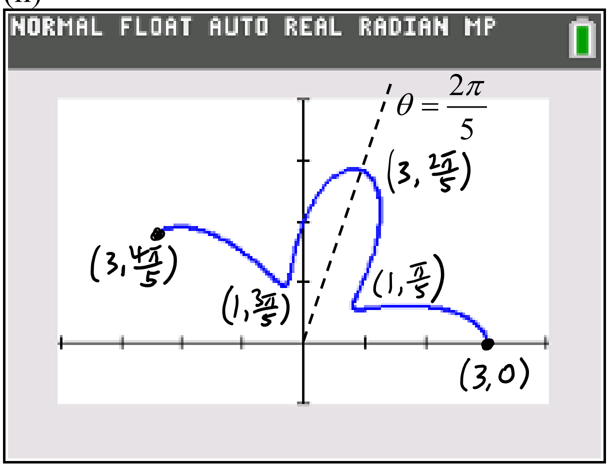
	Solution	
1.	(i) $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\,\mathrm{d}x$ $= \int_1^{\mathrm{e}} \frac{u-1}{(u+1)u}\,\mathrm{d}u$ $u = \mathrm{e}^x$ $\frac{du}{dx} = \mathrm{e}^x = u$ when $x = 0, u = 1$ when $x = 1, u = \mathrm{e}$ $= \int_1^{\mathrm{e}} \frac{-1}{u} + \frac{2}{u+1}\,\mathrm{d}u$ $= \left[2\ln\lvert u+1\rvert - \ln\lvert u\rvert\right]_1^{\mathrm{e}}$ $= 2\ln(\mathrm{e}+1) - \ln \mathrm{e} - 2\ln 2 + \ln 1$ $= 2\ln\left(\frac{\mathrm{e}+1}{2}\right) - 1$ (where $a = 2$ and $b = -1$) (ii) By Simpson's rule, $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x \approx \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left[\mathrm{f}(0) + 4\mathrm{f}\left(\frac{1}{4}\right) + 2\mathrm{f}\left(\frac{2}{4}\right) + 4\mathrm{f}\left(\frac{3}{4}\right) + \mathrm{f}(1)\right]$ $= 0.24023$ (5 d.p.) $2\ln\left(\frac{\mathrm{e}+1}{2}\right) - 1 \approx 0.240233$ $\therefore \ln\left(\frac{\mathrm{e}+1}{2}\right) \approx 0.620$	

	Solution	
2.	$P_{n+1} = \frac{1}{3}\left(5P_n + 2P_{n-1}\right) + 3^{n-1}$ ---(1) Applying the substitution $P_n = Q_n + \frac{1}{10}\left(3^{n+1}\right)$ to (1): $Q_{n+1} + \frac{1}{10}\left(3^{n+2}\right) = \frac{1}{3}\left(5\left(Q_n + \frac{1}{10}\left(3^{n+1}\right)\right) + 2\left(Q_{n-1} + \frac{1}{10}\left(3^n\right)\right)\right) + 3^{n-1}$ $Q_{n+1} + \frac{9}{10}\left(3^n\right) = \frac{5}{3}Q_n + \frac{1}{2}\left(3^n\right) + \frac{2}{3}Q_{n-1} + \frac{1}{15}\left(3^n\right) + \frac{1}{3}\left(3^n\right)$ $Q_{n+1} = \frac{5}{3}Q_n + \frac{2}{3}Q_{n-1}$ $\therefore 3Q_{n+1} - 5Q_n - 2Q_{n-1} = 0$ Auxiliary Equation: $3x^2 - 5x - 2 = 0$ $(3x+1)(x-2) = 0$ $x = -\frac{1}{3}$ or $x = 2$	

	So $Q_n = A\left(-\frac{1}{3}\right)^n + B(2)^n$, where A and B are arbitrary constants $\Rightarrow P_n = A\left(-\frac{1}{3}\right)^n + B(2)^n + \frac{1}{10}\left(3^{n+1}\right)$ $P_0 = 0 \Rightarrow A + B = -\frac{3}{10}$ ---(2) $P_1 = \frac{7}{5} \Rightarrow -\frac{1}{3}A + 2B + \frac{9}{10} = \frac{7}{5}$ $\Rightarrow -\frac{1}{3}A + 2B = \frac{1}{2}$ ---(3) Using GC to solve (2) and (3), $A = -\frac{33}{70}, B = \frac{6}{35}$ $\therefore P_n = -\frac{33}{70}\left(-\frac{1}{3}\right)^n + \frac{6}{35}(2)^n + \frac{1}{10}\left(3^{n+1}\right)$	

