

2024 - 2025



CATHOLIC JUNIOR COLLEGE MATHEMATICS DEPARTMENT

GCE 'A' LEVEL TOPICAL (YEAR 2007 – YEAR 2016)

MODULE I FUNCTIONS AND GRAPHS	
Graphing Techniques	– Tested with other topics –
Functions	Page 1
Transformations	Page 4
Conics	Page 6
Inequalities	Page 7
System of Linear Equations	Page 8
MODULE II SEQUENCES AND SERIES	
Sigma Notation and Recurrence Relations	Page 10
Arithmetic & Geometric Progression	Page 12
MODULE III VECTORS	
Vectors (Basic)	Page 15
Vectors (Lines and Planes)	Page 18
MODULE IV CALCULUS (DIFFERENTIATION AND ITS APPLICATIONS)	
Techniques of Differentiation	Page 22
Applications of Differentiation	Page 23
Maclaurin's Series	Page 29
MODULE IV CALCULUS (INTEGRATION AND ITS APPLICATIONS)	
Techniques of Integration	Page 33
Definite Integrals	Page 35
Parametric Equations	Page 39
Differential Equations	Page 43
MODULE V COMPLEX NUMBERS	
Introduction to Complex Numbers	Page 47
MODULE VI STATISTICS	
Permutations and Combinations	Page 49
Probability	Page 51
Discrete Random Variable	– Not in syllabus for Year 2007 – Year 2016 –
Binomial Distribution	Page 56
Normal Distribution	Page 57
Sampling Distribution	– Tested with other topics –
Hypothesis Testing	Page 60
Correlation and Linear Regression	Page 62

NAME: _____
CLASS: _____



Module I Functions and Graphs

Functions

1 [9740/2008/P2/4]

The function f is defined by

$$f: x \mapsto (x-4)^2 + 1 \text{ for } x \in \mathbb{R}, x > 4.$$

- (i) Sketch the graph of $y = f(x)$. Your sketch should indicate the position of the graph in relation to the origin. [2]
- (ii) Find $f^{-1}(x)$, stating the domain of f^{-1} . [3]
- (iii) On the same diagram as in part (i), sketch the graph of $y = f^{-1}(x)$. [1]
- (iv) Write down the equation of the line in which the graph of $y = f(x)$ must be reflected in order to obtain the graph of $y = f^{-1}(x)$, and hence find the exact solution of the equation $f(x) = f^{-1}(x)$. [5]

2 [9740/2009/P2/3]

The function f is defined by

$$f: x \mapsto \frac{ax}{bx-a}, \text{ for } x \in \mathbb{R}, x \neq \frac{a}{b}$$

where a and b are non zero constants.

- (i) Find $f^{-1}(x)$. Hence or otherwise find $f^2(x)$ and state the range of f^2 . [5]
- (ii) The function g is defined by $g: x \mapsto \frac{1}{x}$ for all real non-zero x . State whether the composite function fg exists, justifying your answer. [2]
- (iii) Solve the equation $f^{-1}(x) = x$. [3]

3 [9740/2010/P2/4]

The function f is defined as follows.

$$f: x \mapsto \frac{1}{x^2-1} \text{ for } x \in \mathbb{R}, x \neq -1, x \neq 1.$$

- (i) Sketch the graph of $y = f(x)$. [1]
- (ii) If the domain of f is further restricted to $x \geq k$, state with a reason the least value of k for which the function f^{-1} exists. [2]

In the rest of the question, the domain of f is $x \in \mathbb{R}, x \neq -1, x \neq 1$, as originally defined.

The function g is defined as follows.

$$g: x \mapsto \frac{1}{x-3} \text{ for } x \in \mathbb{R}, x \neq 2, x \neq 3, x \neq 4.$$

- (iii) Show that $fg(x) = \frac{(x-3)^2}{(4-x)(x-2)}$. [2]
- (iv) Solve the inequality $fg(x) > 0$. [3]
- (v) Find the range of fg . [3]



4 [9740/2011/P2/3]

The function f is defined by

$$f: x \mapsto \ln(2x+1) + 3, \quad x \in \mathbb{R}, \quad x > -\frac{1}{2}$$

- (i) Find $f^{-1}(x)$ and write down the domain and range of f^{-1} . [4]
- (ii) Sketch on the same diagram the graphs of $y = f(x)$ and $y = f^{-1}(x)$, giving the equation of any asymptotes and the exact coordinates of any points where the curves cross the x - and y -axes. [4]
- (iii) Explain why the x -coordinates of the points of intersection of the curves in part (ii) satisfy the equation

$$\ln(2x+1) = x - 3$$

and find the values of these x -coordinates, correct to 4 significant figures. [3]

5 [9740/2012/P1/7]

A function is said to be self-inverse if $f(x) = f^{-1}(x)$ for all x values in the domain of f .

The function g is given by

$$g: x \mapsto \frac{x+k}{x-1}, \quad x \in \mathbb{R}, \quad x \neq 1,$$

where k is a constant, $k \neq 1$.

- (i) Show that g is self-inverse. [2]
- (ii) Given that $k > 0$, sketch the curve $y = g(x)$, stating the equations of any asymptotes and the coordinates of any points where the curve crosses the x - and y -axes. [3]
- (iii) State the equation of line of symmetry of the curve in part (ii) and describe fully a sequence of transformations which would transform the curve $y = \frac{1}{x}$ onto this curve. [4]

6 [9740/2013/P2/1]

Functions f and g are defined by

$$f: x \mapsto \frac{2+x}{1-x}, \quad x \in \mathbb{R}, \quad x \neq 1$$

$$g: x \mapsto 1 - 2x, \quad x \in \mathbb{R}.$$

- (i) Explain why the composite function fg does not exist. [2]
- (ii) Find an expression for $gf(x)$ and hence or otherwise, find $(gf)^{-1}(5)$. [4]

7 [9740/2014/P1/1]

The function f is defined by

$$f: x \mapsto \frac{1}{1-x}, \quad x \in \mathbb{R}, \quad x \neq 1, \quad x \neq 0.$$

- (i) Show that $f^2(x) = f^{-1}(x)$. [4]
- (ii) Find $f^3(x)$ in simplified form. [1]



8 [9740/2015/P2/3] Inequalities

(a) The function f is defined by $f: x \mapsto \frac{1}{1-x^2}$, $x \in \mathbb{R}$, $x > 1$.

(i) Show that f has an inverse. [2]

(ii) Find $f^{-1}(x)$ and state the domain of f^{-1} . [3]

(b) The function g is defined by $g: x \mapsto \frac{2+x}{1-x^2}$, $x \in \mathbb{R}$, $x \neq 1$. Find algebraically the range of g , giving your answer in term of $\sqrt{3}$ as simply as possible. [5]

9 [9740/2016/P1/10]

(a) The function f is given by $f: x \mapsto 1 + \sqrt{x}$, for $x \in \mathbb{R}$, $x \geq 0$.

(i) Find $f^{-1}(x)$ and state the domain of f^{-1} . [3]

(ii) Show that if $ff(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$. Hence find the value of x for which $ff(x) = x$. Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$. [5]

(b) The function g , with domain the set of non-negative integers, is given by

$$g(n) = \begin{cases} 1 & \text{for } n = 0, \\ 2 + g\left(\frac{1}{2}n\right) & \text{for } n \text{ even,} \\ 1 + g(n-1) & \text{for } n \text{ odd.} \end{cases}$$

(i) Find $g(4)$, $g(7)$ and $g(12)$ [3]

(ii) Does g have an inverse? Justify your answer. [2]

Answers

1	(ii) $4 + \sqrt{x-1}$; $(1, \infty)$ (iv) $y = x$; $\frac{9 + \sqrt{13}}{2}$	2	(i) $\frac{ax}{bx-a}$; x ; $R_{f^{-1}} = \left(-\infty, \frac{a}{b}\right) \cup \left(\frac{a}{b}, \infty\right)$ (ii) fg does not exist (iii) $x = 0$ or $\frac{2a}{b}$
3	(ii) 0 (iv) $2 < x < 3$ or $3 < x < 4$ (v) $R_{fg} = (-\infty, -1) \cup (0, \infty)$	4	(i) $\frac{e^{x-3} - 1}{2}$; $D_{f^{-1}} = (-\infty, \infty)$; $R_{f^{-1}} = \left(-\frac{1}{2}, \infty\right)$ (iii) -0.4847 and 5.482
5	(iii) $y = x$; translate 1 unit in positive x -direction, scale parallel to y -axis with scale factor $(k+1)$, translate 1 unit in positive y -direction	6	(ii) $\frac{3+3x}{x-1}$; 4
7	(ii) x	8	(a)(ii) $\sqrt{1 - \frac{1}{x}}$; $(-\infty, 0)$ (b) $R_g = \left(-\infty, 1 - \frac{\sqrt{3}}{2}\right] \cup \left[1 + \frac{\sqrt{3}}{2}, \infty\right)$
9	(a)(i) $(x-1)^2$; $[1, \infty)$ (ii) 2.62 (b)(i) 6, 8, 9 (ii) No		



Transformations

1 [9740/2007/P1/5]

Show that the equation $y = \frac{2x+7}{x+2}$ can be written as $y = A + \frac{B}{x+2}$, where A and B are constants

to be found. Hence state a sequence of transformations which transform the graph of $y = \frac{1}{x}$ to

the graph of $y = \frac{2x+7}{x+2}$. [4]

Sketch the graph of $y = \frac{2x+7}{x+2}$, giving the equations of any asymptotes and the coordinates of any points of intersection with the x - and y -axes. [3]

2 [9740/2010/P1/5]

Functions

The curve with equation $y = x^3$ is transformed by a translation of 2 units in the positive x -direction, followed by a stretch with scale factor $\frac{1}{2}$ parallel to the y -axis, followed by a translation of 6 units in the negative y -direction.

