RAFFLES INSTITUTION
H3 Mathematics 9820

Chapter 8 Sequences and Series

This chapter is concerned with the study of sequences and series. The material has far-reaching applications in various areas of science and engineering and is the cornerstone for many branches of mathematics. We proceed via sequences, before looking at series as a sequence of partial sums.

SYLLABUS INCLUDES

- Sequences and series, general terms, sums, limiting behaviours and bounds

CONTENT

1 Sequences

In everyday usage, the term "sequence" suggests a succession of objects or events in a specified order. Informally, a "sequence" in mathematics refers to an unending succession of numbers. Some examples are

$$1, 2, 3, \ldots$$
$$2, 4, 6, \ldots$$
$$1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots$$
$$1, -1, 1, -1, \ldots$$

In each case, ellipses (the three dots) are used to suggest that the sequence continues indefinitely, following the (hopefully) obvious pattern. The numbers in the sequence are called the *terms* of the sequence, and are described according to the positions they occupy. Thus we speak of the *first term*, *second term* and so forth of a sequence.

The most common way to specify a sequence is to give a formula for the terms, telling us how to compute the n^{th} term in terms of n. For the second example above, we see that the n^{th} is given by the formula $2n$. This can be expressed using the bracket notation by writing $\{2n\}_{n=1}^{\infty}$.

When we want to talk about a sequence without actually specifying the numerical values of the terms, we use a letter with subscripts to denote the terms:

$$a_1, a_2, \ldots, a_n, \ldots \quad \text{or} \quad \{a_n\}_{n=1}^{\infty} \quad \text{or} \quad \{a_n\}.$$

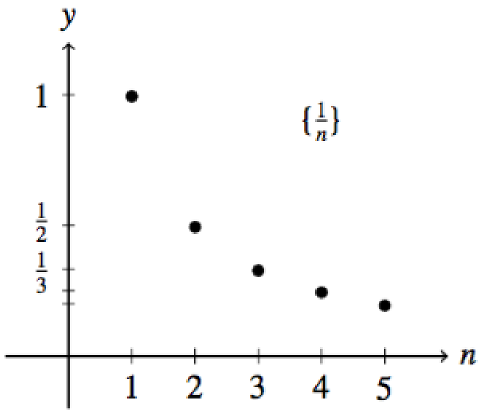
Let us now consider the idea of a sequence more precisely. Notice that in bracket notation, such as $\{2n\}$, we are actually specifying a rule that tells us how to associate a numerical value, $2n$ in this case, with each positive integer. We have a name for such an association – a function! Thus we have the following formal definition.

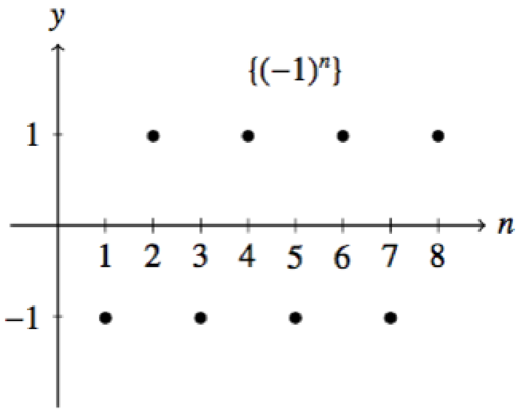
Definition 1. A *sequence* is a function whose domain is the set of positive integers.

The codomain is intentionally left unspecified above to allow for the possibility of sequences of other entities (like complex numbers), but can be taken to be the real numbers for this chapter. Thus a sequence is a function $\mathrm{f}: \mathbb{Z}^+ \to \mathbb{R}$. The notation we have been using is then simply a listing of the functional values

$$\mathrm{f}(1), \mathrm{f}(2), \ldots, \mathrm{f}(n), \ldots$$

Thinking of a sequence as a function also leads us to inquire about the graph of a sequence. Since the domain is $\mathbb{Z}^+$, the graph of a sequence consists of a succession of isolated points. For example, the graphs of $\left\{\frac{1}{n}\right\}$ and $\left\{(-1)^n\right\}$ are as follows:





One of the questions we can ask about a sequence is its behaviour for large values of *n*. This is what we consider in the next section and the graph of a sequence can be a useful visual aid in this pursuit.

1.1 Limit of a Sequence

The three sequences $\{n\}$, $\left\{\frac{1}{n}\right\}$ and $\left\{(-1)^n\right\}$ behave very differently as n gets larger and n larger. In the sequence $\{n\}$ the terms increase without bound; in the sequence $\left\{\frac{1}{n}\right\}$ the terms decrease and approaches a "limiting" value of 0; and in the sequence $\left\{(-1)^n\right\}$ the terms oscillate between 1 and -1. It is the behaviour of $\left\{\frac{1}{n}\right\}$ that we are most interested in, but before making the concept precise, let us consider some more examples.

