$ and $\lvert u+r\rvert$, $\rightarrow \lvert u-r\rvert = \lvert ZW\rvert\sin\frac{1}{2}\theta$ and $\lvert u+r\rvert = \lvert ZW\rvert\cos\frac{1}{2}\theta$ $\therefore \lvert u-r\rvert = 2r\sin\frac{1}{2}\theta$ and $\lvert u+r\rvert = 2r\cos\frac{1}{2}\theta$	This part mainly assesses candidates' ability to ascribe features to shapes on an Argand diagram, but also deals with operations on complex modulus. After drawing the Argand diagram and deducing the length ratio with suitable operations, the relationship between angle θ and the arguments α and β eventually becomes apparent. Once the angle θ is identified, further results can be obtained using standard trigonometric properties on right triangles.

6 (a) [4]	Using the formula in List of Formulae (MF26), $\overrightarrow{OC} = \dfrac{\lambda\mathbf{a} + (1-\lambda)\mathbf{b}}{\lambda + 1 - \lambda} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ Since OP bisects angle AOB, $\cos\angle AOC = \cos\angle BOC$ $\rightarrow \dfrac{\overrightarrow{OC}\cdot\mathbf{a}}{\lvert\overrightarrow{OC}\rvert\lvert\mathbf{a}\rvert} = \dfrac{\overrightarrow{OC}\cdot\mathbf{b}}{\lvert\overrightarrow{OC}\rvert\lvert\mathbf{b}\rvert}$ $\rightarrow \dfrac{\lambda\mathbf{a}\cdot\mathbf{a} + (1-\lambda)\mathbf{b}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert} = \dfrac{\lambda\mathbf{a}\cdot\mathbf{b} + (1-\lambda)\mathbf{b}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert}$ $\rightarrow \dfrac{\lambda\lvert\mathbf{a}\rvert^2 + (1-\lambda)\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\cos\angle AOB}{\lvert\mathbf{a}\rvert} = \dfrac{\lambda\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB + (1-\lambda)\lvert\mathbf{b}\rvert^2}{\lvert\mathbf{b}\rvert}$ $\rightarrow \lambda\lvert\mathbf{a}\rvert + (1-\lambda)\lvert\mathbf{b}\rvert\cos\angle AOB = \lambda\lvert\mathbf{a}\rvert\cos\angle AOB + (1-\lambda)\lvert\mathbf{b}\rvert$ $\rightarrow \lambda\lvert\mathbf{a}\rvert(1-\cos\angle AOB) + \big((1-\lambda)\lvert\mathbf{b}\rvert\big)(\cos\angle AOB - 1) = 0$ $\rightarrow (\lambda\lvert\mathbf{a}\rvert - (1-\lambda)\lvert\mathbf{b}\rvert)(1-\cos\angle AOB) = 0$ Since O, A and B are not collinear, $\cos\angle AOB \neq 1$ $\rightarrow \lambda\lvert\mathbf{a}\rvert - (1-\lambda)\lvert\mathbf{b}\rvert = 0$ $\rightarrow \lambda\lvert\mathbf{a}\rvert - \lvert\mathbf{b}\rvert + \lambda\lvert\mathbf{b}\rvert = 0$ $\rightarrow \lambda(\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert) = \lvert\mathbf{b}\rvert$ $\rightarrow \lambda = \dfrac{\lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}$	This part requires candidates' aptitude in vector product operations. To write an equation regarding the bisection, an expression for $\overrightarrow{OC}$ must be obtained first, which can be done using the ratio theorem given in the List of Formulae (MF27). Subsequently, candidates can use the fact that a bisected angle yields two angles with equal cosine values. The resulting equation eventually simplifies to a product of two separate factors, one of which expressed in terms of the constant λ to be solved. While the other factor $(1-\cos\angle AOB)$ is irrelevant towards finding λ, **candidates must still justify why the root of this factor is rejected**. Details provided in the question that initially seem irrelevant but later turn significant must always be referred to in the answer, lest they become a reason for penalty under a strict marking scheme.