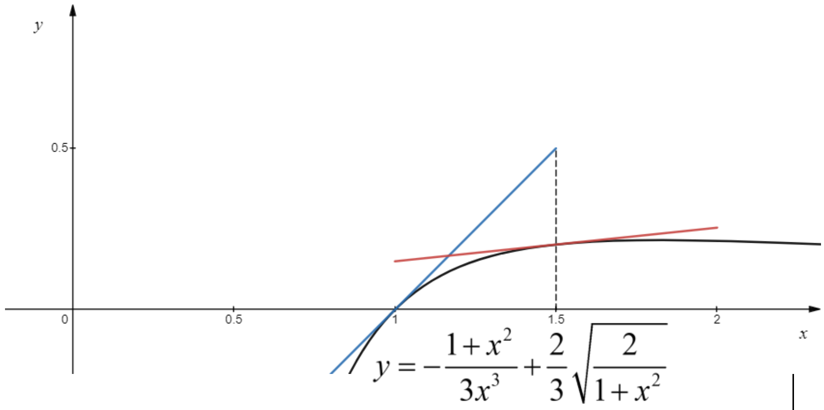
	Solution	
3.	(i) $r = 2 + \cos 5\theta$ $\frac{dr}{d\theta} = -5\sin 5\theta$ $\frac{dr}{d\theta} = 0 \Rightarrow \sin 5\theta = 0$ $\Rightarrow 5\theta = 0, \pi, 2\pi, 3\pi, 4\pi$ $\Rightarrow \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ Hence, the polar coordinates of the required points are $(3,0), \left(1,\frac{\pi}{5}\right), \left(3,\frac{2\pi}{5}\right), \left(1,\frac{3\pi}{5}\right), \left(3,\frac{4\pi}{5}\right)$. (ii) 	

	(iii) Area $= \frac{1}{2}\int_0^{\frac{\pi}{4}} r^2 d\theta$ $= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(2+\cos 5\theta\right)^2 d\theta$ = 1.51 units^2	

	Solution	
4.	**(a)** $\int \frac{\sqrt{1+x^2}}{x^4}\ \mathrm{d}x$ $= \int \frac{\sqrt{1+\tan^2\theta}}{\tan^4\theta}\ \sec^2\theta\ \mathrm{d}\theta$ $= \int \frac{\sec^3\theta}{\tan^4\theta}\ \mathrm{d}\theta$ $= \int \cos\theta\left(\sin\theta\right)^{-4}\ \mathrm{d}\theta$ $= \frac{\left(\sin\theta\right)^{-3}}{-3}$ $= -\frac{\left(1+x^2\right)^{\frac{3}{2}}}{3x^3} + c$ (shown) Let $x = \tan\theta$ $\Rightarrow \frac{dx}{d\theta} = \sec^2\theta$ and $\sin\theta = \frac{x}{\sqrt{1+x^2}}$ **Alternatively**, $\int \frac{\sqrt{1+x^2}}{x^4}\ \mathrm{d}x$ $= \int \frac{1}{x^3}\sqrt{\frac{1+x^2}{x^2}}\ \mathrm{d}x$ $= \int x^{-3}\sqrt{x^{-2}+1}\ \mathrm{d}x$ $= \left(-\frac{1}{2}\right)\frac{\left(\frac{1}{x^2}+1\right)^{\frac{3}{2}}}{\frac{3}{2}} + c$ $= -\frac{\left(1+x^2\right)^{\frac{3}{2}}}{3x^3} + c$ (shown) **(b)(i)** Integrating factor: $\mathrm{e}^{\int \frac{x}{1+x^2}\ \mathrm{d}x} = \mathrm{e}^{\frac{1}{2}\int \frac{2x}{1+x^2}\ \mathrm{d}x}$ $= \mathrm{e}^{\frac{1}{2}\ln\left(1+x^2\right)}$ $= \mathrm{e}^{\ln\sqrt{1+x^2}}$ $= \sqrt{1+x^2}$.	

	Thus, $$y\sqrt{1+x^2}=\int \frac{\sqrt{1+x^2}}{x^4}\,\mathrm{d}x$$ $$=-\frac{\left(1+x^2\right)^{\frac{3}{2}}}{3x^3}+c \text{ (using part (i)'s result)}$$ Sub. $x=1$ and $y=0$: $$0=-\frac{(2)^{\frac{3}{2}}}{3}+c\Rightarrow c=\frac{2\sqrt{2}}{3}.$$ Therefore, we have $$y\sqrt{1+x^2}=-\frac{\left(1+x^2\right)^{\frac{3}{2}}}{3x^3}+\frac{2\sqrt{2}}{3}$$ $$y=-\frac{1+x^2}{3x^3}+\frac{2}{3}\sqrt{\frac{2}{1+x^2}}.$$ **(b)(ii)** Euler method: $$y_1=0+(0.5)\left(\frac{1}{1^4}-0\right)$$ $$=0.5.$$ **(b)(iii)** The actual value of y when $x=1.5$ is $$y=-\frac{1+(1.5)^2}{3(1.5)^3}+\frac{2}{3}\sqrt{\frac{2}{1+(1.5)^2}}$$ $$=0.201988706048.$$ Percentage error $=\frac{0.5-0.201988706048}{0.201988706048}\times 100\%$ $=147.54\%$ which is large. Hence, there is a large discrepancy between the approximation and the actual value. (<u>OR</u> stating that the approximation more than doubled the actual value.) One reason for the large percentage error is that the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x=1.5$ is 0.10430530756 which is much smaller than the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x=1$ which is 1. OR	