(i) Find the equation of the new curve in the form $y = f(x)$ and the exact coordinates of the points where this curve crosses the x - and y -axes. Sketch the new curve. [5]

(ii) On the same diagram, sketch the graph of $y = f^{-1}(x)$, stating the exact coordinates of the points where the graph crosses the x - and y -axes. [3]

3 [9740/2012/P2/3]

It is given that $f(x) = x^3 + x^2 - 2x - 4$.

(i) Sketch the graph of $y = f(x)$. [1]

(ii) Find the integer solution of the equation $f(x) = 4$, and prove algebraically that there are no other real solutions. [3]

(iii) State the integer solution of the equation $(x+3)^3 + (x+3)^2 - 2(x+3) - 4 = 4$. [1]

(iv) Sketch the graph of $y = |f(x)|$. [1]

(v) Write down two different cubic equations which between them give the roots of the equation $|f(x)| = 4$. Hence find all the roots of this equation. [4]



4 [9740/2015/P1/5]

- (i) State a sequence of transformation that will transform the curve with equation $y = x^2$ on to the curve with equation $y = \frac{1}{4}(x-3)^2$. [2]

A curve has equation $y = f(x)$, where

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ \frac{1}{4}(x-3)^2 & \text{for } 1 < x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

- (ii) Sketch the curve for $-1 \leq x \leq 4$. [3]
(iii) On a separate diagram sketch the curve with equation $y = 1 + f\left(\frac{1}{2}x\right)$, for $-1 \leq x \leq 4$. [2]

5 [9740/2016/P1/3] **Differentiation**

The curve $y = x^4$ is transformed onto the curve with equation $y = f(x)$. The turning point on $y = x^4$ corresponds to the point with coordinates (a, b) on $y = f(x)$. The curve $y = f(x)$ also passes through the point with coordinates $(0, c)$. Given that $f(x)$ has the form $k(x-l)^4 + m$ and that a , b and c are positive constants with $c > b$, express k , l and m in terms of a , b and c . [2]

By sketching the curve $y = f(x)$, or otherwise, sketch the curve $y = \frac{1}{f(x)}$. State, in terms of a , b and c , the coordinates of any points where $y = \frac{1}{f(x)}$ crosses the axes and of any turning points. [4]

Answers

1	$A = 2, B = 3$; translate 2 units in negative x -direction, scale parallel to y -axis with scale factor 3, translate 3 units in positive y -direction	2	(i) $y = \frac{1}{2}(x-2)^3 - 6$; $(0, -10)$; $(2 + \sqrt[3]{12}, 0)$
3	(ii) 2 (iii) -1 (v) $x^3 + x^2 - 2x - 8 = 0$, $x^3 + x^2 - 2x = 0$; $x = -2, 0, 1$ and 2	4	(i) translate 3 units in positive x -direction, scale parallel to y -axis with scale factor $\frac{1}{4}$
5	$k = \frac{c-b}{a^4}$, $l = a$, $m = b$; Axial intercepts $\left(0, \frac{1}{c}\right)$, Turning points $\left(a, \frac{1}{b}\right)$		



Conics

1 [9740/2009/P1/6]

The curve C_1 has equation $y = \frac{x-2}{x+2}$. The curve C_2 has equation $\frac{x^2}{6} + \frac{y^2}{3} = 1$.

- (i) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes and the equation of any asymptotes. [4]
- (ii) Show algebraically that the x -coordinates of the points of intersection of C_1 and C_2 satisfy the equation $2(x-2)^2 = (x+2)^2(6-x^2)$. [2]
- (iii) Use your calculator to find these x -coordinates. [2]

Answers

1 (iii) -0.515 and 2.45



Inequalities

1 [9740/2007/P1/1]

Show that $\frac{2x^2 - x - 19}{x^2 + 3x + 2} - 1 = \frac{x^2 - 4x - 21}{x^2 + 3x + 2}$. [1]

Hence, without using a calculator, solve the inequality

$$\frac{2x^2 - x - 19}{x^2 + 3x + 2} > 1. \quad [4]$$

2 [9740/2011/P1/1]

Without using a calculator, solve the inequality

$$\frac{x^2 + x + 1}{x^2 + x - 2} < 0 \quad [4]$$

3 [9740/2013/P1/2]

It is given that $y = \frac{x^2 + x + 1}{x - 1}, x \in \mathbb{R}, x \neq 1$.

Without using a calculator, find the set of values that y can take. [5]

4 [9740/2013/P1/3]

(i) Sketch the curve with equation $y = \frac{x+1}{2x-1}$, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [4]

(ii) Solve the inequality $\frac{x+1}{2x-1} < 1$. [2]

5 [9740/2015/P1/2]

(a) Sketch the curve with equation $y = \left| \frac{x+1}{1-x} \right|$, stating the equations of the asymptotes. On the same diagram sketch the line with equation $y = x + 2$. [3]

(b) Solve the inequality $\left| \frac{x+1}{1-x} \right| < x + 2$. [3]

6 [9740/2016/P1/1]

Express $\frac{4x^2 + 4x - 14}{x - 4} - (x + 3)$ as a single simplified fraction. [2]

Hence, without using a calculator, solve the inequality

$$\frac{4x^2 + 4x - 14}{x - 4} < x + 3. \quad [3]$$

Answers

1	$x < -3$ or $-2 < x < -1$ or $x > 7$	2	$-2 < x < 1$
3	$\{y: y \in \mathbb{R}, y \leq 3 - 2\sqrt{3} \text{ or } y \geq 3 + 2\sqrt{3}\}$	4	(ii) $x < \frac{1}{2}$ or $x > 2$
5	(b) $-1.73 < x < 0.414$ or $x > 1.74$	6	$\frac{3x^2 + 5x - 2}{x - 4}; x < -2$ or $\frac{1}{3} < x < 4$



System of Linear Equations

1 [9740/2007/P2/1]

Four friends buy three different kinds of fruits in the market. When they get home they cannot remember the individual prices per kilogram, but three of them can remember the total amount that they each paid. The weights of fruits and the total amount paid are shown in the following table.

	Suresh	Fandi	Cindy	Lee Lian
Pineapples (kg)	1.15	1.20	2.15	1.30
Mangoes (kg)	0.60	0.45	0.90	0.25
Lychees (kg)	0.55	0.30	0.65	0.50
Total amount paid in \$	8.28	6.84	13.05	

Assuming that, for each variety of fruit, the price per kilogram paid by each of the friends is the same, calculate the total amount that Lee Lian paid. [6]

2 [9740/2009/P1/1]

Sequences and Series

- (i) The first three terms of a sequence are given by $u_1 = 10$, $u_2 = 6$, $u_3 = 5$. Given that u_n is a quadratic polynomial in n , find u_n in terms of n . [4]
- (ii) Find the set of values of n for which u_n is greater than 100. [2]

3 [9740/2012/P1/1]

A cinema sells tickets at three different prices, depending on the age of the customers. The age categories are under 16 years, between 16 and 65 years, and over 65 years. Three groups of people, A , B and C , go to the cinema on the same day. The numbers in each age category for each group, together with the total cost of the tickets for each group, are given in the following table.

Group	Under 16 years	Between 16 and 65 years	Over 65 years	Total cost
A	9	6	4	\$162.03
B	7	5	3	\$128.36
C	10	4	5	\$158.50

Write down and solve equations to find the cost of a ticket for each of the age categories. [4]

4 [9740/2015/P1/1]

Techniques of Differentiation

A curve C has equation

$$y = \frac{a}{x^2} + bx + c$$

where a, b and c are constants. It is given that C passes through the points with coordinates $(1.6, -2.4)$ and $(-0.7, 3.6)$, and that the gradient of C is 2 at the point where $x = 1$.

- (i) Find the values of a, b and c , giving your answers correct to 3 decimal places. [4]
- (ii) Find the x -coordinates of the point where C crosses the x -axis, giving your answer correct to 3 decimal places. [2]
- (iii) One asymptote of C is the line with equation $x = 0$. Write down the equation of the other asymptote of C . [1]



Answers

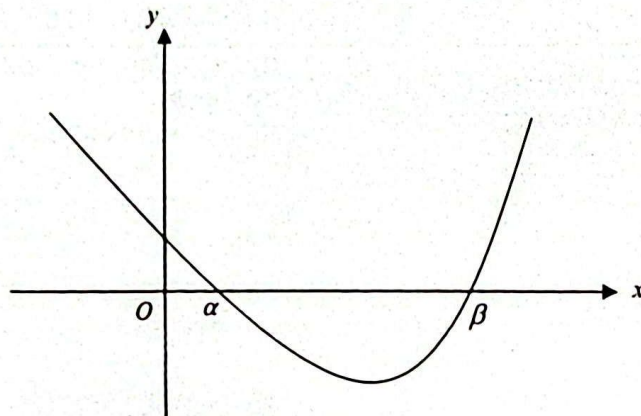
1	\$7.65	2	(i) $u_n = \frac{3}{2}n^2 - \frac{17}{2}n + 17$ (ii) $\{n: n \in \mathbb{Z}, n \geq 11\}$
3	Under 16 years - \$7.65, Bet. 16 and 65 years - \$9.85, Over 65 years - \$8.52	4	(i) $a = -3.593$, $b = -5.187$, $c = 7.303$ (ii) -0.589 (iii) $y = -5.19x + 7.30$



Module II Sequences and Series

Sigma Notation

1 [9740/2007/P1/9]



The diagram shows the graph of $y = e^x - 3x$. The two roots of the equation $e^x - 3x = 0$ are denoted by α and β , where $\alpha < \beta$.