6 (b) [4]	Given that $AC = BD$, $\overrightarrow{OD} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$ Using dot product to find OC^2 and OD^2, $OC^2 = \overrightarrow{OC}\cdot\overrightarrow{OC} = \lambda^2\mathbf{a}\cdot\mathbf{a} + 2\lambda(1-\lambda)\mathbf{a}\cdot\mathbf{b} + (1-\lambda)^2\mathbf{b}\cdot\mathbf{b}$ $OD^2 = \overrightarrow{OD}\cdot\overrightarrow{OD} = \lambda^2\mathbf{b}\cdot\mathbf{b} + 2\lambda(1-\lambda)\mathbf{b}\cdot\mathbf{a} + (1-\lambda)^2\mathbf{a}\cdot\mathbf{a}$ $\therefore OD^2 - OC^2$ $= \left(\lambda^2 - (1-\lambda)^2\right)\mathbf{b}\cdot\mathbf{b} - \left(\lambda^2 - (1-\lambda)^2\right)\mathbf{a}\cdot\mathbf{a}$ $= \left(\lambda^2 - (1-\lambda)^2\right)(\mathbf{b}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{a})$ $= (\lambda + 1 - \lambda)(\lambda - 1 + \lambda)\left(\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2\right)$ $= (1)(2\lambda - 1)\left(\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2\right)$ $= \left(\frac{2\lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert} - \frac{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}\right)\left(\lvert\mathbf{b}\rvert^2 - \lvert\mathbf{a}\rvert^2\right)$ $= \left(\frac{\lvert\mathbf{b}\rvert - \lvert\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert}\right)(\lvert\mathbf{b}\rvert + \lvert\mathbf{a}\rvert)(\lvert\mathbf{b}\rvert - \lvert\mathbf{a}\rvert)$ $= (\lvert\mathbf{b}\rvert - \lvert\mathbf{a}\rvert)^2$	Finding an expression for $\overrightarrow{OD}$ might prove to be a hurdle. A sketch may help illustrate the similarity between $\overrightarrow{OC}$ and $\overrightarrow{OD}$. As hinted by the question, the values of OC^2 and OD^2 can be deduced using dot product. The remaining parts of the proof follow with some algebraic manipulation.

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(a)
(i)
[3]</td>
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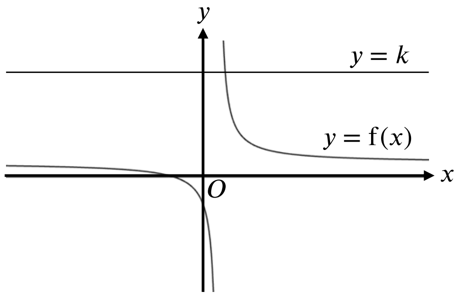
Consider the sketch of the graph of $y = \mathrm{f}(x)$:



In the sketch, the line $y = k$, where $k \in \mathbb{R}$, intersects the graph $y = \mathrm{f}(x)$ at most once.
$\therefore$ f is a one-one function, and hence f has an inverse.

Let $y = \mathrm{f}(x)$.

$$2y = 2\left(\frac{x+1}{2x-1}\right) = \frac{2x+2}{2x-1} = \frac{2x-1+3}{2x-1} = 1 + \frac{3}{2x-1}$$
$$2y - 1 = \frac{3}{2x-1}$$
$$2x - 1 = \frac{3}{2y-1}$$
$$x = \frac{1}{2}\left(1 + \frac{3}{2y-1}\right) = \frac{1}{2}\left(\frac{2y+2}{2y-1}\right) = \frac{y+1}{2y-1}$$

$$\therefore \mathrm{f}^{-1}(x) = \frac{x+1}{2x-1}, \quad x \in \mathbb{R}, \quad x \neq \frac{1}{2}.$$
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Questions on functions and inverses are relatively standard in A–Levels. Once a sketch of $y = \mathrm{f}(x)$ is produced, the explanation for why the inverse of f exists is straightforward, as is finding an expression for this inverse.

As a side, it might be interesting to note that the given function $\mathrm{f}(x)$ is coincidentally a *self-inverse* function, i.e., it satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. Self-inverse functions, or otherwise called *involutions* or *involutory functions*, often appear in A–Levels and school papers, and among the questions that are asked involving them is, most popularly, telescoping (e.g., evaluating $\mathrm{f}^{2024}(x)$).
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<tr>
<td>7
(a)
(ii)
[2]</td>
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$$\mathrm{gf}(x) = \frac{2x-7}{x-2} \to \mathrm{gff}^{-1}(x) = \frac{2\mathrm{f}^{-1}(x) - 7}{\mathrm{f}^{-1}(x) - 2}$$

Using the identity $\mathrm{ff}^{-1}(x) = x$, we have

$$\mathrm{g}(x) = \frac{2\left(\frac{x+1}{2x-1}\right) - 7}{\frac{x+1}{2x-1} - 2} = \frac{2(x+1) - 7(2x-1)}{x+1-2(2x-1)} = \frac{2x+2-14x+7}{x+1-4x+2} = \frac{9-12x}{3-3x} = \frac{4x-3}{x-1}$$
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This part may prove challenging. An intuitive approach to this part requires recognising the identity $\mathrm{ff}^{-1}(x) = x$, which can be used to reduce $\mathrm{gf}(x)$ to $\mathrm{g}(x)$.