Exercise 1

Describe the behaviour of the following sequences, with the aid of your GC if necessary

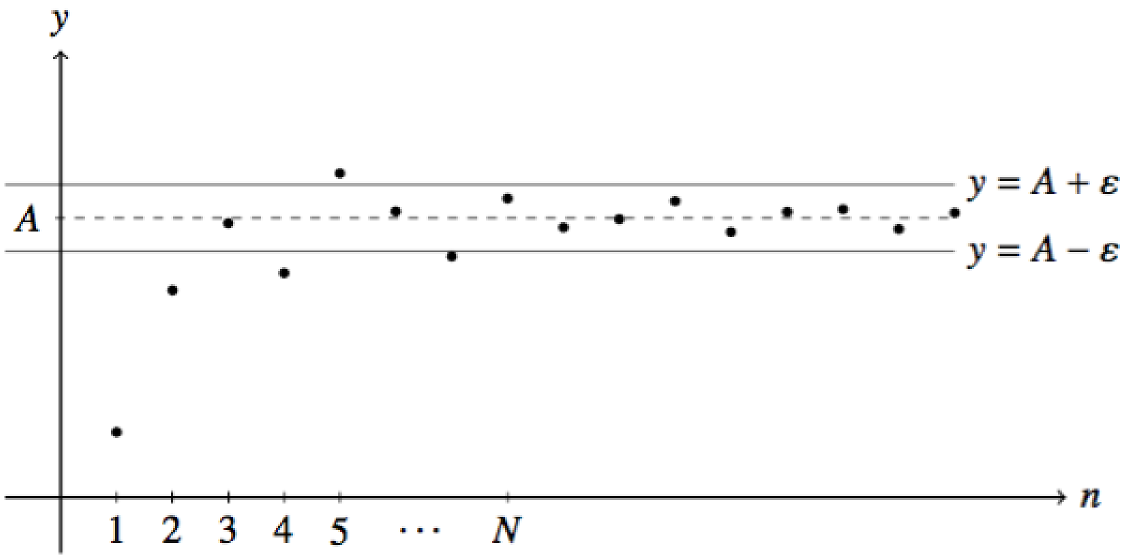
(i) $\left\{-\dfrac{n^3}{3n^2+1}\right\}$;

(ii) $\left\{\dfrac{n}{n+1}\right\}$;

(iii) $\left\{\dfrac{\sin n}{n}\right\}$;

(iv) $\left\{\sin n\right\}$.

Let us try to make the term "limit" precise. To say that the sequence $\{a_n\}$ approaches a limit A as n gets large is meant to convey the idea that, *eventually*, the terms in the sequence become *arbitrarily close* to the number A. This means that, whatever our criterion for closeness is (such as being within a distance of ε of A), all the terms in the sequence *from some point onwards* will be within that distance of A. Geometrically, if we sketch the lines $y = A + \varepsilon$ and $y = A - \varepsilon$ on the graph of the sequence, all the terms in the sequence after some point, say N, will be trapped within the band between these lines. This is illustrated in the diagram below:



If a sequence $\{a_n\}$ has a limit A, we say that the sequence *converges* to A and write

$$\lim_{n\to\infty} a_n = A$$

A sequence that does not have a finite limit is said to *diverge*.

Example 1

Let us look at some common sequences and see if they converge to a finite limit. The aim is to develop an intuitive feel for the idea without getting into the details of the proofs.

(i) A constant sequence, that is, a sequence in which each term a_n is a fixed number, say a, converges and has the limit a.

(ii) The sequence $\left\{\frac{1}{n}\right\}$ converges to the limit 0.

(iii) The sequences $\{n\}$ and $\{2n\}$ increases without bound and diverges. The sequence $\{-n^2\}$ also diverges, but does so by decreasing without bound.

(iv) The sequences $\left\{\frac{n}{n+1}\right\}$ and $\left\{1+\left(-\frac{1}{2}\right)^n\right\}$ both converge to 1. The terms of $\left\{\frac{n}{n+1}\right\}$ are always less than 1 and approaches 1 from below, while those of $\left\{1+\left(-\frac{1}{2}\right)^n\right\}$ are alternately below or above 1, but nevertheless still approach 1.

(v) The terms of the sequence $\left\{(-1)^n\right\}$ oscillates alternately between −1 and 1 and does not converge. The sequence $\left\{\sin n\right\}$ also does not converge, but its terms vary erratically between −1 and 1.

(vi) Finally, the sequence $\left\{\frac{\sin n}{n}\right\}$ converges to 0 even though its terms flips between being positive or negative erratically.

We will not attempt to formally define what it means for a sequence to converge or prove the following properties; this you will do in a course on analysis. Instead we will just list here some results that will allows us to determine if a sequence converges and find its limit when it does.

Theorem 1 If a sequence is convergent, the limit is unique. Also, every subsequence is convergent and has the same limit.

A *subsequence* of $\left\{a_n\right\}$ refers to a sequence obtained by picking some of the terms of $\left\{a_n\right\}$. For example, two subsequences of 1, 2, 3, 4, 5, ..., *n*, ... are

1, 2, 4, 6, ..., 2*n*, ... and 1, 3, 9, 27, 81, ..., 3^n, ...

Similar to our discussion in inequalities, we will also introduce the notion of bounds for sequences. A sequence $\left\{a_n\right\}$ is said to be *bounded above* if there is a real number *M* such that all its terms are less than *M*. In other words, $a_n \le M$ for all *n*. Similarly, $\left\{a_n\right\}$ is said to be *bounded below* if there is a real number *L* such that $a_n \ge L$ for all *n*. Finally, $\left\{a_n\right\}$ is said to be *bounded* if it is both bounded above and bounded below. Since the terms of a convergent sequence must eventually lie close to its limit, it is not hard to see that the following theorem holds.

Theorem 2 A convergent sequence is bounded.

This result is often used in the contrapositive: if a sequence is not bounded, then it is divergent. Thus we see that $\{2n\}$ and $\left\{-n^2\right\}$ are divergent since they are not bounded.

Exercise 2 Determine if the following sequences are bounded

(i) $\left\{1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}\right\}$;

(ii) $\left\{\cos\frac{1}{n}+\sec\frac{1}{n}\right\}$.

So far, we have talked about uniqueness of limits and a condition under which a sequence will converge. In practice, however, we are usually interested in calculating limits, and the following result provides us with some algebraic rules in which to handle the limits.