- (i) Find the values of α and β , each correct to 3 decimal places. [2]

A sequence of real numbers $x_1, x_2, x_3, \dots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3}e^{x_n}$$

for $n \geq 1$.

- (ii) Prove algebraically that, if the sequence converges, then it converges to either α or β . [2]

- (iii) Use a calculator to determine the behaviour of the sequence for each of the cases $x_1 = 0$, $x_1 = 1$, $x_1 = 2$. [3]

- (iv) By considering $x_{n+1} - x_n$, prove that

$$\begin{aligned} x_{n+1} &< x_n & \text{if } \alpha < x_n < \beta, \\ x_{n+1} &> x_n & \text{if } x_n < \alpha \text{ or } x_n > \beta. \end{aligned} \quad [2]$$

- (v) State briefly how the results in part (iv) relate to the behaviours determined in part (iii). [2]

2 [9740/2010/P1/3]

The sum, S_n , of the first n terms of a sequence $u_1, u_2, u_3, \dots$ is given by

$$S_n = n(2n + c),$$

where c is a constant.

- (i) Find u_n in terms of c and n . [3]
(ii) Find a recurrence relation of the form $u_{n+1} = f(u_n)$. [2]



3 [9740/2012/P1/3(part of)]

A sequence $u_1, u_2, u_3, \dots$ is given by

$$u_1 = 2 \text{ and } u_{n+1} = \frac{3u_n - 1}{6} \text{ for } n \geq 1.$$

- (i) Find the exact values of u_2 and u_3 . [2]
(ii) It is given that $u_n \rightarrow l$ as $n \rightarrow \infty$. Showing your working, find the exact value of l .

4 [9740/2014/P1/6(part of)]

- (a) Given that $p_n = \frac{1}{3}(7 - 4^n)$ for $n \geq 1$, find $\sum_{r=1}^n p_r$. [3]

- (b) The sum, S_n , of the first n terms of a sequence $u_1, u_2, u_3, \dots$ is given by

$$S_n = 1 - \frac{1}{(n+1)!}.$$

- (i) Give a reason why the series $\sum u_r$ converges, and write down the sum to infinity. [2]
(ii) Find a formula for u_n in simplified form. [2]

Answers

1	(i) $\alpha = 0.619$; $\beta = 1.51$	2	(i) $u_n = 4n - 2 + c$ (ii) $u_{n+1} = u_n + 4$
3	(i) $u_2 = \frac{5}{6}$; $u_3 = \frac{1}{4}$ (ii) $-\frac{1}{3}$	4	(a) $\frac{7}{3}n - \frac{4}{9}(4^n - 1)$ (b)(i) 1 (ii) $\frac{n}{(n+1)!}$



Arithmetic & Geometric Progression

1 [9740/2008/P1/10]

- (i) A student saves \$10 on 1 January 2009. On the first day of each subsequent month she saves \$3 more than in the previous month, so that she saves \$13 on 1 February 2009, \$16 on 1 March 2009, and so on. On what date will she first have saved over \$2000 in total? [5]
- (ii) A second student puts \$10 on 1 January 2009 into a bank account which pays compound interest at a rate of 2% per month on the last day of each month. She puts a further \$10 into the account on the first day of each subsequent month.
- (a) How much compound interest has her original \$10 earned at the end of 2 years? [2]
- (b) How much in total is in the account at the end of 2 years? [3]
- (c) After how many complete months will the total in the account first exceed \$2000? [4]

2 [9740/2009/P1/8]

Two musical instruments, A and B , consist of metal bars of decreasing lengths.

- (i) The first bar of instrument A has length 20 cm and the lengths of the bars form a geometric progression. The 25th bar has length 5 cm. Show that the total length of all the bars must be less than 357 cm, no matter how many bars there are. [4]

Instrument B consists of only 25 bars which are identical to the first 25 bars of instrument A .

- (ii) Find the total length, L cm, of all the bars of instrument B and the length of the 13th bar. [3]
- (iii) Unfortunately the manufacturer misunderstands the instructions and constructs instrument B wrongly, so that the lengths of the bars are in arithmetic progression with common difference d cm. If the total length of the 25 bars is still L cm and the length of the 25th bar is still 5 cm, find the value of d and the length of the longest bar. [4]

3 [9740/2011/P1/9]

- (i) A company is drilling for oil. Using machine A , the depth drilled on the first day is 256 metres. On each subsequent day, the depth drilled is 7 metres less than the previous day. Drilling continues daily up to and including the day when a depth of less than 10 metres is drilled. What depth is drilled on the 10th day, what is the total depth when drilling is completed? [6]
- (ii) Using machine B , the depth drilled on the first day is also 256 metres. On each subsequent day, the depth drilled is $\frac{8}{9}$ of the depth on the previous day. How many days does it take for the depth drilled to exceed 99% of the theoretical maximum total depth? [4]

4 [9740/2012/P2/4]

On 1 January 2001 Mrs A put \$100 into a bank account, and on the first day of each subsequent month she put in \$10 more than in the previous month. Thus on 1 February she put \$110 into the account and on 1 March she put \$120 into the account, and so on. The account pays no interest.

- (i) On what date did the value of Mrs A 's account become greater than \$5000? [5]

On 1 January 2001 Mr B put \$100 into a savings account, and on the first day of each subsequent month he put another \$100 into the account. The interest rate was 0.5% per month, so that on the last day of each month the amount in the account on that day was increased by 0.5%.

- (ii) Use the formula for sum of a geometric progression to find an expression for the value of Mr B 's account on the last day of the n th month (where January 2001 was the 1st month, February 2001 was the 2nd month, and so on). Hence find in which month the value of Mr B 's account first became greater than \$5000. [5]
- (iii) Mr B wanted the value of his account to be \$5000 on 2 December 2003. What interest rate per month, applied from January 2001, would achieve this? [3]



5 [9740/2013/P1/7]

A gardener is cutting off pieces of string from a long roll of string. The first piece he cuts off is 128 cm long and each successive piece is $\frac{2}{3}$ as long as the preceding piece.

(i) The length of n th piece of string cut off is p cm. Show that

$$\ln p = (An + B)\ln 2 + (Cn + D)\ln 3,$$

for constants A , B , C and D to be determined. [3]

(ii) Show that the total length of the string cut off can never be greater than 384 cm. [2]

(iii) How many pieces must be cut off before the total length cut off is greater than 380 cm? You must show sufficient working to justify your answer. [4]

4 [9740/2014/P2/3]

In a training exercise, athletes run from a starting point O to and from a series of points $A_1, A_2, A_3, \dots$, increasingly far away in a straight line. In the exercise, athletes starts at O and run stage 1 from O to A_1 and back to O , then stage 2 from O to A_2 and back to O , and so on.

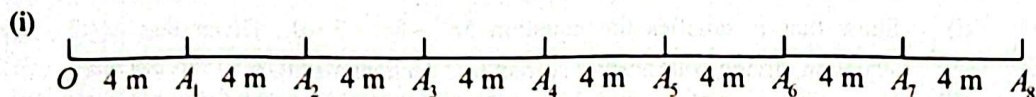


Fig. 1

In Version 1 of the exercise, the distance between adjacent points are all 4 m (see Fig. 1).

(a) Find the distance run by an athlete who completes the first 10 stages of Version 1 of the exercise. [2]

(b) Write down an expression for the distance run by an athlete who completed n stages of Version 1. Hence find the least number of stages that the athlete needs to complete to run at least 5 km. [4]

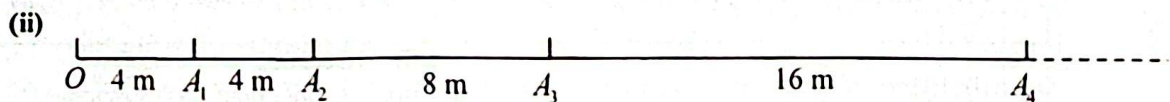


Fig. 2

In version 2 of the exercise, the distance between the points are such that $OA_1 = 4\text{m}$, $A_1A_2 = 4\text{m}$, $A_2A_3 = 8\text{m}$ and $A_nA_{n+1} = 2A_{n-1}A_n$ (see Fig. 2). Write down an expression for the distance run by an athlete who completes n stages for Version 2. Hence find the distance from O , and the direction of travel, of the athlete after he has run exactly 10 km using Version 2. [5]



5 [9740/2015/P1/8]

Two athletes are to run 20 km by running 50 laps around a circular track of 400 m. They aim to complete the distance in between $1\frac{1}{2}$ hours and $1\frac{1}{4}$ hours inclusive.

- Athlete A runs the first lap in T seconds and each subsequent lap takes 2 seconds longer than the previous lap. Find the set of values of T that will enable A to complete the distance within the required time interval. [4]
- Athlete B runs the first lap in t seconds and the time for each subsequent lap is 2% more than the time for the previous lap. Find the set of values of t which will enable B to complete the distance within the required time interval. [4]
- Assuming that each athlete completes the 20 km run exactly in $1\frac{1}{2}$ hours, find the difference in the athletes' times for their 50th laps, giving your answer to the nearest second. [3]

6 [9740/2016/P1/4]

An arithmetic series has first term a and common difference d , where a and d are non-zero. A geometric series has first term b and common ratio r , where b and r are non-zero. It is given that the 4th, 9th and 12th terms of the arithmetic series are equal to the 5th, 8th and 15th terms of the geometric series respectively.