Alternatively, since it is previously discovered that f is self-inverse, it is possible to reduce $\mathrm{gf}(x)$ by writing $\mathrm{gff}(x)$ to obtain $\mathrm{g}(x)$.
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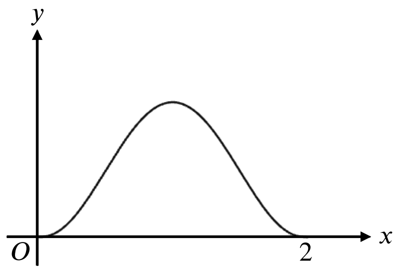
7 (b) (i) [3]	$x = 2 \to \mathrm{h}(2) - 2\mathrm{h}(0.5) + \mathrm{h}(-1) = \frac{9}{2}$ $x = 0.5 \to \mathrm{h}(0.5) - 2\mathrm{h}(-1) + \mathrm{h}(2) = -\frac{9}{2}$ $x = -1 \to \mathrm{h}(-1) - 2\mathrm{h}(2) + \mathrm{h}(0.5) = 0$ Using GC, $\mathrm{h}(2) = 0.5$, $\mathrm{h}(0.5) = -1$ and $\mathrm{h}(-1) = 2$.	The values of $\mathrm{h}(2)$, $\mathrm{h}(0.5)$ and $\mathrm{h}(-1)$ can be found most directly using a system of linear equations. The relevant equations should become apparent after substituting the x values of interest into the given functional equation.
7 (b) (ii) [2]	$\mathrm{h}(2) = 0.5$ $\mathrm{hh}(2) = \mathrm{h}(0.5) = -1$ $\mathrm{hhh}(2) = \mathrm{h}(-1) = 2$ $\therefore \mathrm{hhh}(x) = x$ $\mathrm{hhh}(x) = \mathrm{h}\left(\frac{1}{1-x}\right) = x$ Let $u = \frac{1}{1-x}$ $\therefore 1 - x = \frac{1}{u} \to x = 1 - \frac{1}{u}$ $\therefore \mathrm{h}(u) = 1 - \frac{1}{u}$ Rewriting variable $u \to x$, we have $\mathrm{h}(x) = 1 - \frac{1}{x}$	It should be rather evident from **(b)(i)** that self-composing h enough times will yield the input of the function. Afterwards, candidates may encounter a function h with its input being a complicated expression in x, which can be simplified using a substitution – essentially similar to the standard method of finding inverses. As a side, it might be interesting to note that, like the function f in **(a)**, h is also “self-inverse” – or rather more accurately an *iterated function* with period 3 (and involutions like f are said to be iterated functions with period 2).

8 **(a)** **[4]**	$x = a\sin^2\theta$ $\rightarrow \mathrm{d}x = 2a\sin\theta\cos\theta\,\mathrm{d}\theta = a\sin 2\theta$ $\rightarrow$ When $x = 0$, $\theta = 0$ $\rightarrow$ When $x = a$, $\theta = \frac{1}{2}\pi$ $\therefore I_0$ $= \int_0^a \sqrt{ax - x^2}\,\mathrm{d}x$ $= \int_0^{\frac{1}{2}\pi} \sqrt{a^2\sin^2\theta - a^2\sin^4\theta}\,(a\sin 2\theta)\,\mathrm{d}\theta$ $= \int_0^{\frac{1}{2}\pi} \sqrt{a^2\sin^2\theta\,(1 - \sin^2\theta)}(a\sin 2\theta)\,\mathrm{d}\theta$ $= \int_0^{\frac{1}{2}\pi} (a\sin\theta\cos\theta)(a\sin 2\theta)\,\mathrm{d}\theta$ $= \frac{1}{2}a^2 \int_0^{\frac{1}{2}\pi} \sin^2 2\theta\,\mathrm{d}\theta$ $= \frac{1}{2}a^2 \int_0^{\frac{1}{2}\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right)\mathrm{d}\theta$ $= \frac{1}{2}a^2 \left[\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_0^{\frac{1}{2}\pi}$ $= \frac{1}{8}\pi a^2$	Integration by trigonometric substitution is fairly routine in A–Levels. Like most integration questions involving trigonometric functions, this one involves the use of several identities to arrive at the exact answer.