Theorem 3 Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences with limits A and B respectively, and c is a constant. Then

(i) $\lim\limits_{n\to\infty} ca_n = cA$;

(ii) $\lim\limits_{n\to\infty}\left(a_n \pm b_n\right) = \left(A \pm B\right)$;

(iii) $\lim\limits_{n\to\infty}\left(a_n b_n\right) = AB$;

(iv) $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{A}{B}$, if $B \neq 0$ and $b_n = 0$ for finitely many n;

(v) If f is a continuous function, then $\lim\limits_{n\to\infty} \mathrm{f}(a_n) = \mathrm{f}\left(\lim\limits_{n\to\infty} a_n\right) = \mathrm{f}(A)$.

Let us see how the above results allow us to conclude that $\left\{\dfrac{n}{n+1}\right\}$ converge to 1. Dividing both the numerator and denominator by n, we have

$$\lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{\lim\limits_{n\to\infty}1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)} = \frac{\lim\limits_{n\to\infty}1}{\lim\limits_{n\to\infty}1+\lim\limits_{n\to\infty}\frac{1}{n}} = \frac{1}{1+0} = 1.$$

Alternatively, we have also

$$\lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = \lim_{n\to\infty}1 - \lim_{n\to\infty}\frac{1}{n+1} = 1-0 = 1.$$

Exercise 3 For each of the following sequences, determine if it converges and find the limit if it does.

(i) $\left\{\dfrac{n^2}{2n^2-n+1}\right\}$;

(ii) $\left\{\cos\dfrac{3}{n}\right\}$;

(iii) $\left\{\dfrac{1}{\sqrt{n+1}-\sqrt{n}}\right\}$.

1.2 Monotone Sequences.

This section deals with monotone sequences and their convergence. The results are relatively intuitive and easy to apply. Their proofs, however, rely on fundamental properties of the real numbers and is beyond the scope of this course.

We begin with some terminology.

Definition 2 A sequence $\{a_n\}$ is called

- *increasing* if $a_n \le a_{n+1}$ for all *n*;
- *strictly increasing* if $a_n < a_{n+1}$ for all *n*;
- *decreasing* if $a_n \ge a_{n+1}$ for all *n*;
- *strictly decreasing* if $a_n > a_{n+1}$ for all *n*.

A sequence that is increasing or decreasing is called *monotone* while a sequence that is strictly increasing or strictly decreasing is called *strictly monotone*.

A simple way to determine if a sequence is monotone is to check the sign of $a_{n+1} - a_n$, as the next example illustrates.

Example 2

We have seen, at least intuitively that the sequence $\left\{\dfrac{n}{n+1}\right\}$ is increasing. In order to show this formally, consider

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)}$$

Thus $a_{n+1} - a_n > 0$ and the sequence is strictly increasing.

In situations where all the terms of the sequence are of the same sign, we can also consider the ratio $\dfrac{a_{n+1}}{a_n}$.

Example 3

Let us consider the sequence $a_n = \dfrac{2^n}{(n+1)!}$. Since all the terms are positive, let us consider $\dfrac{a_{n+1}}{a_n}$.

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+2)!}\frac{(n+1)!}{2^n} = \frac{2}{n+2} < 1$$

for all positive integers *n*. Hence $a_n > a_{n+1}$ and the sequence is strictly decreasing.
Remark: Can we consider $a_{n+1} - a_n$? What if the terms of the sequence are negative?

Exercise 4 Classify the following sequences according to whether they are increasing, strictly increasing, decreasing, strictly decreasing.

(i) $\left\{n-2^{n}\right\}$;

(ii) $\left\{n\mathrm{e}^{-n}\right\}$;

(iii) $\left\{\dfrac{\mathrm{e}^{n}}{n!}\right\}$.

We come now to the main result of this section, which provides a convenient way to check if monotone sequences are convergent. However, it provides no means to determine the limit. This result is also known as the Monotone Convergence Theorem. The proof, is of course, out of the limits of the H3 syllabus.

Theorem 4 (Monotone Convergence Theorem)

A monotone sequence of real numbers is convergent if and only if it is bounded.

In particular, if $\{a_n\}$ is an increasing sequence and is bounded above by *M*, then the sequence $\{a_n\}$ converges and we can also show that $\lim_{n\to\infty} a_n \le M$.
Similarly, if $\{a_n\}$ is a decreasing sequence and is bounded below by *L*, then the sequence $\{a_n\}$ converges and we can also show that $\lim_{n\to\infty} a_n \ge L$.

Example 4

We have seen that $\left\{\dfrac{n}{n+1}\right\}$ is an increasing sequence. It is also bounded above by 1, since

$$\frac{n}{n+1}=1-\frac{1}{n+1}<1$$

for all *n*. Thus it converges to a limit no larger than 1. Note that the Monotone Convergence Theorem only ensures the existence of the limit, but does not specify how to calculate the limit. Similarly, the sequence $\left\{\frac{1}{n}\right\}$ is decreasing and bounded below by 0, so it converges to a limit not smaller than 0.

When investigating the convergence of a sequence, it is only the eventual behaviour that matters. Thus we can disregard the behaviour of finitely many terms of the sequence. We can make use of this observation when a sequence is monotone from some point onwards, as the following exercise illustrates.

Exercise 5

Is $\left\{\dfrac{100^{n}}{n!}\right\}$ a monotone sequence? Is it convergent?

Exercise 6

Consider the sequence defined by $a_n = \left(1+\frac{1}{n}\right)^n$.

(i) Show that a_n is increasing by using the Binomial Theorem.