- Show that r satisfies the equation $5r^{10} - 8r^3 + 3 = 0$. Given that $|r| < 1$, solve this equation, giving your answer correct to 2 decimal places. [4]
- Using this value of r , find, in terms of b and n , the sum of the terms of the geometric series after, but not including, the n th term, simplifying your answer. [3]

Answers

1	(i) 1Oct2011 (ii)(a) \$6.08 (b) \$310.30 (c) 81	2	(ii) 272cm (iii) -0.491; 16.8cm
3	(i) 193m (ii) 4810m; 40m	4	(i) 1Dec2002 (ii) $20100(1.005^n - 1)$; Sep2004 (iii) 1.80%
5	(i) $A=1, B=6, C=-1, D=1$ (iii) 12	6	(i)(a) 440m (b) 35 (ii) 1816m from O , running away from O
7	(i) $\{T: T \in \mathbb{R}, 59 \leq t \leq 77\}$ (ii) $\{t: t \in \mathbb{R}, 63.9 \leq t \leq 74.4\}$ (iii) 11s	8	(i) 0.74 (ii) $\frac{(0.74^n)b}{0.26}$



Module III Vectors

Vectors (Basic)

1 [9740/2007/P1/6]

Referred to the origin O , the position vectors of points A and B are $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ respectively.

- (i) Show that OA is perpendicular to OB . [2]
- (ii) Find the position vector of the point M on the line segment AB such that $AM : MB = 1 : 2$. [3]
- (iii) The point C has position vector $-4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Use a vector product to find the exact area of triangle OAC . [4]

2 [9740/2008/P1/3]

Points O, A, B are such that $\overrightarrow{OA} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{OB} = 5\mathbf{i} - \mathbf{j}$, and the point P is such that $OAPB$ is a parallelogram.

- (i) Find $\overrightarrow{OP}$. [1]
- (ii) Find the size of angle AOB . [3]
- (iii) Find the exact area of the parallelogram $OAPB$. [2]

3 [9740/2009/P2/2]

Relative to the origin O , two points A and B have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

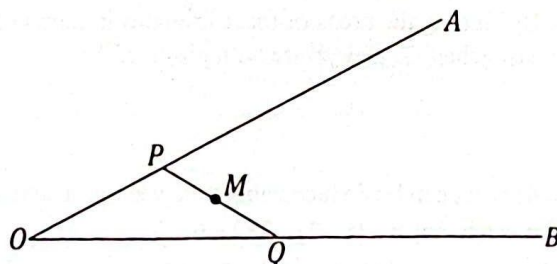
- (i) The point P divides the line AB in the ratio $2 : 1$. Find the coordinates of P . [2]
- (ii) Show that AB and OP are perpendicular. [2]
- (iii) The vector $\mathbf{c}$ is a unit vector in the direction of $\overrightarrow{OP}$. Write $\mathbf{c}$ as a column vector, and give the geometric meaning of $|\mathbf{a} \cdot \mathbf{c}|$. [2]
- (iv) Find $\mathbf{a} \times \mathbf{p}$, where $\mathbf{p}$ is the vector $\overrightarrow{OP}$, and give the geometric meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence write down the area of triangle OAP . [4]

4 [9740/2010/P1/1]

The position vectors $\mathbf{a}$ and $\mathbf{b}$ are given by $\mathbf{a} = 2p\mathbf{i} + 3p\mathbf{j} + 6p\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ where $p > 0$. It is given that $|\mathbf{a}| = |\mathbf{b}|$.

- (i) Find the exact value of p . [2]
- (ii) Show that $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$. [3]

5 [9740/2011/P1/7]





Referred to the origin O , the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on OA is such that $OP:PA = 1:2$, and the point Q on OB is such that $OQ:QB = 3:2$.

The mid-point of PQ is M (see diagram).

- (i) Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and show that area of triangle OPM can be written as $k|\mathbf{a} \times \mathbf{b}|$, where k is a constant to be found. [6]

- (ii) The vectors $\mathbf{a}$ and $\mathbf{b}$ are now given by

$$\mathbf{a} = 2p\mathbf{i} - 6p\mathbf{j} + 3p\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

where p is a positive constant. Given that $\mathbf{a}$ is a unit vector,

- (a) find the exact value of p , [2]

- (b) Give a geometrical meaning of $|\mathbf{a} \cdot \mathbf{b}|$, [1]

- (c) evaluate $\mathbf{a} \times \mathbf{b}$. [2]

6 [9740/2012/P1/5]

Referred to the origin O , the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ such that

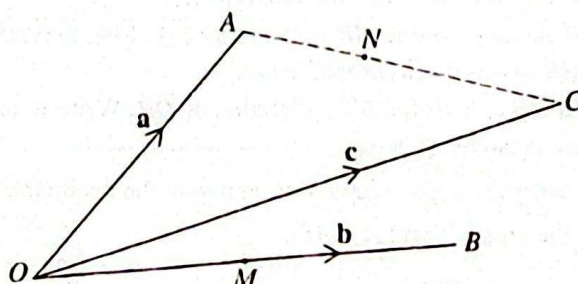
$$\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} + 2\mathbf{j}.$$

The point C has position vector $\mathbf{c}$ given by $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, where λ and μ are positive constants.

- (i) Given that the area of OAC is $\sqrt{126}$, find μ . [4]

- (ii) Given instead that $\mu = 4$ and $OC = 5\sqrt{3}$, find the possible coordinates of C . [4]

7 [9740/2013/P1/6]



The origin O and the points A , B and C lies in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$ (see diagram).

- (i) Explain why $\mathbf{c}$ can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ . [1]

The point N is on AC such that $AN:NC = 3:4$.

- (ii) Write down the position vector of N in terms of $\mathbf{a}$ and $\mathbf{c}$. [1]

- (iii) It is given that the area of ONC is equal to the area of triangle OMC , where M is the mid-point of OB . By finding the areas of these triangles in terms of $\mathbf{a}$ and $\mathbf{b}$, find λ in terms of μ in the case where λ and μ are both positive. [5]

8 [9740/2014/P1/3]

- (i) Given that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, what can be deduced about the vectors $\mathbf{a}$ and $\mathbf{b}$? [2]

- (ii) Find a unit vector $\mathbf{n}$ such that $\mathbf{n} \times (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \mathbf{0}$. [2]

- (iii) Find the cosine of the acute angle between $\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and the z -axis. [1]



9 [9740/2016/P1/5]

The vectors $\mathbf{u}$ and $\mathbf{v}$ are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$, where a and b are constants.

- (i) Find $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a and b . [2]
 (ii) Given that the $\mathbf{i}$ - and $\mathbf{k}$ -components of the answer to part (i) are equal, express $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ in terms of a only. Hence find, in exact form, the possible values of a for which $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$ is a unit vector. [4]
 (iii) Given instead that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$, find the numerical value of $|\mathbf{v}|$. [2]

Answers

1	(ii) $\frac{1}{3} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$ (iii) $\sqrt{35}$	2	(i) $\begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$ (ii) 87.8° (iii) $15\sqrt{3} \text{ units}^2$
3	(i) $(12, -4, 6)$ (iii) $\frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$; length of projection of $\overrightarrow{OA}$ on $\overrightarrow{OP}$ (iv) area of parallelogram formed by vectors $\overrightarrow{OA}$ on $\overrightarrow{OP}$; $98\sqrt{2} \text{ units}^2$	4	(i) $\frac{3}{7}$
5	(i) $\frac{1}{6}a + \frac{3}{10}b$, $k = \frac{1}{20}$ (ii)(a) $\frac{1}{7}$ (b) length of projection of $\underline{b}$ on $\underline{a}$ (c) $\frac{1}{7} \begin{pmatrix} 9 \\ 7 \\ 8 \end{pmatrix}$	6	(i) 6 (ii) $\left(\frac{17}{3}, \frac{19}{3}, \frac{5}{3}\right), (5, 7, 1)$
7	(ii) $\frac{1}{7}(3c + 4a)$ (iii) $\lambda = \frac{8}{7}\mu$	8	(i) $\underline{a}$ and $\underline{b}$ are parallel or either $\underline{a}$ or $\underline{b}$ is $\underline{0}$ (ii) $\frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ or $-\frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ (iii) $\frac{2}{3}$
9	(i) $\begin{pmatrix} 2b \\ 4b - 4a \\ -2a \end{pmatrix}$ (ii) $\begin{pmatrix} -2a \\ -8a \\ -2a \end{pmatrix}$ (iii) $\pm \frac{1}{6\sqrt{2}}$		



Vectors (Lines & Planes)

1 [9740/2007/P1/8]

The line l passes through the points A and B with coordinates $(1, 2, 4)$ and $(-2, 3, 1)$ respectively. The plane p has equation $3x - y + 2z = 17$. Find

- (i) the coordinates of the point of intersection of l and p , [5]
- (ii) the acute angle between l and p , [3]
- (iii) the perpendicular distance from A to p . [3]

2 [9740/2008/P1/11(modified)(part of)]

The equations of two planes p_1 and p_2 are

$$2x - 5y + 3z = 3,$$

$$3x + 2y - 5z = -5,$$

respectively.

The plane p_1 and p_2 intersect in a line l .

- (i) Find a vector equation of l . [2]
- (ii) Find the Cartesian equation of the plane which contains l and the point $(1, -1, 3)$. [4]

3 [9740/2009/P1/10]

The planes p_1 and p_2 have equations $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 1$ and $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 2$ respectively, and meet in a line

l .