8 **(b)** **[4]**	Using integration by parts, I_{n+1} $= \int_0^a x^{n+1}\sqrt{ax-x^2}\,\mathrm{d}x$ $= \int_0^a x^{n+\frac{3}{2}}(a-x)^{\frac{1}{2}}\,\mathrm{d}x$ $= \left[\left(x^{n+\frac{3}{2}}\right)\left(-\frac{2}{3}(a-x)^{\frac{3}{2}}\right)\right]_0^a - \int_0^a \left(\frac{2n+3}{2}x^{n+\frac{1}{2}}\right)\left(-\frac{2}{3}(a-x)^{\frac{3}{2}}\right)\mathrm{d}x$ $= \left[\left(a^{\frac{2n+3}{2}}\right)(0) - (0)(-\frac{2}{3}a^{\frac{3}{2}})\right] + \frac{2n+3}{3}\int_0^a \left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right](a-x)\,\mathrm{d}x$ $= \frac{2n+3}{3}\int_0^a a\left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right] - x\left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right]\mathrm{d}x$ $= \frac{2n+3}{3}\left[a\int_0^a \left[\left(x^{n+\frac{1}{2}}\right)(a-x)^{\frac{1}{2}}\right]\mathrm{d}x - \int_0^a \left[\left(x^{n+\frac{3}{2}}\right)(a-x)^{\frac{1}{2}}\right]\mathrm{d}x\right]$ $= \frac{2n+3}{3}\left[a\int_0^a x^n\sqrt{ax-x^2}\,\mathrm{d}x - \int_0^a x^{n+1}\sqrt{ax-x^2}\,\mathrm{d}x\right]$ $= \frac{2n+3}{3}\left[aI_n - I_{n+1}\right]$ $\rightarrow 3I_{n+1} = (2n+3)aI_n - (2n+3)I_{n+1}$ $\rightarrow (2n+6)I_{n+1} = (2n+3)aI_n$ $\therefore I_{n+1} = \left(\frac{2n+3}{2n+6}\right)aI_n$	Integral equations of this form are known as *reduction formula* of an integral. While reduction formula is not formally tested in A–Levels (and in fact explicitly excluded as per 2025), it has often appeared in past preliminary exams as an application of integration by parts. Admittedly, this part of the question may prove to be challenging. To proceed, there are key observations to be recognised: • Both I_{n+1} and I_n are in the equation → the term x^n will undergo integration or differentiation, signalling the need for **integration by parts**. • The constant a is in the equation → ***a* must somehow be extracted out** from the integrand, either by algebraic manipulation, calculus, or any other way. This suggested answer begins with I_{n+1} and rewriting it to obtain the term $x^{n+\frac{3}{2}}$ to be differentiated, and the term $(a-x)^{\frac{1}{2}}$ to be integrated. It is also possible to begin with I_n and integrating x^n to obtain I_{n+1} – both approaches will nonetheless require some degree of manipulation.

8 (c) [4]

Consider the sketch of $y = \sqrt{(2x - x^2)^5}$ below:



$\therefore$ Required area

$$= \int_0^2 \sqrt{(2x - x^2)^5}\,\mathrm{d}x$$
$$= \int_0^2 \left(2x - x^2\right)^2 \sqrt{2x - x^2}\,\mathrm{d}x$$
$$= \int_0^2 \left(4x^2 - 4x^3 + x^4\right) \sqrt{2x - x^2}\,\mathrm{d}x$$
$$= 4\int_0^2 x^2\sqrt{2x - x^2}\,\mathrm{d}x - 4\int_0^2 x^3\sqrt{2x - x^2}\,\mathrm{d}x + \int_0^2 x^4\sqrt{2x - x^2}\,\mathrm{d}x$$
$$= 4I_2 - 4I_3 + I_4$$
$$= 4I_2 - 4\times\left(\frac{2(2)+3}{2(2)+6}\right)(2)\times I_2 + \left(\frac{2(3)+3}{2(3)+6}\right)(2)\times\left(\frac{2(2)+3}{2(2)+6}\right)(2)\times I_2$$
$$= \left(4 - \frac{28}{5} + \frac{63}{30}\right)\times I_2$$
$$= \left(4 - \frac{28}{5} + \frac{63}{30}\right)\times\left(\frac{2(1)+3}{2(1)+6}\right)(2)\times\left(\frac{2(0)+3}{2(0)+6}\right)(2)\times\frac{1}{8}\pi(2)^2$$
$$= \frac{1}{2}\times\frac{5}{4}\times 1\times\frac{1}{2}\pi$$
$$= \frac{5}{16}\pi \text{ units}^2$$

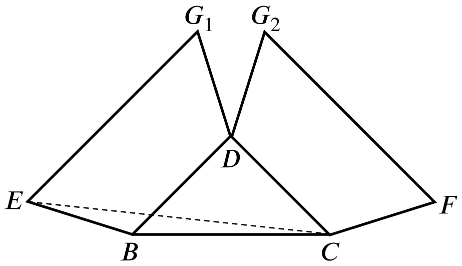
The most appropriate way to start finding the area is to first sketch the region of interest. This will first and foremost reveal the boundary values of the integrals.

When integrating, there may be an initial temptation to factor out some power of x from $\sqrt{(2x - x^2)^5}$ to obtain $x^n\sqrt{(2x - x^2)}$, which is not clearly feasible and may potentially lead to a dead-end. A more feasible approach is to interpret the expression as being some power away from the root function $\sqrt{2x - x^2}$, yielding the form $\left(2x - x^2\right)^n\sqrt{2x - x^2}$ which is far more straightforward to execute.