(ii) Show also that a_n is bounded. In fact, show that $2 \le a_n \le 3$ for all n using the expansion in (i) and also the result (which you should prove) that $\frac{1}{k!} \le \frac{1}{2^{k-1}}$.

(iii) Hence conclude that a_n converges to a value between 2 and 3. This is the number e.

1.3 Recurrence Relations

Whenever we talk about sequences, it is fairly common we discuss about their monotonicity and boundedness. This is the result of one the most important theorems in real analysis, the monotone convergence theorem, which you have just seen.

In this section, we will look at sequences which are defined recursively, and study properties (bounded, monotonicity, convergence) about these sequences as well as their limits.

Recurrence relations provide another way to define an infinite sequence. Instead of giving an explicit formula for the n^{th} term of the sequence, we describe how it can be computed from previously known terms. For example, we may start by declaring that the first term, u_1 is 1, and that each subsequent term is obtained by adding 2 to the previous term, i.e.

$$u_{n+1} = u_n + 2$$

The formula that allows us to compute a term from previously known terms is called a *recurrence relation* as it allows us to recursively compute the next term. The above example, of course, defines an arithmetic sequence with first term 1 and common difference 2.

Another example is the Fibonacci sequence, defined by

$$F_0 = F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0.$$

The main goal, of course, is to find a formula for u_n defined by a recurrence relation. While there are techniques that can be applied in certain situations to obtain a closed form formula for u_n which we may cover later, the general problem is usually prohibitively difficult to solve. For instance, it is not obvious if we can obtain a closed formula for u_n defined by

$$u_0 = u_1 = 1, u_{n+2} = \frac{1}{\sqrt{u_{n+1}}} + u_n \text{ for } n \geq 0.$$

Let us look at an example of a sequence where we can find a closed form for the general term.

Example 5

Let $u_1 = a, u_{k+1} = \frac{1}{2}(u_k + 1)$ for $k \geq 1$.

To find a closed form for the general term, let us first start by considering a small case, say u_4. To find u_4, we first need to find u_3, which in turns requires u_2, so we need to "unroll" the recurrence relation:

$$\begin{aligned}
u_4 &= \tfrac{1}{2}u_3 + \tfrac{1}{2} \\
&= \tfrac{1}{2}\left(\tfrac{1}{2}u_2 + \tfrac{1}{2}\right) + \tfrac{1}{2} \\
&= \left(\tfrac{1}{2}\right)^2 u_2 + \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} \\
&= \left(\tfrac{1}{2}\right)^2 \left[\tfrac{1}{2}u_1 + \tfrac{1}{2}\right] + \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} \\
&= \left(\tfrac{1}{2}\right)^3 u_1 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2}
\end{aligned}$$

From this the pattern should be clear and for a general term, we have

$$\begin{aligned}
u_k &= \tfrac{1}{2}u_{k-1} + \tfrac{1}{2} \\
&= \tfrac{1}{2}\left(\tfrac{1}{2}u_{k-2} + \tfrac{1}{2}\right) + \tfrac{1}{2} \\
&= \left(\tfrac{1}{2}\right)^2 u_{k-2} + \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} \\
&= \left(\tfrac{1}{2}\right)^2 \left[\tfrac{1}{2}u_{k-3} + \tfrac{1}{2}\right] + \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} \\
&= \left(\tfrac{1}{2}\right)^3 u_{k-3} + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2} \\
&= \ \vdots \\
&= \left(\tfrac{1}{2}\right)^{k-1} u_1 + \left(\tfrac{1}{2}\right)^{k-1} + \left(\tfrac{1}{2}\right)^{k-2} + \ldots + \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{2}
\end{aligned}$$

The first term is just $a\left(\tfrac{1}{2}\right)^{k-1}$, and the remaining terms form a geometric progression whose sum is easy to calculate.

Hence, we have

$$u_k = a\left(\frac{1}{2}\right)^{k-1} + \frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^{k-1}\right]}{1-\frac{1}{2}} = a\left(\frac{1}{2}\right)^{k-1} + 1 - \left(\frac{1}{2}\right)^{k-1} = 1+(a-1)\left(\frac{1}{2}\right)^{k-1}.$$

From here it is easy to see that the sequence converges to a limit of 1.

Remark If we are only interested in finding the limit of the sequence, we could have argued as follows. By taking limits on both sides of the recurrence relation, we have

$$\lim_{k\to\infty} u_{k+1} = \lim_{k\to\infty} \frac{1}{2}\left(u_k + 1\right).$$

Hence if the limit of the sequence is L, then $L = \frac{1}{2}\left(L+1\right) \Rightarrow L = 1.$

However, the above procedure does not ensure that the sequence converges. All it says is that if it converges, it must converge to 1.

Example 6

Define a sequence by $u_1 = 0, u_k = \sqrt{u_{k-1}+2}.$ Find the possible limit(s) of the sequence if the sequence converges.

If the sequence converges to a limit L, then $L = \sqrt{L+2}.$ Solving for L, we have $L = -1$ or $L = 2$. However, since the terms of the sequence are all nonnegative, $L = 2$.

Remark. It must be emphasized again that the above procedure *in no way proves that the sequence converges*. All it says is that if it does converge, then the limit must be one of the values found. It can also happen that which value, if any, the sequence converges to depends on the starting value of u_1.

To prove that the sequence does converge, one of the usual techniques is to use the Monotone Convergence Theorem. That is to say, we prove that the sequence is monotone (increasing/decreasing) and bounded, thus converges. To illustrate, in the above exercise, let us show that the sequence is monotone increasing and bounded above by 2.