One reason for the large percentage error is that there is a rapid decrease in gradient value in the interval $[1,1.5]$ as can be seen in the sketch below:



Solution

5. (i) $\begin{pmatrix} 2 & -3 & 6 & 2 \\ -2 & 3 & -3 & -3 \\ 4 & -6 & 9 & 5 \\ 2 & -3 & 3 & a \end{pmatrix} \xrightarrow[\substack{R_3-2R_1\to R_3 \\ R_4-R_1\to R_4}]{R_2+R_1\to R_2} \begin{pmatrix} 2 & -3 & 6 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -3 & a-2 \end{pmatrix}$

$$\xrightarrow[R_4+R_2\to R_4]{R_3+R_2\to R_3} \begin{pmatrix} 2 & -3 & 6 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-3 \end{pmatrix}$$

$$\xrightarrow{R_3\leftrightarrow R_4} \begin{pmatrix} 2 & -3 & 6 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & a-3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\substack{\frac{1}{2}\times R_1\to R_1 \\ \frac{1}{3}\times R_2\to R_2 \\ \frac{1}{a-3}\times R_3\to R_3}} \begin{pmatrix} 1 & -\frac{3}{2} & 3 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

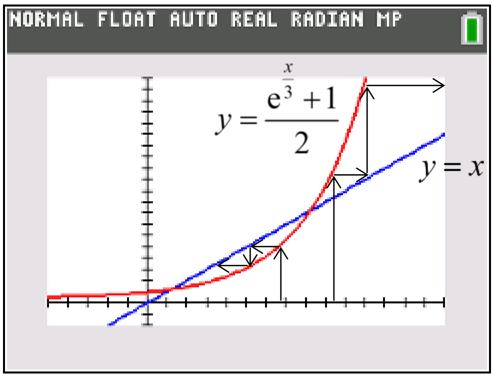
Since there are 3 non-zero rows in the row echelon form of **M**, $\dim(R)=\text{rank}(\mathbf{M})=3$. Also, the 1st, 3rd and 4th columns are pivot columns, thus the corresponding columns of **M** namely $\begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 6 \\ -3 \\ 9 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \\ a \end{pmatrix}$ are linearly independent.

	Hence $\left\{\begin{pmatrix}2\\-2\\4\\2\end{pmatrix},\begin{pmatrix}6\\-3\\9\\3\end{pmatrix},\begin{pmatrix}2\\-3\\5\\a\end{pmatrix}\right\}$ is a basis of R. (ii) Since $\begin{pmatrix}w\\x\\y\\z\end{pmatrix}$ belongs to R, $\begin{pmatrix}w\\x\\y\\z\end{pmatrix}$ can be expressed as a linear combination of $\begin{pmatrix}2\\-2\\4\\2\end{pmatrix}$, $\begin{pmatrix}6\\-3\\9\\3\end{pmatrix}$ and $\begin{pmatrix}2\\-3\\5\\a\end{pmatrix}$. i.e. $\begin{pmatrix}w\\x\\y\\z\end{pmatrix}=c_1\begin{pmatrix}2\\-2\\4\\2\end{pmatrix}+c_2\begin{pmatrix}6\\-3\\9\\3\end{pmatrix}+c_3\begin{pmatrix}2\\-3\\5\\a\end{pmatrix}$, where c_1, c_2 and $c_3 \in \mathbb{R}$. Consider the augmented matrix $\left(\begin{array}{ccc\|c}2&6&2&w\\-2&-3&-3&x\\4&9&5&y\\2&3&a&z\end{array}\right)\xrightarrow[R_4-R_1\to R_4]{R_2+R_1\to R_2,\ R_3-2R_1\to R_3}\left(\begin{array}{ccc\|c}2&6&2&w\\0&3&-1&x+w\\0&-3&1&y-2w\\0&-3&a-2&z-w\end{array}\right)$ $\xrightarrow[R_4+R_2\to R4]{R_3+R_2\to R_3}\left(\begin{array}{ccc\|c}2&6&2&w\\0&3&-1&x+w\\0&0&0&x+y-w\\0&0&a-3&x+z\end{array}\right)$ For the system to have consistent solution, we must have $x+y-w=0 \Rightarrow w=x+y$ (shown) (iii) Consider $\mathbf{Mx}=\mathbf{0}$ where $\mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$. Using GC, $\left(\begin{array}{cccc\|c}2&-3&6&2&0\\-2&3&-3&-3&0\\4&-6&9&5&0\\2&-3&3&3&0\end{array}\right)\xrightarrow{\text{RREF}}\left(\begin{array}{cccc\|c}1&-\frac{3}{2}&0&2&0\\0&0&1&-\frac{1}{3}&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)$	