It is eventually apparent after several manipulation that the area can be expressed as a combination of I_2, I_3 and I_4. The remaining calculations follow, with further simplification done by collecting I_2 first before collapsing further to I_0.

9 (a) [2]	Since BCD and EFG are parallel, EG is parallel to BD. $\rightarrow \overrightarrow{EG} = k\overrightarrow{BD}$, $k \in \mathbb{R}$. $\rightarrow \begin{pmatrix} 4 \\ -1 \\ 10 \end{pmatrix} = k \begin{pmatrix} 3 \\ a-7 \\ b-2 \end{pmatrix}$ From the **i**–component, $4 = 3k \rightarrow k = \frac{4}{3}$ From the **j**–component, $-1 = \frac{4}{3}(a-7) \rightarrow a = 7 - \frac{3}{4} = \frac{25}{4} = 6.25$ From the **k**–component, $10 = \frac{4}{3}(b-2) \rightarrow b = 2 + \frac{15}{2} = \frac{19}{2} = 9.5$	This "show" part helps reveal to candidates the relationship between relevant parallel edges. After further deducing that EG is a scalar multiple of BD, the remaining results will eventually follow.
9 (b) [6]	From **(i)**, it can be deduced that $BD = \frac{3}{4}EG$ and $CD = \frac{3}{4}FG$ Area of face $BDGE$ $= \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert}{\lvert\overrightarrow{EG}\rvert}\right)\left(\lvert\overrightarrow{BD}\rvert + \lvert\overrightarrow{EG}\rvert\right) = \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert}{\lvert\overrightarrow{EG}\rvert}\right)\left(\frac{7}{4}\lvert\overrightarrow{EG}\rvert\right) = \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert$ Area of face $CDFG$ $= \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert}{\lvert\overrightarrow{FG}\rvert}\right)\left(\lvert\overrightarrow{CD}\rvert + \lvert\overrightarrow{FG}\rvert\right) = \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert}{\lvert\overrightarrow{FG}\rvert}\right)\left(\frac{7}{4}\lvert\overrightarrow{FG}\rvert\right) = \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert$ Area of face BCD $= \frac{1}{2}\lvert\overrightarrow{BD}\times\overrightarrow{CD}\rvert = \frac{1}{2}\left\lvert\frac{3}{4}\overrightarrow{EG}\times\frac{3}{4}\overrightarrow{FG}\right\rvert = \frac{9}{32}\lvert\overrightarrow{EG}\times\overrightarrow{FG}\rvert$ $\therefore$ Area of carboard used $= \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert + \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert + \frac{9}{32}\lvert\overrightarrow{EG}\times\overrightarrow{FG}\rvert$ $= \frac{7}{8}\left\lvert\begin{pmatrix} -1 \\ -6.25 \\ -1.5 \end{pmatrix}\times\begin{pmatrix} -5 \\ -2 \\ 7 \end{pmatrix}\right\rvert + \frac{7}{8}\left\lvert\begin{pmatrix} -1 \\ -6.25 \\ -1.5 \end{pmatrix}\times\begin{pmatrix} 4 \\ -1 \\ 10 \end{pmatrix}\right\rvert + \frac{9}{32}\left\lvert\begin{pmatrix} 4 \\ -1 \\ 10 \end{pmatrix}\times\begin{pmatrix} -5 \\ -2 \\ 7 \end{pmatrix}\right\rvert$ $= \frac{7}{8}\left\lvert\begin{pmatrix} -46.75 \\ 14.5 \\ -29.25 \end{pmatrix}\right\rvert + \frac{7}{8}\left\lvert\begin{pmatrix} -64 \\ 4 \\ 26 \end{pmatrix}\right\rvert + \frac{9}{32}\left\lvert\begin{pmatrix} 13 \\ -78 \\ -13 \end{pmatrix}\right\rvert$ $\approx 132.98 \text{ units}^2 \approx 133 \text{ units}^2$	This part on the whole tests on the ability to perform cross product to calculate areas defined by vertices on a three-dimensional space. The cardboard area consists of three faces: two trapezoids $BDGE$ and $CDFG$, and one triangle BCD. When finding the trapezoid areas, note that, after establishing in **(i)** the scale factor relating the parallel faces in, the sum of parallel edges can be further simplified. The altitude can be found using cross product, and the area expression for the trapezoids soon follow. Finding an area expression for triangle BCD should not pose as much of an issue. The second part on numerically evaluating the cross products may prove tedious, but nonetheless can be done correctly with careful calculation.