Exercise 7

Define a sequence by $u_1 = 0, u_k = \sqrt{u_{k-1}+2}, k \geq 1$.

(i) Show by mathematical induction that $u_k < 2$ for all positive integers k.

(ii) Show also by mathematical induction that the sequence is strictly increasing.

(iii) Conclude that the limit of the sequence is 2.

2 Series

The purpose of this section is to discuss sums

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + ...$$

that contain infinitely many terms. These are known as *infinite series*. From H2 Maths, you should already be familiar with examples like power series and geometric series.

The expression $\sum_{k=1}^{\infty} u_k$ directs us to obtain the "sum" of the terms $u_1, u_2, u_3, \ldots$. However, it is impossible to "add up" infinitely many numbers, so we shall deal with the infinite sum by means of a limiting process. We start by adding finitely many terms from the series and ask what happens as we keep adding more terms. This leads us naturally to consider the *partial sums*:

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ &\vdots \\ s_n &= u_1 + u_2 + ... + u_n \end{aligned}$$

The sequence $\{s_n\}$ is called the *sequence of partial sums* and its limit (if it exists) as $n \to \infty$, will be taken to be the sum of all the terms in the series.

Definition 3 Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{k=1}^{\infty} u_k$. If the sequence $\{s_n\}$ converges to a limit *S*, then the series is said to converge and *S* is called the *sum* of the series. We write $S = \sum_{k=1}^{\infty} u_k$ to denote this. If the sequence of partial sums diverge, then the series is said to diverge. A divergent series has no sum.

Exercise 8 Show that the following series diverge.

(i) $\sum_{k=1}^{\infty}(-1)^k = -1+1-1+1-...;$

(ii) $\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+...;$.

[Hint: Show that the partial sums $s_2, s_4, s_8, s_{16}, ...,$ … increases without bound.]

2.1 Computing Partial Sums

Since the sum of a series is defined to be the limit of its sequence of partial sums, we naturally have to compute the partial sums. In this section we look at some series for which this can be done and develop some techniques for computing their partial sums.

Arithmetic and Geometric Series

These should be familiar from H2 Math and we will only recall the main results.

<u>Arithmetic Series</u>

An arithmetic series is a series $\sum_{k=1}^{\infty} u_k$ where $u_k = a+(k-1)d$ for some constants *a* and *d*. The partial sums are $s_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}\left(u_1+u_n\right)$.

Exercise 9

For what values of *a* and *d* do the partial sums of an arithmetic series converge?

<u>Geometric Series</u>

A geometric series is a series $\sum_{k=1}^{\infty} u_k$ where $u_k = ar^{k-1}$ for some constants *a* and *r*. The partial sums are $s_n = \frac{a\left(r^n-1\right)}{r-1}$.

Exercise 10

For what values of *a* and *r* do the partial sums of a geometric series converge?

Method of Difference

Consider a series $\sum_{k=1}^{\infty} u_k$, where $u_k = \mathrm{f}(k) - \mathrm{f}(k-1)$, i.e. the general term can be written as a difference. Then it is a simple matter to find the partial sums of the series, as

$$\begin{aligned} s_n &= u_1 + u_2 + \dots + u_n \\ &= [\mathrm{f}(1) - \mathrm{f}(2)] + [\mathrm{f}(2) - \mathrm{f}(3)] + \dots + [\mathrm{f}(n-1) - \mathrm{f}(n)] + [\mathrm{f}(n) - \mathrm{f}(n+1)] \\ &= \mathrm{f}(1) - \mathrm{f}(n+1) \end{aligned}$$

since all the middle terms cancel. For this reason, such a sum is also called a *telescoping sum*, as it collapses (like a telescope) into two terms.

Exercise 11

Find a closed form for the n^{th} partial sum of the series

$$\ln\frac{1}{3} + \ln\frac{2}{4} + \ln\frac{3}{5} + \dots + \ln\frac{n}{n+2} \dots$$

and determine if it converges or diverges.

Exercise 12

In this question the notation $\prod_{k=1}^{n} u_k$ denotes the product $u_1 u_2 \dots u_n$. By considering the "partial products" $\prod_{k=2}^{n}\left(1 - \frac{1}{k}\right)$, determine whether the infinite product $\prod_{k=2}^{n}\left(1 - \frac{1}{k}\right)$ converges. Find the limit if it does. Repeat the question for $\prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right)$.

Maclaurin Series

The Maclaurin Series can also be used to evaluate certain series. We just need to be careful about the range of validity. Recall the following standard series expansions which can be found in the MF26:

(a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$ for $|x|<1$

(b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$ for all x

(c) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$ for all x

(d) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$ for all x

(e) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n!} + \ldots$ for $-1 < x \le 1$

Using the expansions above, we can see that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$, $\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$.

2.2 Tests of Convergence

Except for the limited situations where techniques from the previous sections apply, the partial sums of a series are usually very difficult to compute. As such, it is often futile to try to determine if a series converges directly from the definition. In this section we consider a number of observations that might allow us to determine if a series converges without having to compute the partial sums.

The first test is actually a test for divergence, and stems from the following basic result about convergent series.

Theorem 5 If the series $\sum_{k=1}^{\infty} u_k$ converge, then $\lim_{n\to\infty} u_n = 0$.

This is actually one theorem we can easily prove. In fact if the series converge, let the limit be S. Then $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(S_{n+1} - S_n\right) = S - S = 0$.

Remark. Note that this result says nothing about the series if $\lim_{n\to\infty} u_n = 0$. In fact, the above result is most useful in its contrapositive. That is, if $\lim_{n\to\infty} u_n \ne 0$, then the series $\sum_{k=1}^{\infty} u_k$ diverge.