- (i) Find the acute angle between p_1 and p_2 . [3]
- (ii) Find a vector equation of l . [4]
- (iii) The plane p_3 has equation $2x + y + 3z - 1 + k(-x + 2y + z - 2) = 0$. Explain why l lies in p_3 for any constant k . Hence, or otherwise, find a cartesian equation of the plane in which both l and the point $(2, 3, 4)$ lie. [5]

4 [9740/2010/P1/10]

The line l has equation $\frac{x-10}{-3} = \frac{y+1}{6} = \frac{z+3}{9}$, and the plane p has equation $x - 2y - 3z = 0$.

- (i) Show that l is perpendicular to p . [2]
- (ii) Find the coordinates of the point of intersection of l and p . [4]
- (iii) Show that the point A with coordinates $(-2, 23, 33)$ lies on l . Find the coordinates of the point B which is the mirror image of A in p . [3]
- (iv) Find the area of triangle OAB where O is the origin, giving your answer to the nearest whole number. [3]



5 [9740/2011/P1/11]

The plane p passes through the points with coordinates $(4, -1, -3)$, $(-2, -5, 2)$ and $(4, -3, -2)$.

- (i) Find the cartesian equation of p . [4]

The line l_1 has equation $\frac{x-1}{2} = \frac{y-2}{-4} = \frac{z+3}{1}$ and the line l_2 has equation $\frac{x+2}{1} = \frac{y-1}{5} = \frac{z-3}{k}$, where k is a constant. It is given that l_1 and l_2 intersect.

- (ii) Find the value of k . [4]

- (iii) Show that l_1 lies in p and find the coordinates of the point at which l_2 intersects p . [4]

- (iv) Find the acute angle between l_2 and p . [3]

6 [9740/2012/P1/9]

- (i) Find a vector equation of the line through the points A and B with position vectors $7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$ and $-\mathbf{i} - 8\mathbf{j} + \mathbf{k}$ respectively. [3]

- (ii) The perpendicular to this line from the point C with position vector $\mathbf{i} + 8\mathbf{j} + 3\mathbf{k}$ meets the line at the point N . Find the position vector of N and the ratio of $AN : NB$. [5]

- (iii) Find a cartesian equation of the line which is a reflection of the line AC in the line AB . [4]

7 [9740/2014/P1/9]

Planes p and q are perpendicular. Plane p has equation $x + 2y - 3z = 12$. Plane q contains

line l with equation $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$. The point A on l has coordinates $(1, -1, 3)$.

- (i) Find a cartesian equation of q . [4]

- (ii) Find a vector equation of the line m where p and q meet. [4]

- (iii) B is a general point on m . Find an expression for the square of the distance AB . Hence, or otherwise, find the coordinates of the point on m which is nearest to A . [5]

8 [9740/2015/P1/7]

Referred to the origin O , points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA , between O and A , such that $OC : CA = 3 : 2$. Point D lies on OB , between O and B , such that $OD : DB = 5 : 6$.

- (i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

- (ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, where λ is a parameter. Find in a similar form the vector equation of the line AD in terms of parameter μ . [3]

- (iii) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet and find the ratio $AE : ED$. [5]



9 [9740/2015/P2/2]

The line L equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$.

(i) Find the acute angle between L and the x -axis. [2]

The point P has position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.

(ii) Find the points on L which are a distance of $\sqrt{33}$ from P . Hence or otherwise find the point on L which is closest to P . [5]

(iii) Find a Cartesian equation of the plane that includes the line L and the point P . [3]

10 [9740/2016/P1/11]

The plane p has equation $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} a \\ 4 \\ -2 \end{pmatrix}$ and the line l has equation

$\mathbf{r} = \begin{pmatrix} a-1 \\ a \\ a+1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, where a is a constant and λ , μ and t are parameters.

(i) In the case where $a = 0$,

(a) show that l is perpendicular to p and find the values of λ , μ and t which give the coordinates of the point at which l and p intersect, [5]

(b) find the cartesian equations of the planes such that the perpendicular distance from each plane to p is 12. [5]

(ii) Find the value of a such that l and p do not meet in a unique point. [3]



Answers

1	(i) $\left(\frac{5}{2}, \frac{3}{2}, \frac{11}{2}\right)$ (ii) 78.8° (iii) $\frac{8}{\sqrt{14}}$	2	(i) $r = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ (ii) $3x - y - 2z = -2$
3	(i) 70.9° (ii) $r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ (iii) $x - y = -1$	4	(ii) $\left(\frac{17}{2}, 2, \frac{3}{2}\right)$ (iii) $(19, -19, -30)$ (iv) 348 units^2
5	(i) $x + y + 2z = -3$ (ii) -7 (iii) $(-1, 6, -4)$ (iv) 22.2°	6	(i) $r = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ (ii) $\begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix}; 1:3$ (iii) $x - 7 = \frac{8 - y}{4} = z - 9$
7	(i) $-x + 2y + z = 0$ (ii) $r = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$ (iii) $\left(\frac{18}{7}, \frac{15}{7}, -\frac{12}{7}\right)$	8	(i) $\overline{OC} = \frac{3}{5}a, \overline{OD} = \frac{5}{11}b$ (ii) $r = \frac{5}{11}\mu b + (1 - \mu)a, \mu \in \mathbb{R}$ (iii) $11:9$
9	(i) 73.4° (ii) $\begin{pmatrix} 3 \\ 1 \\ -10 \end{pmatrix}, \begin{pmatrix} \frac{13}{7} \\ -\frac{5}{7} \\ -\frac{46}{7} \end{pmatrix}; \begin{pmatrix} \frac{17}{7} \\ \frac{1}{7} \\ -\frac{58}{7} \end{pmatrix}$ (iii) $36x - 2y + 11z = -4$	10	(i)(a) $\lambda = -\frac{8}{9}, \mu = \frac{19}{8}, t = -\frac{5}{9}$ (b) $-2x + y + 2z = 3, -2x + y + 2z = 37$ (ii) $\frac{9}{2}$



Module IV Calculus (Differentiation and its Applications)

Techniques of Differentiation

1 [9740/2011/P1/2] System of Linear Equations

It is given that $f(x) = ax^2 + bx + c$, where a, b and c are constants.

- (i) Given that the curve with equation $y = f(x)$ passes through the points with coordinates $(-1.5, 4.5)$, $(2.1, 3.2)$ and $(3.4, 4.1)$, find the values of a, b and c . Give your answers correct to 3 decimal places. [3]
- (ii) Find the set of values of x for which $f(x)$ is an increasing function. [2]

2 [9740/2014/P1/2]

The curve C has equation $x^2y + xy^2 + 54 = 0$. Without using a calculator, find the coordinates of the point on C at which the gradient is -1 , showing that there is only one such point. [6]

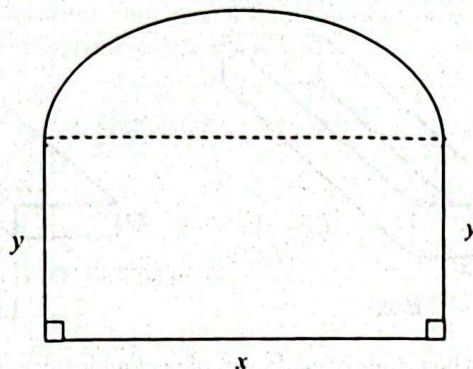
Answers

1	(i) $a = 0.215$, $b = -0.490$, $c = 3.281$ (ii) $\{x: x \in \mathbb{R}, x > 1.14\}$	2	$(-3, -3)$
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Applications of Differentiation

1 [9740/2008/P1/7]



A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flower-bed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area. [10]

2 [9740/2008/P1/9(part of)(modified)]

It is given that

$$f(x) = \frac{ax+b}{cx+d},$$

for non-zero constants a , b , c and d .

- (i) Given that $ad - bc \neq 0$, show by differentiation that the graph of $y = f(x)$ has no turning points. [3]
- (ii) What can be said about the graph of $y = f(x)$ when $ad - bc = 0$? [2]
- (iii) Deduce from part (i) that the graph of

$$y = \frac{3x-7}{2x+1}$$

has a positive gradient at all points of the graph. [1]

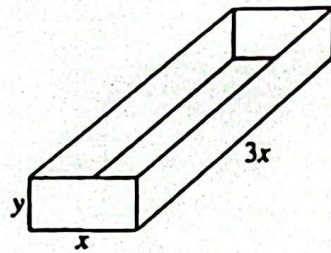
- (iv) Sketch the graph of $y = \frac{3x-7}{2x+1}$ including the coordinates of the points where the graphs cross the axes and the equations of any asymptotes. [2]

3 [9740/2010/P1/4]

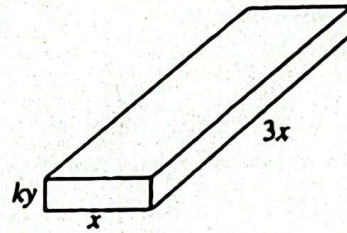
- (i) Given that $x^2 - y^2 + 2xy + 4 = 0$, find $\frac{dy}{dx}$ in terms of x and y . [4]
- (ii) For the curve with equation $x^2 - y^2 + 2xy + 4 = 0$, find the coordinates of each point at which the tangent is parallel to the x -axis. [4]



4 [9740/2010/P1/9]



Box



Lid

A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The length of the box is $3x \text{ cm}$, the width is $x \text{ cm}$ and the height is $y \text{ cm}$. The lid has depth $ky \text{ cm}$, where $0 < k \leq 1$ (see diagram).