	So we have $x_1 = \frac{3}{2}x_2 - 2x_4$, $x_3 = \frac{1}{3}x_4$. $\therefore \mathbf{x} = \begin{pmatrix} \frac{3}{2}x_2 - 2x_4 \\ x_2 \\ \frac{1}{3}x_4 \\ x_4 \end{pmatrix} = \frac{1}{2}x_2 \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3}x_4 \begin{pmatrix} -6 \\ 0 \\ 1 \\ 3 \end{pmatrix}$ A basis for the null space of T is $\left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}$. Observe that $\begin{pmatrix} 2 & -3 & 6 & 2 \\ -2 & 3 & -3 & -3 \\ 4 & -6 & 9 & 5 \\ 2 & -3 & 3 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$. So $\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is a particular solution of $\mathbf{Mx} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$. $\therefore \mathbf{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 0 \\ 1 \\ 3 \end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$.	

	Solution	
6.	Let $\mathrm{f}(x) = \mathrm{e}^{\frac{x}{3}} - 2x + 1$. $\mathrm{f}(1) = 0.396 > 0$ $\mathrm{f}(6) = -3.61 < 0$ $\mathrm{f}(9) = 3.09 > 0$ Hence, one root is between 1 and 6, while the other is between 6 and 9. (shown) $\mathrm{f}'(x) = \frac{1}{3}\mathrm{e}^{\frac{x}{3}} - 2$ $x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$ $x_0 = 6$ $x_1 = 13.7987$ $x_2 = 11.4600$ $\mathrm{f}'(x_0) = \mathrm{f}'(6) = 0.463$	

The gradient of the curve $y = \mathrm{f}(x)$ at x_0 is relatively gentle, resulting in the next approximation x_1 (where tangent at x_0 cuts the x-axis) being further away from the root α than x_0.



Using the iterative formula $x_{n+1} = \dfrac{e^{\frac{x_n}{3}} + 1}{2}$, approximations diverge from α as seen in the diagram, regardless of whether the initial approximation is greater than or less than α.

Using instead the iterative formula $x_{n+1} = 3\ln(2x_n - 1)$,

$x_1 = 7.5$

$x_2 = 7.9172$

$x_3 = 8.0908$

$x_4 = 8.1603$

$x_5 = 8.1876$

$x_6 = 8.1983$

$x_7 = 8.2024$

$x_8 = 8.2040$

$x_9 = 8.2047 \approx 8.205$

$x_{10} = 8.2049 \approx 8.205$

$\therefore \alpha = 8.205$ (to 3 dp)

	Solution	
7.	(i) $p=0,\ q=\dfrac{3}{5}$. (ii) $\lambda\mathbf{I}-\mathbf{M}=\begin{pmatrix}\lambda-\frac{4}{5} & -\frac{1}{10} & -\frac{3}{10}\\ 0 & \lambda-\frac{7}{10} & -\frac{1}{10}\\ -\frac{1}{5} & -\frac{1}{5} & \lambda-\frac{3}{5}\end{pmatrix}$ $\det(\lambda\mathbf{I}-\mathbf{M})=0$ $\left(\lambda-\frac{4}{5}\right)\left[\left(\lambda-\frac{7}{10}\right)\left(\lambda-\frac{3}{5}\right)-\frac{1}{50}\right]+\frac{1}{10}\left[-\frac{1}{50}\right]-\frac{3}{10}\left[\frac{1}{5}\left(\lambda-\frac{7}{10}\right)\right]=0$ $\left(\lambda^2-\frac{3}{2}\lambda+\frac{28}{50}\right)\left(\lambda-\frac{3}{5}\right)-\frac{1}{50}\lambda+\frac{4}{250}-\frac{1}{500}-\frac{3}{50}\lambda+\frac{21}{500}=0$ $\lambda^3-\frac{21}{10}\lambda^2+\frac{69}{50}\lambda-\frac{7}{25}=0$ $50\lambda^3-105\lambda^2+69\lambda-14=0$ Using GC, the three eigenvalues of **M** are $\lambda_1=1,\ \lambda_2=\frac{7}{10},\ \lambda_3=\frac{2}{5}$. When $\lambda=1$: $\left(\begin{array}{ccc\|c}\frac{1}{5} & -\frac{1}{10} & -\frac{3}{10} & 0\\ 0 & \frac{3}{10} & -\frac{1}{10} & 0\\ -\frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & 0\end{array}\right)\xrightarrow{\text{RREF}}\left(\begin{array}{ccc\|c}1 & 0 & -\frac{5}{3} & 0\\ 0 & 1 & -\frac{1}{3} & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ Hence an eigenvector is $\begin{pmatrix}\frac{5}{3}\\ \frac{1}{3}\\ 1\end{pmatrix}$. When $\lambda=\frac{7}{10}$: $\left(\begin{array}{ccc\|c}-\frac{1}{10} & -\frac{1}{10} & -\frac{3}{10} & 0\\ 0 & 0 & -\frac{1}{10} & 0\\ -\frac{1}{5} & -\frac{1}{5} & \frac{1}{10} & 0\end{array}\right)\xrightarrow{\text{RREF}}\left(\begin{array}{ccc\|c}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ Hence an eigenvector is $\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}$. When $\lambda=\frac{2}{5}$: $\left(\begin{array}{ccc\|c}-\frac{2}{5} & -\frac{1}{10} & -\frac{3}{10} & 0\\ 0 & -\frac{3}{10} & -\frac{1}{10} & 0\\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 0\end{array}\right)\xrightarrow{\text{RREF}}\left(\begin{array}{ccc\|c}1 & 0 & \frac{2}{3} & 0\\ 0 & 1 & \frac{1}{3} & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ Hence an eigenvector is $\begin{pmatrix}-\frac{2}{3}\\ -\frac{1}{3}\\ 1\end{pmatrix}$.	