9 (c) [5]



Considering the shape of the flat surface, the required distance is $|\overrightarrow{CE}|$. Using cosine rule,

$$|\overrightarrow{CE}| = \sqrt{|\overrightarrow{BE}|^2 + |\overrightarrow{BC}|^2 - 2|\overrightarrow{BE}||\overrightarrow{BC}|\cos(\angle EBD + \angle DBC)}$$

Finding $\angle EBD$,

$$\angle EBD = \cos^{-1}\left(\frac{\overrightarrow{BE}\cdot\overrightarrow{BD}}{|\overrightarrow{BE}||\overrightarrow{BD}|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}-2\\-6\\-4\end{pmatrix}\cdot\begin{pmatrix}3\\-0.75\\7.5\end{pmatrix}}{\left|\begin{pmatrix}-2\\-6\\-4\end{pmatrix}\right|\left|\begin{pmatrix}3\\-0.75\\7.5\end{pmatrix}\right|}\right) = \cos^{-1}\left(\frac{-31.5}{|\sqrt{56}||\sqrt{65.8125}|}\right) = 121.26^\circ$$

Finding $\angle DBC$, and considering $\Delta BCD \cong \Delta EFG$ on the original folded shape,

$$\angle DBC = \angle GEF = \cos^{-1}\left(\frac{\overrightarrow{EG}\cdot\overrightarrow{EF}}{|\overrightarrow{EG}||\overrightarrow{EF}|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}4\\-1\\10\end{pmatrix}\cdot\begin{pmatrix}9\\1\\3\end{pmatrix}}{\left|\begin{pmatrix}4\\-1\\10\end{pmatrix}\right|\left|\begin{pmatrix}9\\1\\3\end{pmatrix}\right|}\right) = \cos^{-1}\left(\frac{65}{|\sqrt{117}||\sqrt{91}|}\right) = 50.95^\circ$$

$\therefore$ Required distance

$$= \sqrt{|\overrightarrow{BE}|^2 + \left(\frac{3}{4}|\overrightarrow{EF}|\right)^2 - 2|\overrightarrow{BE}|\left(\frac{3}{4}|\overrightarrow{EF}|\right)\cos(121.26^\circ + 50.95^\circ)}$$

$$= \sqrt{56 + \frac{9}{16}(91) - \frac{3}{2}\left(\sqrt{56}\right)\left(\sqrt{91}\right)\cos(172.21^\circ)}$$

≈ 14.60407 units ≈ 14.6 units

When approaching this part, a preliminary diagram of the flattened surface may prove helpful. The diagram should sufficiently hint at applying the cosine rule to the flat triangle BCE to find the flattened length of CE.

The cosine rule calls for the values of angles and edge lengths on faces $BDGE$ and BCD, which can be deduced by proxy of the original folded shape, since these angles and edges are preserved in both states of the shape.

To find angles and edges involving the vertex C, which has not been asked to be determined, a suggested method is considering similar triangles in the folded shape. This method is reasonably efficient as it relieves the need of discovering a vertex whose coordinates may turn out to be complicated. Still, proceeding by finding C beforehand is acceptable, and in fact there may be other alternatives that can be considered apart from these two.

Throughout the working, some vectors may need to be calculated beforehand, such as $\overrightarrow{BE}$, $\overrightarrow{BD}$ and $\overrightarrow{EF}$, but the rest can be recalled from previous parts. Equally important to note in this answer is the scalar relationship $\overrightarrow{BC} = \frac{3}{4}|\overrightarrow{EF}|$, which is relevant in the final calculation.

10 (a) [1]	$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{a}{t}$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{a}{2}\left(1-\frac{1}{t^2}\right)$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{a}{2}\left(1-\frac{1}{t^2}\right)}{\frac{a}{t}}=\frac{\frac{a}{2t^2}(t^2-1)}{\frac{a}{2t^2}(2t)}=\frac{t^2-1}{2t}$	Finding the gradient function of a curve defined parametrically is standard in A–Levels. Candidates should be able to arrive at the result with little to no issue.
10 (b) [3]	For the lowest point on the rope, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{t^2-1}{2t}=0 \rightarrow t=\pm 1$ Finding the horizontal position x of the lowest point with $t=1$ (since $\ln t$ is only defined for $t>1$), $x=a(\ln 1)=0$ Since the lowest point is horizontally in the middle of both persons standing $2a$ units apart, Person 1 is at $x=-a \rightarrow t_1=\mathrm{e}^{-1}$ Person 2 is at $x=a \rightarrow t_2=\mathrm{e}$	Finding the lowest point on the rope is a classic minima problem. It follows that $x=0$ for the lowest point. The subsequent part on finding t_1 and t_2 requires proper interpretation of the question. It should be relatively easy to visualise, without a sketch, that the two persons can only stand on $x=-a$ and $x=a$ for them to stand $2a$ units apart with the low point in the middle at $x=0$.
10 (c) [2]	Angle at $t_1=\mathrm{e}$ $=\tan^{-1}\left(\frac{\mathrm{e}^2-1}{2\mathrm{e}}\right)\approx 49.6°$ Angle at $t_2=\mathrm{e}^{-1}$ $=\tan^{-1}\left(\frac{\mathrm{e}^{-2}-1}{2\mathrm{e}^{-1}}\right)\approx -49.6°$ Both ends of the rope make an angle of 49.6° with the horizontal.	It is becoming a trend in A–Levels to inquire about angles given the gradient of a line. Although not squarely within the syllabus, the relationship between the **gradient $\boldsymbol{m}$** of a line and the principal **angle $\boldsymbol{\alpha}$** it makes with the horizontal satisfies $\boldsymbol{\alpha=\tan^{-1} m}$**.** In the context of this question, the angle subtended by both ends of the rope with the horizontal is equal due to the symmetry of the curve.