- Use differentiation to find, in terms of k , the value of x which gives a minimum total external surface area of the box and the lid. [6]
- Find also the ratio of the height to the width, $\frac{y}{x}$, in this case, simplifying your answer. [2]
- Find the values between which $\frac{y}{x}$ must lie. [2]
- Find the value of k for which the box has square ends. [2]

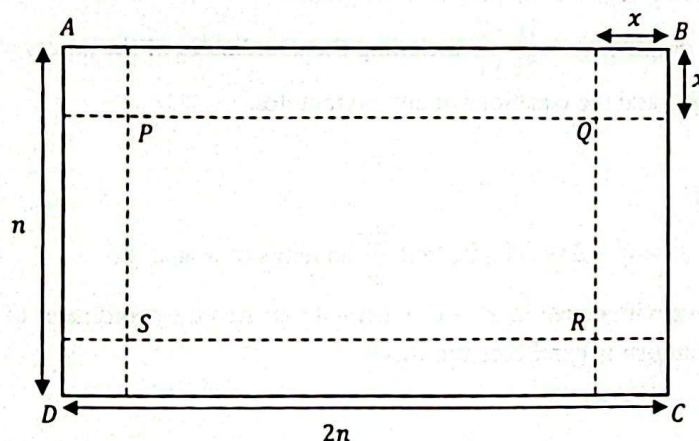
5 [9740/2010/P2/3(part of)(modified)]

- Given that $y = x\sqrt{x+2}$, find $\frac{dy}{dx}$, expressing your answer as a single algebraic fraction.

Hence show that there is only one value of x for which the curve $y = x\sqrt{x+2}$ has a turning point, and state this value. [5]

- Sketch the graph of $y = f'(x)$, where $f(x) = x\sqrt{x+2}$. State the equations of any asymptotes. [2]

6 [9740/2011/P2/2]



The diagram shows a rectangular piece of cardboard $ABCD$ of sides n metres and $2n$ metres, where n is a positive constant. A square side x metres is removed from each corner of $ABCD$. The remaining shape is now folded along PQ, QR, RS and SP to form an open rectangle box of height x metres.



- (i) Show that the volume V cubic metres of the box is given by $V = 2n^2x - 6nx^2 + 4x^3$. [3]
(ii) Without using a calculator, find in surd form the value of x that gives a stationary value of V , and explain why there is only one value. [6]

7 [9740/2012/P1/8]

The curve C has equation

$$x - y = (x + y)^2.$$

It is given that C has only one turning point.

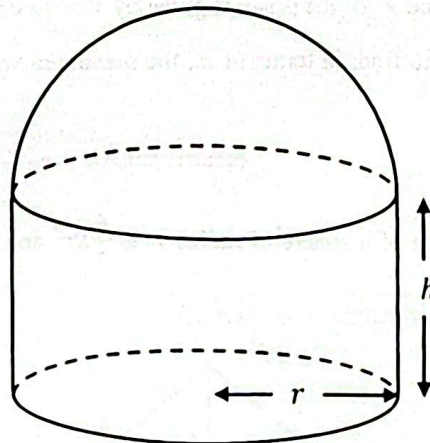
(i) Show that $1 + \frac{dy}{dx} = \frac{2}{2x + 2y + 1}$. [4]

(ii) Hence, or otherwise, show that $\frac{d^2y}{dx^2} = -\left(1 + \frac{dy}{dx}\right)^3$. [3]

(iii) Hence state, with a reason, whether the turning point is a maximum or a minimum. [2]

8 [9740/2012/P1/10]

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A model of a concert hall is made up of three parts.

- The roof is modelled by the curved surface of a hemisphere of radius r cm.
- The walls are modelled by the curved surface of a cylinder of radius r cm and height h cm.
- The floor is modelled by a circular disc of radius r cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

- (i) It is given that the volume of the model is a fixed value $k \text{ cm}^3$, and the external surface area is a minimum. Use differentiation to find the values of r and h in terms of k . Simplify your answers. [7]
(ii) It is given instead that the volume of the model is 200 cm^3 and its external surface area is 180cm^2 . Show that there are two possible values of r . Given also that $r < h$, find the value of r and the value of h . [5]



9 [9740/2013/P2/2]

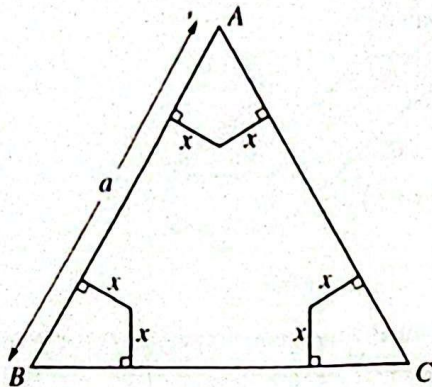


Fig. 1

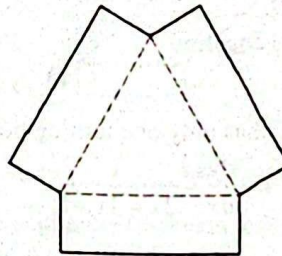


Fig. 2

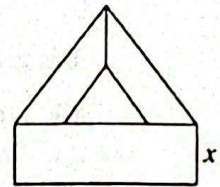


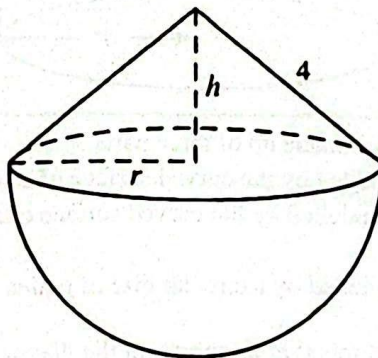
Fig. 3

Fig. 1 shows a piece of card, ABC , in the form of an equilateral triangle of side a . A kite shape is cut from each corner, to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines, to form the open triangular prism of height x shown in Fig. 3.

- (i) Show that the volume V of the prism is given by $V = \frac{1}{4}x\sqrt{3}(a - 2x\sqrt{3})^2$. [3]
- (ii) Use differentiation to find, in terms of a , the maximum value of V , proving that it is a maximum. [6]

10 [9740/2014/P1/11]

[It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A toy manufacturer makes a toy which consists of a hemisphere of radius r cm joined to a circular cone of base radius r cm and height h cm (see diagram). The manufacturer determines that the length of slant edge of the cone must be 4 cm and that the total volume of the toy, V cm³, should be as large as possible.

- (i) Find a formula for V in terms of r . Given that $r = r_1$ is the value of r which gives the maximum value of V , show that r_1 satisfies the equation $45r^4 - 768r^2 + 1024 = 0$. [6]



- (ii) Find the two solutions to the equation in part (i) for which $r > 0$, giving your answers correct to 3 decimal places. [2]
- (iii) Show that one of the solutions found in part (ii) does not give a stationary value of V . Hence write down the value of r_1 and find the corresponding value of h . [3]
- (iv) Sketch the graph showing the volume of the toy as the radius of the hemisphere varies. [3]

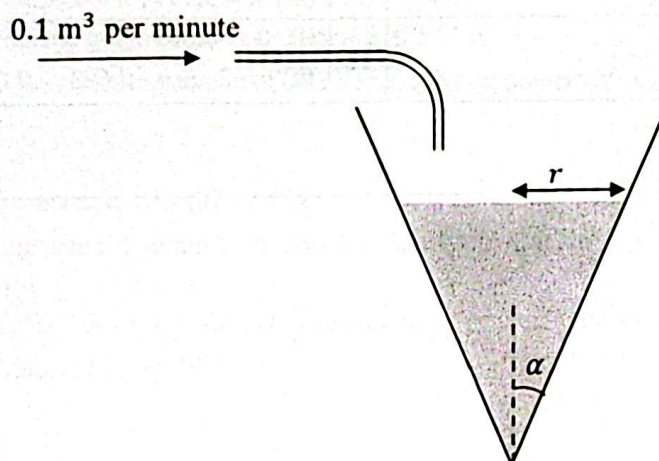
11 [9740/2015/P1/4]

A piece of wire of fixed length d m is cut into two parts. One part is bent into the shape of a rectangle with sides x m and y m. The other part is bent into the shape of a semicircle, including its diameter. The radius of the semicircle is x m. Show that the maximum value of the total area of the two shapes can be expressed as $kd^2 \text{ m}^2$, where k is a constant to be found. [6]

12 [9740/2016/P1/2]

- (i) Use your calculator to find the gradient of the curve $y = 2^{\cos x}$ at the points where $x = 0$ and $x = \frac{1}{2}\pi$. [2]
- (ii) Find the equations of the tangents to this curve at the points where $x = 0$ and $x = \frac{1}{2}\pi$ and find the coordinates of the points where these tangents meet. [3]

13 [9740/2016/P2/1]



Water is poured at a rate of 0.1 m^3 per minute into a container in the form of an open cone. The semi-vertical angle of the cone is α , where $\tan \alpha = 0.5$. At time t minutes after the start, the radius of the water surface is r m (see diagram). Find the rate of increase of the depth of water when the volume of water in the container is 3 m^3 . [7]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$]



Answers

1	$x = 6.09 \text{ m}, y = 12.6 \text{ m}$	2	(ii) f is a horizontal line parallel to x -axis
3	(i) $\frac{x+y}{y-x}$ (ii) $(-\sqrt{2}, \sqrt{2})(\sqrt{2}, -\sqrt{2})$	4	(i) $x = \sqrt[3]{\frac{200}{3}}(1+k)$ (ii) $\frac{3}{2(1+k)}$ (iii) $\frac{3}{4} \leq \frac{3}{2(1+k)} < \frac{3}{2}$ (iv) $\frac{1}{2}$
5	(i) $\frac{3x+4}{\sqrt{x+2}}, -\frac{4}{3}$ (ii) $x = -2$	6	(ii) $\frac{3n - \sqrt{3}n}{6}$
7	(iii) maximum	8	(i) $r = \sqrt[3]{\frac{3k}{5\pi}}, h = \sqrt[3]{\frac{3k}{5\pi}}$ (ii) $r = 3.04, h = 4.88$
9	(ii) $\frac{a^3}{54}$	10	(i) $V = \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 \sqrt{16 - r^2}$ (ii) $r = 1.207$ or 3.951 (iii) $r_1 = 3.951, h = 0.625$
11	$k = \frac{1}{32}$	12	(i) $0, -0.693$ (ii) $y = 2, y = -0.693x + 2.09; (0.128, 2)$
13	$\frac{2}{5\pi} \left(\frac{\pi}{36} \right)^{\frac{1}{3}}$		