(iii)

Let $\mathbf{D}=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{7}{10} & 0\\ 0 & 0 & \frac{2}{5}\end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix}\frac{5}{3} & -1 & -\frac{2}{3}\\ \frac{1}{3} & 1 & -\frac{1}{3}\\ 1 & 0 & 1\end{pmatrix}$.

$\mathbf{M}=\mathbf{PDP}^{-1}$

$\Rightarrow \mathbf{M}^n=\mathbf{PD}^n\mathbf{P}^{-1}$

$$\Rightarrow \mathbf{M}^n=\begin{pmatrix}\frac{5}{3} & -1 & -\frac{2}{3}\\ \frac{1}{3} & 1 & -\frac{1}{3}\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1^n & 0 & 0\\ 0 & \left(\frac{7}{10}\right)^n & 0\\ 0 & 0 & \left(\frac{2}{5}\right)^n\end{pmatrix}\begin{pmatrix}\frac{5}{3} & -1 & -\frac{2}{3}\\ \frac{1}{3} & 1 & -\frac{1}{3}\\ 1 & 0 & 1\end{pmatrix}^{-1}$$

As $n\to\infty$, $\left(\frac{7}{10}\right)^n\to 0$ and $\left(\frac{2}{5}\right)^n\to 0$.

So

$$\mathbf{M}^n\to\begin{pmatrix}\frac{5}{3} & -1 & -\frac{2}{3}\\ \frac{1}{3} & 1 & -\frac{1}{3}\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{5}{3} & -1 & -\frac{2}{3}\\ \frac{1}{3} & 1 & -\frac{1}{3}\\ 1 & 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}\frac{5}{9} & \frac{5}{9} & \frac{5}{9}\\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}$$

$\mathbf{v}_{n+1}=\mathbf{Mv}_n=\mathbf{M}^2\mathbf{v}_{n-1}=\ldots=\mathbf{M}^n\mathbf{v}_1=\mathbf{M}^{n+1}\mathbf{v}_0$

Hence $\mathbf{v}_{n+1}\to\begin{pmatrix}\frac{5}{9} & \frac{5}{9} & \frac{5}{9}\\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9}\\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}7.6\\ 4.8\\ 5.6\end{pmatrix}=\begin{pmatrix}10\\ 2\\ 6\end{pmatrix}$ as $n\to\infty$.

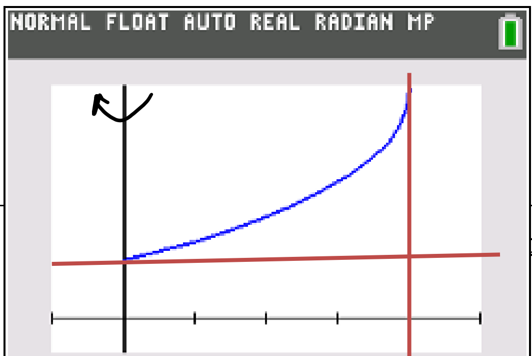
In the long run, the population of the cities will stabilise according to the following figures.