Exercise 13 Using the above result, show that the following series diverge:

(i) $\sum_{k=1}^{\infty}(-1)^k$;

(ii) $\sum_{k=1}^{\infty}\frac{2k+1}{3k-2}$.

Exercise 14 Find a divergent series for which $\lim_{n\to\infty}u_n = 0$ and a convergent series for which $\lim_{n\to\infty}u_n = 0$. This shows that the condition $\lim_{n\to\infty}u_n = 0$ does not determine whether a series is convergent or divergent.

There are many tests to determine if a given series $\sum_{k=1}^{\infty}u_k$ is convergent. Amongst them are the ratio and root tests, which you can easily read about from any text on real analysis. However, for the H3 syllabus, these are not required. If those results are to be used, then they will be given and you just need to apply and check the conditions of convergence accordingly. However, the most basic test of convergence or divergence which you will need to know is the comparison test.

As the name suggests, this allows us to infer the behaviour of a series by comparing it to one whose behaviour we know.

The Comparison Test

Theorem 6 Let $\sum_{k=1}^{\infty}u_k$ and $\sum_{k=1}^{\infty}v_k$ be two series with positive terms, and suppose that $u_k \le v_k$ for all k[1].

(i) If $\sum_{k=1}^{\infty}v_k$ is convergent, then $\sum_{k=1}^{\infty}u_k$ is also convergent.

(ii) If $\sum_{k=1}^{\infty}u_k$ is divergent, then $\sum_{k=1}^{\infty}v_k$ is also divergent.

Note that the result in (i) above, can also be interpreted as a consequence of the Monotone Convergence Theorem. Since the terms of the series $\sum_{k=1}^{\infty}u_k$ are positive, the series is strictly increasing and if $\sum_{k=1}^{\infty}v_k$ is convergent, then we know that $\sum_{k=1}^{\infty}u_k$ is thus bounded above using the condition $u_k \le v_k$.

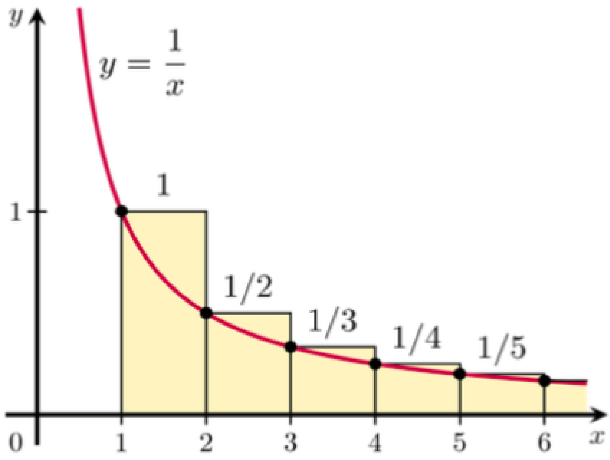
[1] Note that this is not necessary. It suffices that there is a k_0 such that $u_k \le v_k$ for all $k \ge k_0$. The behaviour of the series is determined by its tail end, and not the first finite number of terms.

Exercise 15 Determine if the following series converge or diverge by comparing them with suitable series.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k}$;

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{k-2}{k^2}$.

Another useful form of comparison is the **Integral Test**, where we compare each series with a suitable corresponding integral. This is best illustrated with the following example. Consider the Harmonic Series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$, which we have shown to be divergent in Exercise 8(ii).



From the diagram above, we can see that $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\, \mathrm{d}x = \ln(n+1)$. Since $\ln(n+1)$ diverges, we conclude that the Harmonic Series diverges.

Exercise 16 For any real number *p*, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

is known as the *p*-series. When $p = 1$, we have the Harmonic Series and we have shown (twice!) that the series diverge. Show that the *p*-series converge if $p > 1$ and that the *p*-series diverge if $p \le 1$.

Tutorial

1. [FM/June 1995/I/3(b), modified]

The sequence of numbers $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = \frac{u_n^2 + 4}{u_n + 2},$$

where $u_1 = 1$. By considering $2 - u_{n+1}$ or otherwise, prove by induction that $0 < u_n < 2$, for all positive integers *n*. Hence show that the sequence is strictly increasing and conclude that the sequence converges and find its limit. **[8]**

2. [RI/2014/Lecture Test 1/Q3]

Let $A_0, A_1, A_2, \ldots$ be a sequence of points on the Cartesian plane defined by

- A_0 is the origin;
- A_1 is the point (1, 3);
- for all $n \geq 0$, the point A_{n+2} is the midpoint of the line segment $A_n A_{n+1}$.

For all $n \geq 0$, we denote by a_n the *x*-coordinate of the point A_n.

(i) Show by mathematical induction that for all $n \geq 0$, $a_{n+1} = 1 - \dfrac{a_n}{2}$. **[5]**

(ii) By considering the sequence $v_n = a_n - \dfrac{2}{3}$, or otherwise, show that the sequence $a_0, a_1, a_2, \ldots$ converge and determine the point in the Cartesian plane which the sequence of points $A_0, A_1, A_2, \ldots$ approach. **[3]**

3. [RI/2014/Lecture Test 1/Q4]

The geometric progression U has terms $u_1, u_2, u_3, \ldots,$ with common ratio r, where $|r| < 1$.
It is given that $v_i = u_i^{\,2}$ for $i = 1, 2, 3, \ldots$.

(i) Show that $\displaystyle\sum_{i=1}^{n} v_i = \frac{u_1}{1+r} \sum_{i=1}^{2n} u_i$ **[3]**

It is given further that $w_i = v_i - v_{i+1}$ for $i = 1, 2, 3, \ldots$.