10 (d) [2]	$x = a\ln t \to t = \mathrm{e}^{\frac{x}{a}}$ $\therefore y = \frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \frac{1}{\mathrm{e}^{\frac{x}{a}}}\right) = \frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right)$ Using t_1 (or t_2) to find the height at which the rope is being held, $k = \frac{a}{2}\left(\mathrm{e} + \mathrm{e}^{-1}\right)$	Finding the cartesian equation of this curve should be straightforward, as there is a simple relationship between x and t. Finding k in terms of a afterwards is routine
10 (e) [5]	Considering transformation, volume required $= \pi \int_{-a}^{a} (y-k)^2 \, \mathrm{d}x$ $= \pi \int_{-a}^{a} \left[\frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right) - \frac{a}{2}\left(\mathrm{e} + \mathrm{e}^{-1}\right)\right]^2 \mathrm{d}x$ $= \frac{1}{4}\pi a^2 \int_{-a}^{a} \left[\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right)^2 - 2\left(\mathrm{e} + \mathrm{e}^{-1}\right)\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right) + \left(\mathrm{e} + \mathrm{e}^{-1}\right)^2\right] \mathrm{d}x$ $= \frac{1}{4}\pi a^2 \int_{-a}^{a} \left[\mathrm{e}^{\frac{2x}{a}} + 2 + \mathrm{e}^{-\frac{2x}{a}} - 2\left(\mathrm{e} + \mathrm{e}^{-1}\right)\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right) + \left(\mathrm{e} + \mathrm{e}^{-1}\right)^2\right] \mathrm{d}x$ $= \frac{1}{4}\pi a^2 \left[\frac{a}{2}\mathrm{e}^{\frac{2x}{a}} + 2x - \frac{a}{2}\mathrm{e}^{-\frac{2x}{a}} - 2\left(\mathrm{e} + \mathrm{e}^{-1}\right)\left(a\mathrm{e}^{\frac{x}{a}} - a\mathrm{e}^{-\frac{x}{a}}\right) + \left(\mathrm{e} + \mathrm{e}^{-1}\right)^2 x\right]_{-a}^{a}$ $= \frac{1}{4}\pi a^2 \left[a\mathrm{e}^2 + 4a - a\mathrm{e}^{-2} - 4a\left(\mathrm{e} + \mathrm{e}^{-1}\right)\left(\mathrm{e} - \mathrm{e}^{-1}\right) + 2a\left(\mathrm{e} + \mathrm{e}^{-1}\right)^2\right]$ $= \frac{1}{4}\pi a^3 \left[\mathrm{e}^2 + 4 - \mathrm{e}^{-2} - 4e^2 + 4e^{-2} + 2e^2 + 4 + 2e^{-2}\right]$ $= \frac{1}{4}\pi a^3 \left(8 - e^2 + 5e^{-2}\right)$ $\therefore A = \frac{1}{4}\pi, \quad B = 8, \quad C = -1 \text{ and } D = 5$	This part may prove challenging. There are some key observations to be made. Most importantly, it must be noted that **the axis of rotation for the volume of revolution is not the x–axis as per usual**, but instead the line $y = k$. Most of the heavy lifting in this question comes from integrating $(y-k)^2$, which unfortunately simplifies to a myriad of terms to be integrated – though it *is* straightforward to integrate these terms individually. When evaluating the integral, candidates may simplify their working should they recognise that the integral expression returns the same value for both the upper bound and the lower bound, and thus the integral value is simply double of the integral expression. This is indeed possible as the integrand is in the form of an *even function*, which satisfies $\mathrm{f}(x) = \mathrm{f}(-x)$ for all x (and this can easily be proven for this curve). The context of this question also lends itself to the evenness of this integrand. With careful manipulation, the remaining steps follow.