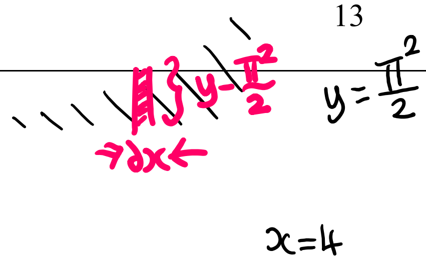
City	Population (in millions)
A	10
B	2
C	6

	Source: TMJC Promo 2025 Qn 8
8(i)	Given $x=4\sqrt{\sin\theta}$, $\int x\sqrt{1-\left(\frac{x}{4}\right)^4}\,\mathrm{d}x$ $=\int 4\sqrt{\sin\theta}\sqrt{1-\left(\sqrt{\sin\theta}\right)^4}\,4\left(\frac{1}{2}\right)(\sin\theta)^{-\frac{1}{2}}\cos\theta\,\mathrm{d}\theta$

	$=\int 8\sqrt{1-(\sin\theta)^2}\cos\theta\,\mathrm{d}\theta$ $=\int 8(\cos\theta)^2\,\mathrm{d}\theta$ $=4\int \cos 2\theta+1\,\mathrm{d}\theta$ $=4\left(\frac{\sin 2\theta}{2}+\theta\right)+c$ $=4(\sin\theta\cos\theta+\theta)+c$ $=4\left(\frac{x^2}{16}\right)\left(\frac{\sqrt{256-x^4}}{16}\right)+4\sin^{-1}\left(\frac{x^2}{16}\right)+c$ $=\frac{x^2\sqrt{256-x^4}}{64}+4\sin^{-1}\left(\frac{x^2}{16}\right)+c$
(ii)	Volume of air gap $=2\pi\int_0^4 x\left[\left(2\left(\frac{x}{2}\right)^2+\left(\frac{x}{4}\right)^4+1\right)-\left(10\left(1-\sqrt{1-\left(\frac{x}{4}\right)^4}\right)\right)\right]\mathrm{d}x$ $=2\pi\int_0^4\left(\frac{x^3}{2}+\frac{x^5}{256}-9x+10x\sqrt{1-\left(\frac{x}{4}\right)^4}\right)\mathrm{d}x$ $=2\pi\int_0^4\left(\frac{x^3}{2}+\frac{x^5}{256}-9x\right)\mathrm{d}x+20\pi\int_0^4\left(x\sqrt{1-\left(\frac{x}{4}\right)^4}\right)\mathrm{d}x$ $=2\pi\left[\frac{x^4}{8}+\frac{x^6}{1536}-\frac{9x^2}{2}\right]_0^4+20\pi\left[\frac{x^2\sqrt{256-x^4}}{64}+4\sin^{-1}\left(\frac{x^2}{16}\right)\right]_0^4$ $=2\pi\left[32+\frac{8}{3}-72\right]+20\pi\left[4\sin^{-1}1\right]$ $=\left(40\pi-\frac{224}{3}\right)\pi\ \text{cm}^3$
(iii)	Maximum volume of liquid that the mug can hold $=\pi\int_0^{10}\left[4\left(1-\left(\frac{y}{10}-1\right)^2\right)^{\frac{1}{4}}\right]^2\mathrm{d}y-\left(40\pi-\frac{224}{3}\right)\pi$ $= 235\ \text{cm}^3$ (3 s.f) Can also rotate C1 about the y-axis Alternatively (using shell method) Maximum volume of liquid that the mug can hold $=\pi(4)^2(10)-2\pi\int_0^4 x\left[\left(2\left(\frac{x}{2}\right)^2+\left(\frac{x}{4}\right)^4+1\right)\right]\mathrm{d}x$ $= 235\ \text{cm}^3$ (3 s.f)

	Using Shell Method $\text{Vol} = \int_{x=0}^{x=4} 2\pi x\left[10-2\left(\frac{x}{2}\right)^2-\left(\frac{x}{4}\right)^4-1\right]\mathrm{d}x$ $= 234.57 \simeq 235\,\text{unit}^3$
(iv)	$y = 2\left(\frac{x}{2}\right)^2+\left(\frac{x}{4}\right)^4+1=\frac{1}{2}x^2+\frac{1}{4^4}x^4+1$ $\frac{\mathrm{d}y}{\mathrm{d}x} = x+\frac{1}{4^3}x^3$ Surface area of inner glass layer $= \int_{x=0}^{x=4} 2\pi x\sqrt{1+\left(x+\frac{x^3}{4^3}\right)^2}\,\mathrm{d}x \simeq 164\,\text{unit}^2$