(ii) Show that $\sum_{i=1}^{n} w_i = u_1 (1 - r) \sum_{i=1}^{2n} u_i$. **[4]**

Let $S_U = \sum_{i=1}^{\infty} u_i$, $S_V = \sum_{i=1}^{\infty} v_i$ and $S_W = \sum_{i=1}^{\infty} w_i$. Show that

(iii) $\dfrac{S_U}{S_V} + \dfrac{1}{S_U} = \dfrac{2}{u_1}$, **[3]**

(iv) $S_W = u_1^{\,2}$. **[2]**

4. [RI/2014/Prelims/Q1]

The sequence $x_1, x_2, x_3, ...$ is defined by $x_1 = x_2 = \frac{3}{2}$ and $x_n = \frac{x_{n-1}x_{n-2}+1}{x_{n-1}+x_{n-2}}$, $n \geq 3$. Another sequence $y_1, y_2, y_3, ...$ is defined by $y_n = x_n - 1$, $n \geq 1$.

(i) Show that $y_n = \frac{y_{n-1}y_{n-2}}{y_{n-1}+y_{n-2}+2}$ for $n \geq 3$. **[1]**

(ii) Prove by mathematical induction that $0 < y_n < 1$ for $n \geq 1$. **[5]**

(iii) Using **(i)** and **(ii)**, show that $y_n < \frac{1}{2}y_{n-1}$ for $n \geq 3$. **[1]**

(iv) Hence show that the series $y_1 + y_2 + y_3 + ...$ converges. **[3]**

(v) The proof of the convergence of the sequence $z_n = \sqrt[n]{x_1 x_2 ... x_n}$, $n \geq 1$ is as follows:

Let $n \geq 3$ be a positive integer. Then,

$$1 < \sqrt[n]{x_1 x_2 ... x_n} \qquad (1)$$

$$< \frac{x_1 + x_2 + ... + x_n}{n} \qquad (2)$$

$$< 1 + \frac{3}{2n} \qquad (3)$$

Therefore, the sequence $z_1, z_2, z_3, ...$ converges to 1.

Justify steps (1), (2) and (3) in the proof above. **[3]**

5. [RI/2015/Lecture Test 2/Q5]

For a positive integer n, let

$$S_n = \int_0^1 \frac{1-(-x)^n}{1+x}\, dx \quad \text{and} \quad T_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k(k+1)}.$$

(i) Show that $\left| S_n - \int_0^1 \frac{1}{1+x}\, dx \right| \leq \frac{1}{n+1}$. **[3]**

(ii) By expressing $\frac{1-(-x)^n}{1+x}$ as a geometric series or otherwise, show that $S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$. **[3]**

(iii) Hence or otherwise, show that $T_n - 2S_n = \frac{(-1)^n}{n+1} - 1$ and evaluate $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(k+1)}$, leaving your answer in exact form. **[6]**

6. **[RI/2015/Lecture Test 3/Q4]**

Let $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a bijection. Define the sequence S_n by

$$S_n = \sum_{k=1}^{n} \frac{\sigma(k)}{k^2}.$$

Part of a proof that the sequence $S_1, S_2, S_3, \ldots$ diverges is as follows:

(1) Let n be a positive integer. Then

(2) $S_{2n} - S_n = \sum_{k=n+1}^{2n} \frac{\sigma(k)}{k^2}$

(3) $\qquad \geq \frac{1}{(2n)^2}(1+2+\ldots+n)$

(4) $\qquad > \frac{1}{8}$

(5) Therefore, the sequence $S_1, S_2, S_3, \ldots$ diverges.

Explain the third, fourth and fifth lines of the proof. **[5]**

7. **(a)** A sequence $x_1, x_2, x_3, \ldots$ of nonzero real numbers is such that $x_{n+2} = \frac{x_{n+1}x_n}{2x_n - x_{n+1}}$ for $n \geq 1$. By considering the sequence $y_n = \frac{1}{x_n}$ or otherwise, determine necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n. **[4]**

(b) A sequence $t_0, t_1, t_2, \ldots$ given by $t_0 = a, t_1 = b,$, and $t_{n+1} = \frac{1}{2}\left(t_{n-1} + \frac{1}{t_n}\right)$ is periodic. By considering the sequence $s_n = t_n t_{n-1} - 1$ for positive integers n or otherwise, prove that $ab = 1$. **[4]**

8. **[9824/2013/Q3]**

(i) For $i \geq 2$, show that $\frac{1}{i^4} < \frac{1}{i^3} - \frac{1}{(i+1)^3}$. **[4]**

(ii) Hence for $n \geq 2$, prove that $\sum_{i=n}^{\infty} \frac{1}{i^4} < \frac{1}{n^3}$. **[2]**

(iii) It is given that $\sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{N}$, where N is a positive integer. Find N. **[6]**

9. [9824/2015/Q1]

A sequence $t_1, t_2, t_3, \ldots$ of positive numbers is such that $t_{n+2}^2 = t_n t_{n+1}$, for $n \geq 1$. It is given that the terms of this sequence converge to 1.