11 (a) (i) [2]	$\dfrac{\mathrm{d}R}{\mathrm{d}t} = 0.6R - 0.4RW$ $\dfrac{\mathrm{d}W}{\mathrm{d}t} = -0.8W + 0.2RW$	Writing down differential equations in context is classic in A–Levels. Such contextual questions usually allocate a few marks in the initial parts to ensure proper interpretation before any mathematical work is done further in subsequent parts.
11 (a) (ii) [5]	$\dfrac{\mathrm{d}W}{\mathrm{d}R} = \dfrac{\frac{\mathrm{d}W}{\mathrm{d}t}}{\frac{\mathrm{d}R}{\mathrm{d}t}} = \dfrac{-0.8W + 0.2RW}{0.6R - 0.4RW} = \dfrac{W(-0.8 + 0.2R)}{R(0.6 - 0.4W)}$ $\int \left(\dfrac{0.6 - 0.4W}{W}\right) \mathrm{d}W = \int \left(\dfrac{-0.8 + 0.2R}{R}\right) \mathrm{d}R$ $\int \left(\dfrac{0.6}{W} - 0.4\right) \mathrm{d}W = \int \left(\dfrac{-0.8}{R} + 0.2\right) \mathrm{d}R$ $0.6 \ln W - 0.4W = -0.8 \ln R + 0.2R + C, \quad C \in \mathbb{R}$ Substituting initial population of $W = 1$ and $R = 1$, $0.6 \ln(1) - 0.4(1) = -0.8 \ln(1) + 0.2(1) + C$ $C = -0.4 - 0.2 = -0.6$ $0.6 \ln W - 0.4W = -0.8 \ln R + 0.2R - 0.6$ $-3 \ln W + 2W = 4 \ln R - R + 3$ $2W + R - 3 \ln W - 4 \ln R = 3$	Lotka–Volterra equations make use of a system of non-homogeneous differential equations – such differential equations are a form that hardly yields a 'nice' expression in terms of for each population (and it would also require mathematical tools outside of the syllabus). It is instead much more manageable to find an implicit relationship governing R and W. The absence of t in the equation should sufficiently hint at **connected rates of change**. After obtaining an expression for $\frac{\mathrm{d}W}{\mathrm{d}R}$ (or $\frac{\mathrm{d}R}{\mathrm{d}W}$) in terms of R and W, subsequent steps involving separable differential equations follow as per usual. As the time variable t is no longer relevant to the equation, the initial populations $R = 1$ and $W = 1$ can be directly substituted to solve for the constant of integration C. This part also serves as a hindsight checkpoint. If the differential equations in **(a)** are indeed correct, they should yield an implicit equation that is consistent with what needs to be shown.

10 (b) (i) [4]	Consider the equations in **(i)(a)** to find the extreme values of R and W. For smallest and largest rabbit population size, $\frac{dR}{dt} = 0 \rightarrow R(0.6 - 0.4W) = 0$ Since $R \neq 0$ as seen from the graph, $0.6 - 0.4W = 0 \rightarrow W = 1.5$ Substituting $W = 1.5$ into the equation in **(i)(b)**, $3 + R - 3\ln 1.5 - 4\ln R = 3$ $R - 4\ln R = 3\ln 1.5$. Using GC, $R = 0.915 \approx 0.9$ or $R = 10.696 \approx 10.7$ $\therefore$ The smallest rabbit population size is 900, and the largest rabbit population size is 10,700. For smallest and largest wolf population size, $\frac{dW}{dt} = 0 \rightarrow W(-0.8 + 0.2R) = 0$ Since $W \neq 0$ as seen from the graph, $-0.8 + 0.2R = 0 \rightarrow R = 4$ Substituting $R = 4$ into the equation in **(i)(b)**, $2W + 4 - 3\ln W - 4\ln 4 = 3$. $2W - 3\ln W = 4\ln 4 - 1$ Using GC, $W = 0.262 \approx 0.3$ or $W = 4.543 \approx 4.5$ $\therefore$ The smallest wolf population size is 300, and the largest wolf population size is 4,500.	The given sketch of $2W + R - 3\ln W - 4\ln R = 3$ shows that the curve contains turning points which correspond to the minimum and maximum values of R and W respectively. Once recognised, this question is reduced to a standard question on maxima-minima. Again, care must be taken to **justify rejecting roots** from certain factors, as such display of awareness are likely to be considered for a few marks. Upon substituting relevant values into the equation in **(a)(ii)**, a calculator can be used subsequently to obtain two values corresponding to the maxima and minima. **Care must be taken to convert these values into the level of precision that is demanded by the question.**
10 (b) (ii) [2]	Considering the equations in **(i)(a)**, and substituting the initial population $R = 1$ and $W = 1$, we have $\frac{dR}{dt} = 0.6 - 0.4 = 0.2 > 0$ $\frac{dW}{dt} = -0.8 + 0.2 = -0.6 < 0$ $\therefore$ After the initial year, R increases while W decreases. $\therefore$ Point B_2 corresponds to the population in the year after the initial.	This question is pertaining rates of change, particularly the ability to apply rate equations to predict, in context, the behaviour of a relevant variable. Substituting the initial population into the rate of change of R and W respectively in **(a)(i)** will yield sufficient information to help determine the behaviour of each population size in the upcoming year (unit time t).