	Solution	
8.	(a) $x = -4\cos t,\quad y = 2t^2$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 4\sin t,\quad \frac{\mathrm{d}y}{\mathrm{d}t} = 4t$ Surface area generated $= 2\pi\int_{\frac{\pi}{2}}^{\pi} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ $= 2\pi\int_{\frac{\pi}{2}}^{\pi} 2t^2\sqrt{(4\sin t)^2+(4t)^2}\,\mathrm{d}t$ $= 1186.07\ \text{units}^2$ (b)(i) $\int t^2\sin t\cos t\ \mathrm{d}t$ $= \int \frac{1}{2}t^2\sin 2t\ \mathrm{d}t$ $= \frac{1}{2}\left[-\frac{1}{2}t^2\cos 2t-\int -\frac{1}{2}(2t)\cos 2t\ \mathrm{d}t\right]$ $= -\frac{1}{4}t^2\cos 2t+\frac{1}{2}\left(\frac{1}{2}t\sin 2t-\int\frac{1}{2}\sin 2t\ \mathrm{d}t\right)$ $= -\frac{1}{4}t^2\cos 2t+\frac{1}{4}t\sin 2t+\frac{1}{8}\cos 2t+D$ 	

	 (ii) Volume generated $=2\pi\int_0^4 x\left(y-\frac{\pi^2}{2}\right)\,\mathrm{d}x$ $=2\pi\int_{\frac{\pi}{2}}^{\pi} -4\cos t\left(2t^2-\frac{\pi^2}{2}\right)(4\sin t)\ \mathrm{d}t$ $=32\pi\int_{\frac{\pi}{2}}^{\pi} -\sin t\cos t\left(2t^2-\frac{\pi^2}{2}\right)\ \mathrm{d}t$ $=32\pi\int_{\frac{\pi}{2}}^{\pi} \frac{\pi^2}{2}\sin t\cos t-2t^2\sin t\cos t\ \mathrm{d}t$ $=8\pi^3\int_{\frac{\pi}{2}}^{\pi}\sin 2t\ \mathrm{d}t-64\pi\int_{\frac{\pi}{2}}^{\pi} t^2\sin t\cos t\ \mathrm{d}t$ $=4\pi^3\left[-\cos 2t\right]_{\frac{\pi}{2}}^{\pi}-64\pi\left[-\frac{1}{4}t^2\cos 2t+\frac{1}{4}t\sin 2t+\frac{1}{8}\cos 2t\right]_{\frac{\pi}{2}}^{\pi}$ $=-8\pi^3-64\pi\left(-\frac{1}{4}\pi^2+\frac{1}{8}-\left(\frac{\pi^2}{16}-\frac{1}{8}\right)\right)$ $=12\pi^3-16\pi\ \text{units}^3$	

	Solution	
9.	**(i)** $\frac{\mathrm{d}k}{\mathrm{d}t}=sAk^{\alpha}-\delta k$. (1)	

At equilibrium, $\frac{\mathrm{d}k}{\mathrm{d}t}=0$.

$$sAk^{\alpha}-\delta k=0$$

$$sA-\delta k^{1-\alpha}=0 \quad (\because k>0)$$

$$k=\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

(ii) From $u=Ak^{1-\alpha}$,

$$\frac{\mathrm{d}u}{\mathrm{d}t}=A(1-\alpha)k^{-\alpha}\frac{\mathrm{d}k}{\mathrm{d}t}$$

$$\frac{\mathrm{d}k}{\mathrm{d}t}=\frac{k^{\alpha}}{A(1-\alpha)}\frac{\mathrm{d}u}{\mathrm{d}t}.$$

Substituting into (1):

$$\frac{k^{\alpha}}{A(1-\alpha)}\frac{\mathrm{d}u}{\mathrm{d}t}=sAk^{\alpha}-\delta k$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=(1-\alpha)\left(sA^2-\delta Ak^{1-\alpha}\right)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t}=(1-\alpha)\left(sA^2-\delta u\right) \quad \text{(shown)}$$

$$\frac{1}{sA^2-\delta u}\frac{\mathrm{d}u}{\mathrm{d}t}=1-\alpha$$

$$\int\frac{1}{sA^2-\delta u}\,\mathrm{d}u=\int 1-\alpha\,\mathrm{d}t$$

$$-\frac{1}{\delta}\ln\left|sA^2-\delta u\right|=(1-\alpha)t+B$$

$$\ln\left|sA^2-\delta u\right|=\delta(\alpha-1)t-\delta B$$

$$sA^2-\delta u=\pm\mathrm{e}^{\delta(\alpha-1)t-\delta B}$$

$$=C\mathrm{e}^{\delta(\alpha-1)t} \quad (\text{where } C=\pm\mathrm{e}^{-\delta B})$$

$$\delta u=sA^2-C\mathrm{e}^{\delta(\alpha-1)t}$$

$$u=\frac{sA^2}{\delta}+D\mathrm{e}^{\delta(\alpha-1)t} \quad \left(\text{where } D=-\frac{C}{\delta}\right)$$