(i) Prove that $t_{n+1}\sqrt{t_n} = t_{n+2}\sqrt{t_{n+1}}$, for $n \geq 1$. **[2]**

(ii) Hence explain why $t_{n+1} = \dfrac{1}{\sqrt{t_n}}$, for $n \geq 1$. **[2]**

(iii) Given that $t_1 \neq 1$, find the sum to infinity of the series $\dfrac{\ln t_1}{\ln t_1} + \dfrac{\ln t_2}{\ln t_1} + \dfrac{\ln t_3}{\ln t_1} + \ldots$. **[4]**

10. [RI/9824/2016/Q1]

(i) A sequence $a_1, a_2, a_3, \ldots$ satisfies $a_n = \dfrac{n-1}{n+1} a_{n-1}$ for $n \geq 2$ and $a_1 = 1$. Express a_n in terms of *n*. **[2]**

(ii) A sequence $b_1, b_2, b_3, \ldots$ satisfies $b_n = \dfrac{n^2+n+1}{n^2-n+1} b_{n-1}$ for $n \geq 2$ and $b_1 = 1$.

(a) Explain why the sequence defined by $\dfrac{b_n}{n^2+n+1}$ is constant for $n \geq 1$. **[1]**

(b) Hence express b_n in terms of *n*. **[2]**

(iii) A sequence $c_1, c_2, c_3, \ldots$ satisfies $c_n = \dfrac{n^3-1}{n^3+1} c_{n-1}$ for $n \geq 2$ and $c_1 = k$, where *k* is a real number.

(a) If *k* = 1, express c_n in terms of *n*. **[2]**

(b) Show that if the series $\sum_{m=1}^{n} c_m$ converge, then the sequence $c_1, c_2, c_3, \ldots$ converges to 0. Hence determine all values of *k* such that the series $\sum_{m=1}^{n} c_m$ converge. **[3]**

11. [9824/Specimen Paper]

(i) Use the formula for the sum of an infinite geometric series to prove that the sum of the series

$$\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$

is less than 2.

(ii) Let a and b be positive numbers and let n be a positive integer. Explain why

$$\sum_{i=2^{n-1}+1}^{2^n}\frac{1}{a+ib}\geq\frac{1}{a+2b}.$$

(iii) Part of a proof of divergence in a textbook is as follows.

Let a and b be positive numbers and let n be a positive integer.
Then $\displaystyle\sum_{i=1}^{2^n}\frac{1}{a+ib}\geq\frac{1}{a+b}+\frac{n}{a+2b}$.
Therefore the series $\displaystyle\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\dots$ is divergent.
Explain the second and third lines of the proof.

Assignment

1. (i) The sequence of numbers $u_0,u_1,u_2,\dots$ is given by $u_0=u$ and, for $n\geq 0$,

$$u_{n+1}=4u_n\left(1-u_n\right). \qquad (*)$$

In the case $u=\sin^2\theta$ for some given angle θ, write down and simplify expressions for u_1 and u_2 in terms of θ. Conjecture and expression for u_n and prove your conjecture.

(ii) The sequence of numbers $v_0,v_1,v_2,\dots$ is given by $v_0=v$ and, for $n\geq 0$,

$$v_{n+1}=-pv_n^2+qv_n+r,$$

where p, q and r are given numbers with $p\neq 0$. Show that a substitution of the form $v_n=\alpha u_n+\beta$, where α and β are suitably chosen, results in the sequence (*) provided that $4pr=8+2q-q^2$.

Hence find the general term of the sequence satisfying $v_0=1$ and, for $n\geq 0$, $v_{n+1}=-v_n^2+2v_n+2$.

2. By simplifying $\sin\left(r+\frac{1}{2}\right)x-\sin\left(r-\frac{1}{2}\right)x$ or otherwise show that, for $\sin\frac{x}{2}\neq 0$,

$$\cos x+\cos 2x+\ldots+\cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}.$$

The functions S_n, for $n = 1, 2, \ldots$ are defined by

$$S_n(x)=\sum_{r=1}^{n}\frac{1}{r}\sin rx \quad (0\leq x\leq \pi).$$

(i) Find the stationary points of $S_2(x)$ for $0\leq x\leq \pi$, and sketch this function.

(ii) Show that if $S_n(x)$ has a stationary point at $x=x_0$, where $0<x_0<\pi$, then

$$\sin nx_0 = (1-\cos nx_0)\tan\frac{1}{2}x_0$$

and hence that $S_n(x_0)\geq S_{n-1}(x_0)$. Deduce that if $S_{n-1}(x)>0$ for all in the interval $0<x<\pi$, then $S_n(x)>0$ for all x in this interval.

(iii) Prove that $S_n(x)\geq 0$ for $n\geq 1$ and $0\leq x\leq \pi$.

3. In this question, you may assume that the infinite series

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n+1}\frac{x^n}{n}+\ldots$$

is valid for $|x|<1$.

(i) Let n be an integer greater than 1. Show that for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}}<\frac{1}{kn^k}.$$

Hence show that $\ln\left(1+\frac{1}{n}\right)<\frac{1}{n}$. Deduce that

$$\left(1+\frac{1}{n}\right)^n<\mathrm{e}.$$

(ii) Show, using an expression in powers of $\frac{1}{y}$, that $\ln\left(\frac{2y+1}{2y-1}\right)>\frac{1}{y}$ for $y>\frac{1}{2}$.

Deduce that, for any positive integer n,

$$\mathrm{e}<\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}.$$

(iv) Use parts (i) and (ii) to show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\mathrm{e}.$

Additional Practice Questions

Refer to the compilation of 2010 to 2019 STEP I and II problems. A graphing calculator should not be used for all these questions.

1. **2010/STEP III/1**

2. **2011/STEP I/6**

3. **2011/STEP II/7**

4. **2012/STEP I/7**

5. **2013/STEP II/6**

6. **2015/STEP II/1**

7. **2016/STEP I/8**

8. **2017/STEP I/8**

9. **2017/STEP II/2**

10. **2019/STEP II/5**