Maclaurin's Series

1 [9740/2007/P2/3]

- (i) By successive differentiating $(1+x)^n$, find Maclaurin's series for $(1+x)^n$, up to and including the term in x^3 . [4]
- (ii) Obtain the expansion of $(4-x)^{\frac{1}{2}}(1+2x^2)^{\frac{1}{2}}$ up to and including the term in x^3 . [5]
- (iii) Find the set of values of x for which the expansion in part (ii) is valid. [2]

2 [9740/2008/P1/6]

- (a) In the triangle ABC , $AB=1$, $BC=3$ and angle $ABC=\theta$ radians. Given that θ is a sufficiently small angle, show that

$$AC \approx (4+3\theta^2)^{\frac{1}{2}} \approx a+b\theta^2,$$

for constants a and b to be determined. [5]

- (b) Given that $f(x) = \tan\left(2x + \frac{1}{4}\pi\right)$, find $f(0)$, $f'(0)$ and $f''(0)$. Hence find the first 3 terms in the Maclaurin series of $f(x)$. [5]

3 [9740/2008/P2/1]

Let $f(x) = e^x \sin x$.

- (i) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$. [2]
- (ii) Find the series expansion of $f(x)$ in ascending powers of x , up to and including the term in x^3 . [3]

Denote the answer to part (ii) by $g(x)$.

- (iii) On the same diagram as in part (i), sketch the graph of $y = g(x)$. Label the two graphs clearly. [1]
- (iv) Find, for $-3 \leq x \leq 3$, the set of values of x for which the value of $g(x)$ is within ± 0.5 of the value of $f(x)$. [3]

4 [9740/2009/P1/7]

- (i) Given that $f(x) = e^{\cos x}$, find $f(0)$, $f'(0)$, $f''(0)$. Hence write down the first two non-zero terms in the Maclaurin series for $f(x)$. Give the coefficients in terms of e . [5]
- (ii) Given that the first two non-zero terms in the Maclaurin series for $f(x)$ are equal to the first two non-zero terms in the series expansion of $\frac{1}{a+bx^2}$, where a and b are constants, find a and b in terms of e . [4]



5 [9740/2010/P1/2]

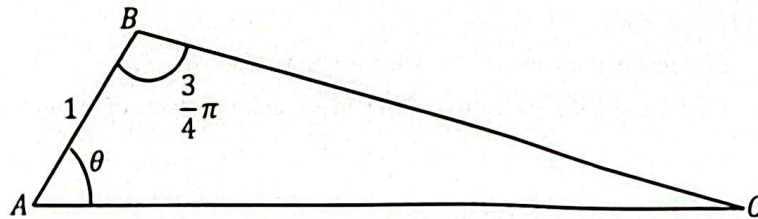
- (i) Find the first three terms of the Maclaurin series for $e^x(1 + \sin 2x)$. [You may use standard results given in the List of Formulae (MF15).] [3]
- (ii) It is given that the first two terms of this series are equal to the first two terms in the series expansion, in ascending powers of x , of $\left(1 + \frac{4}{3}x\right)^n$. Find n and show that the third terms in each of these series are equal. [3]

6 [9740/2011/P1/4]

Integration

- (i) Use the first three non-zero terms of the Maclaurin series for $\cos x$ to find the Maclaurin series for $g(x)$, where $g(x) = \cos^6 x$, up to and including the term in x^4 . [3]
- (ii) (a) Use your answer to part (i) to give an approximation for $\int_0^a g(x) dx$ in terms of a , and evaluate this approximation in the case where $a = \frac{\pi}{4}$. [3]
- (b) Use your calculator to find an accurate value for $\int_0^{\frac{\pi}{4}} g(x) dx$. Why is the approximation in part (ii) (a) not very good? [2]

7 [9740/2012/P1/4]



In the triangle ABC , $AB=1$, angle $BAC=\theta$ radians and angle $ABC=\frac{3}{4}\pi$ radians (see diagram).

- (i) Show that $AC = \frac{1}{\cos \theta - \sin \theta}$. [4]
- (ii) Given that θ is sufficiently small angle, show that $AC \approx 1 + a\theta + b\theta^2$, for constants a and b to be determined. [4]

8 [9740/2013/P2/3]

- (i) Given that $f(x) = \ln(1 + 2 \sin x)$, find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$. Hence write down the first three non-zero terms in the Maclaurin series for $f(x)$. [7]
- (ii) The first two non-zero terms in the Maclaurin series for $f(x)$ are equal to the first two non-zero terms in the series expansion of $e^{ax} \sin nx$. Using appropriate expansions from the List of Formulae (MF26), find the constants a and n . Hence find the third non-zero term of the series expansion $e^{ax} \sin nx$ for these values of a and n . [5]



9 [9740/2014/P1/8]

It is given that $f(x) = \frac{1}{\sqrt{9-x^2}}$, where $-3 < x < 3$.

- (i) Write down $\int f(x) dx$. [1]
- (ii) Find the binomial expansion for $f(x)$, up to and including the term in x^6 . Give the coefficients as exact fractions in their simplest form. [4]
- (iii) Hence, or otherwise, find the first four non-zero terms of the Maclaurin series for $\sin^{-1}\left(\frac{1}{3}x\right)$. Give the coefficients as exact fractions in their simplest form. [4]

10 [9740/2015/P1/6]

- (i) Write down the first three non-zero terms in the Maclaurin series for $\ln(1+2x)$, where $-\frac{1}{2} < x \leq \frac{1}{2}$, simplifying the coefficients. [2]
- (ii) It is given that the three terms found in part (i) are equal to the first three terms in the series expansion $ax(1+bx)^c$ for small x . Find the exact values of the constants a, b and c and use these values to find the coefficient of x^4 in the expansion $ax(1+bx)^c$, giving your answer as a simplified rational number. [6]

11 [9740/2016/P1/8]

It is given that $y = f(x)$, where $f(x) = \tan(ax+b)$ for constants a and b .

- (i) Show that $f'(x) = a + ay^2$. Use this result to find $f''(x)$ and $f'''(x)$ in terms of a and y . [5]
- (ii) In the case where $b = \frac{1}{4}\pi$, use your results from part (i) to find the Maclaurin series for $f(x)$ in terms of x , up to and including the term in x^3 . [3]
- (iii) Find the first two non-zero terms in the Maclaurin series for $\tan 2x$. [3]



Answers

1	(i) $1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \dots$ (ii) $8 - 3x + \frac{387}{16}x^2 - \frac{1151}{1128}x^3 + \dots$ (iii) $\left\{x: x \in \mathbb{R}, -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}\right\}$	2	(a) $a = 2, b = \frac{3}{4}$ (b) $f(0) = 1, f'(0) = 4, f''(0) = 16, 1 + 4x + 8x^2 + \dots$
3	(ii) $x + x^2 + \frac{x^3}{3} + \dots$ (iv) $\{x: x \in \mathbb{R}, -1.96 \leq x \leq 1.5\}$	4	(i) $f(0) = e, f'(0) = 0, f''(0) = -e; e - \frac{e}{2}x^2 + \dots$ (ii) $a = \frac{1}{e}, b = \frac{1}{2e}$
5	(i) $1 + 3x + \frac{5}{2}x^2 + \dots$ (ii) $n = \frac{9}{4}$	6	(i) $1 - 3x^2 + 4x^4 + \dots$ (ii)(a) $a - a^3 + \frac{4}{5}a^5 + \dots; 0.540$ (b) 0.475
7	(ii) $a = 1, b = \frac{3}{2}$	8	(i) $f(0) = 0, f'(0) = 2, f''(0) = -4, f'''(0) = 14; 2x - 2x^2 + \frac{7}{3}x^3 + \dots$ (ii) $a = -1, n = 2; -\frac{1}{3}x^3$
9	(i) $\sin^{-1}\left(\frac{x}{3}\right) + C$ (ii) $\frac{1}{3} + \frac{1}{54}x^2 + \frac{1}{378}x^4 + \frac{5}{34992}x^6 + \dots$ (iii) $\frac{1}{3}x + \frac{1}{162}x^3 + \frac{1}{3240}x^5 + \frac{5}{244944}x^7 + C$	10	(i) $2x - 2x^2 + \frac{8}{3}x^3 + \dots$ (ii) $a = 2, b = \frac{5}{3}, c = -\frac{3}{5}, -\frac{104}{27}$
11	(i) $f'(x) = 2a^2y + 2a^2y^3, f''(x) = 2a^3 + 8a^3y^2 + 6a^3y^4$ (ii) $1 + 2ax + 2a^2x^2 + \frac{8}{3}a^3x^3 + \dots$ (iii) $2x + \frac{8}{3}x^3 + \dots$		



Module IV Calculus (Integration and its Applications)

Techniques of Integration

1 [9740/2008/P1/5]

(i) Find the exact value of $\int_0^{\frac{1}{3}} \frac{1}{1+9x^2} dx$. [3]

(ii) Find, in terms of n and e , $\int_1^e x^n \ln x dx$, where $n \neq -1$. [4]

2 [9740/2009/P1/2]

Find the exact value of p such that

$$\int_0^1 \frac{1}{4-x^2} dx = \int_0^{\frac{1}{p}} \frac{1}{\sqrt{1-p^2x^2}} dx. \quad [5]$$

3 [9740/2011/P1/5]

It is given that $f(x) = 2 - x$.