$$Ak^{1-\alpha}=\frac{sA^2}{\delta}+D\mathrm{e}^{\delta(\alpha-1)t}.$$

When $t=0$, $k=k_0$. Thus,

$$Ak_0^{\,1-\alpha}=\frac{sA^2}{\delta}+D$$

$$D=Ak_0^{\,1-\alpha}-\frac{sA^2}{\delta}.$$

	Therefore, we have $$Ak^{1-\alpha}=\frac{sA^2}{\delta}+\left(Ak_0^{\ 1-\alpha}-\frac{sA^2}{\delta}\right)\mathrm{e}^{\delta(\alpha-1)t}$$ $$k=\left[\frac{sA}{\delta}+\left(k_0^{\ 1-\alpha}-\frac{sA}{\delta}\right)\mathrm{e}^{\delta(\alpha-1)t}\right]^{\frac{1}{1-\alpha}}.$$ **(iii)** As $t\to\infty$, $\mathrm{e}^{\delta(\alpha-1)t}\to 0$ ($\because \alpha<1$). Therefore, $$k=\left[\frac{sA}{\delta}+\left(k_0^{\ 1-\alpha}-\frac{sA}{\delta}\right)\mathrm{e}^{\delta(\alpha-1)t}\right]^{\frac{1}{1-\alpha}}\to\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$ which is the equilibrium value found in part **(i)**. Since $k\to\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$ in the long run regardless of the value of $k_0>0$, this equilibrium is stable. **(iv)** Denote consumption per capita by *c*. Then, we have $$c=q-sq$$ $$=Ak^{\alpha}-\delta k \quad \text{(at the steady state).}$$ Now, to find the value of *k* that maximises *c*, set $$\frac{\mathrm{d}c}{\mathrm{d}k}=\alpha Ak^{\alpha-1}-\delta=0$$ to obtain $$k=\left(\frac{\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}}=\left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1-\alpha}}.$$ Also, at the steady state, $k=\left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$ (by **(i)**). Thus, we have $$\left(\frac{s_G A}{\delta}\right)^{\frac{1}{1-\alpha}}=\left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1-\alpha}}$$ $$\frac{s_G A}{\delta}=\frac{\alpha A}{\delta}$$ $$s_G=\alpha.$$	

	Solution	
10.	(i) $T_{n+1}=0.9T_n+k$. (ii) $T_{n+1}=0.9T_n+k$	

$= 0.9\left(0.9T_{n-1}+k\right)+k$

$= \left(0.9\right)^2 T_{n-1}+0.9T_k+k$

$= \ldots$

$= \left(0.9\right)^n T_1+\left(0.9\right)^{n-1}k+\left(0.9\right)^{n-2}k+\ldots+k$

$= 250\left(0.9\right)^n+k\left[\dfrac{1-\left(0.9\right)^n}{1-0.9}\right]$

$= \left(250-10k\right)\left(0.9\right)^n+10k$

$\therefore T_n=\left(250-10k\right)\left(0.9\right)^{n-1}+10k$

<u>Alternative method</u>

Since $T_{n+1}=0.9T_n+k$ is a first order non-homogenous linear recurrence relation, the general solution is given by $T_n=a\left(0.9\right)^{n-1}+b$, where a and b are constants.

$T_1=250:\ \ a+b=250$ ---(1)

$T_2=0.9T_1+k=225+k$

$\Rightarrow 0.9a+b=225+k$ ---(2)

$(1)-(2):\ \ 0.1a=25-k \Rightarrow a=250-10k$

Sub. $a=250-10k$ into (1), we have $b=10k$.

$\therefore T_n=\left(250-10k\right)\left(0.9\right)^{n-1}+10k$

(iii) We need $T_{12}\geq 350$.

$\Rightarrow \left(250-10k\right)\left(0.9\right)^{11}+10k\geq 350$

Using GC (table),

k	$\left(250-10k\right)\left(0.9\right)^{11}+10k$
39	346.07
40	352.93
41	359.79

Least $k=40$.

	(iv) Profit made at the end of the mth month $= \text{Total Revenue} - \text{Total Cost}$ $= \sum_{n=1}^{m} \left\{ 10\left[(250-500)(0.9)^{n-1} + 500 \right] - 4000 \right\}$ $= \sum_{n=1}^{m} \left\{ 2500\left[2-(0.9)^{m-1} \right] - 4000 \right\}$ Using GC (table), | m | $2500\left[2-(0.9)^{m-1}\right]-4000$ | | 22 | –538.10 | | 23 | 215.73 | $\therefore m = 23$ (v) As $n \to \infty$, $(0.9)^{n-1} \to 0$. $\therefore T_n \to 10k = 10(50) = 500$ The number of subscribers will increase and approach 500 in the long-term. This behaviour is not appropriate as this implies that the number of customers will not grow beyond 500, which is not favourable for the company to be successful in the long-run.	