(i) On separate diagram, sketch the graphs of $y = f(|x|)$ and $y = |f(x)|$, giving the coordinates of any points where the graphs meet the x - and y -axes. You should label the graphs clearly. [3]

(ii) State the set of values of x for which $f(|x|) = |f(x)|$. [1]

(iii) Find the exact value of the constant a for which $\int_{-1}^1 f(|x|) dx = \int_1^a |f(x)| dx$. [3]

4 [9740/2012/P1/2]

(i) Find $\int \frac{x^3}{1+x^4} dx$. [2]

(ii) Use the substitution $u = x^2$ to find $\int \frac{x}{1+x^4} dx$. [3]

(iii) Evaluate $\int_0^1 \left(\frac{x}{1+x^4} \right)^2 dx$, giving the answer correct to 3 decimal places. [1]

5 [9740/2013/P1/5]

It is given that

$$f(x) = \begin{cases} \sqrt{1 - \frac{x^2}{a^2}} & \text{for } -a \leq x \leq a, \\ 0 & \text{for } a < x < 2a, \end{cases}$$

and that $f(x+3a) = f(x)$ for all real values of x , where a is a real constant.

(i) Sketch the graph of $y = f(x)$ for $-4a \leq x \leq 6a$. [3]

(ii) Use the substitution $x = a \sin \theta$ to find the exact area of $\int_{\frac{1}{2}a}^{\frac{\sqrt{3}}{2}a} f(x) dx$ in terms of a and π . [5]



- 6 [9740/2014/P2/2]
Using partial fractions, find

$$\int_0^2 \frac{9x^2 + x - 13}{(2x-5)(x^2+9)} dx.$$

Give your answer in the form $a \ln b + c \tan^{-1} d$, where a, b, c and d are rational numbers to be determined. [9]

- 7 [9740/2015/P1/3]

(i) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right\}$$

is $\int_0^1 f(x) dx$. [2]

(ii) Hence evaluate the $\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \sqrt[3]{1} + \sqrt[3]{2} + \dots + \sqrt[3]{n} \right\}$. [3]

Answers

1	(i) $\frac{\pi}{9}$ (ii) $\frac{1+ne^{n+1}}{(n+1)^2}$	2	$\frac{2\pi}{3 \ln 3}$
3	(ii) $\{x: x \in \mathbb{R}, 0 \leq x \leq 2\}$ (iii) $2 + \sqrt{5}$	4	(i) $\frac{1}{4} \ln(1+x^4) + C$ (ii) $\frac{1}{2} \tan^{-1}(x^2) + C$ (iii) 0.186
5	(ii) $\frac{\pi a}{12}$	6	$\frac{3}{2} \ln \frac{13}{45} + \frac{8}{3} \tan^{-1}\left(\frac{2}{3}\right)$
7	(ii) $\frac{3}{4}$		

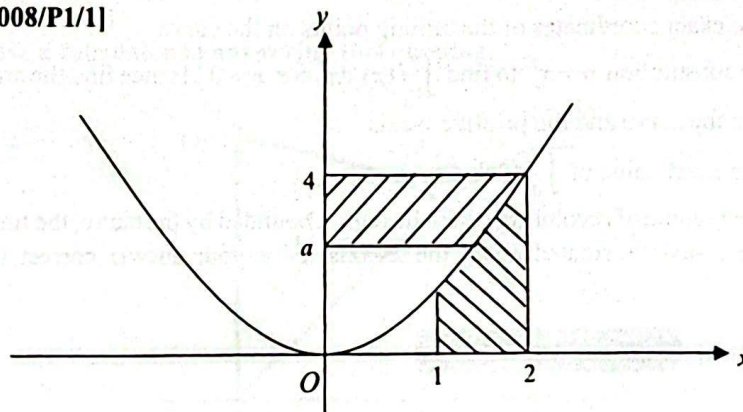


Definite Integrals

1 [9740/2007/P2/4]

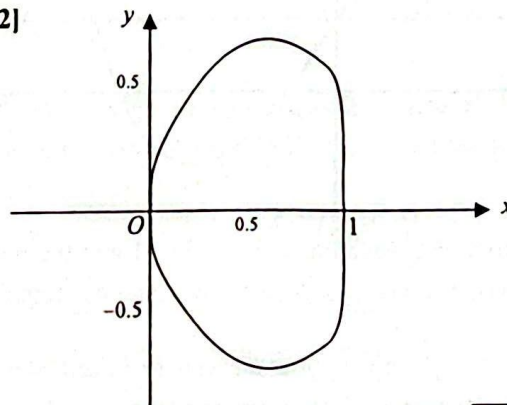
- (i) Find the exact value of $\int_0^{\frac{5}{3}\pi} \sin^2 x \, dx$. Hence find the exact value of $\int_0^{\frac{5}{3}\pi} \cos^2 x \, dx$. [6]
- (ii) The region R is bounded by the curve $y = x^2 \sin x$, the line $x = \frac{1}{2}\pi$ and the part of the x -axis between 0 and $x = \frac{1}{2}\pi$. Find
- (a) the exact area of R , [5]
- (b) the numerical value of the volume of revolution formed when R is rotated completely about the x -axis, giving your answer correct to 3 decimal places. [2]

2 [9740/2008/P1/1]



The diagram shows the curve with equation $y = x^2$. The area of the region bounded by the curve, the lines $x = 1$, $x = 2$ and the x -axis is equal to the area of the region bounded by the curve, the lines $y = a$, $y = 4$ and the y -axis, where $a < 4$. Find the value of a . [4]

3 [9740/2008/P2/2]



The diagram shows the curve C with equation $y^2 = x\sqrt{1-x}$. The region enclosed by C is denoted by R .

- (i) Write down an integral that gives the area of R , and evaluate this integral numerically. [3]
- (ii) The part of R above the x -axis is rotated through 2π radians about the x -axis. By using the substitution $u = 1 - x$, or otherwise, find the exact value of the volume obtained. [3]
- (iii) Find the exact x -coordinate of the maximum point of C . [3]



- 4 [9740/2009/P1/4]
It is given that

$$f(x) = \begin{cases} 7 - x^2 & \text{for } 0 < x \leq 2, \\ 2x - 1 & \text{for } 2 < x \leq 4, \end{cases}$$

and that $f(x) = f(x+4)$ for all real values of x .

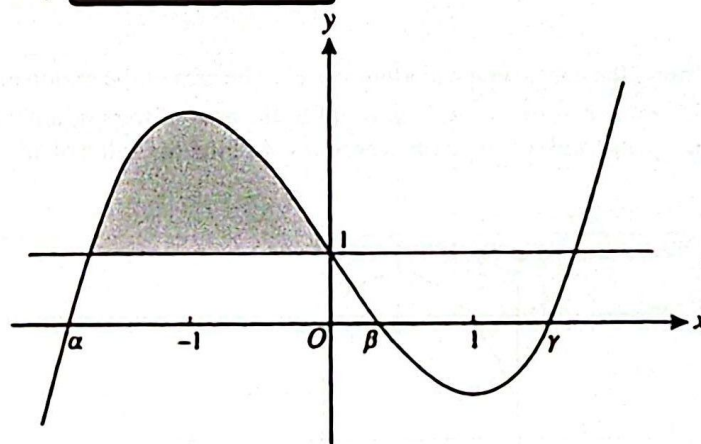
- (i) Evaluate $f(27) + f(45)$. [2]
- (ii) Sketch the graph of $y = f(x)$ for $-7 \leq x \leq 10$. [3]
- (iii) Find $\int_{-4}^3 f(x) dx$. [3]

- 5 [9740/2009/P1/11]

The curve C has equation $y = f(x)$, where $f(x) = xe^{-x^2}$.

- (i) Sketch the curve C . [2]
- (ii) Find the exact coordinates of the turning points on the curve. [4]
- (iii) Use the substitution $u = x^2$ to find $\int_0^n f(x) dx$, for $n > 0$. Hence find the area of the region between the curve and the positive x -axis. [4]
- (iv) Find the exact value of $\int_{-2}^2 |f(x)| dx$. [2]
- (v) Find the volume of revolution when the region bounded by the curve, the lines $x = 0$, $x = 1$ and the x -axis is rotated about the x -axis. Give your answer correct to 3 significant figures. [2]

- 6 [9740/2010/P1/6] **Graphing Techniques**



The diagram shows the curve with equation $y = x^3 - 3x + 1$ and the line with equation $y = 1$. The curve crosses the x -axis at $x = \alpha$, $x = \beta$ and $x = \gamma$ and has turning points at $x = -1$ and $x = 1$.

- (i) Find the values of β and γ , giving your answers to 3 decimal places. [2]
- (ii) Find the area of the region bounded by the curve and the x -axis between $x = \beta$ and $x = \gamma$. [2]
- (iii) Use a non-calculator method to find the area of the shaded region between the curve and the line. [4]
- (iv) Find the set of values of k for which the equation $x^3 - 3x + 1 = k$ has three real distinct roots. [2]



7 [9740/2011/P2/4]

(a) (i) Obtain a formula for $\int_0^n x^2 e^{-2x} dx$ in terms of n , where $n > 0$. [5]

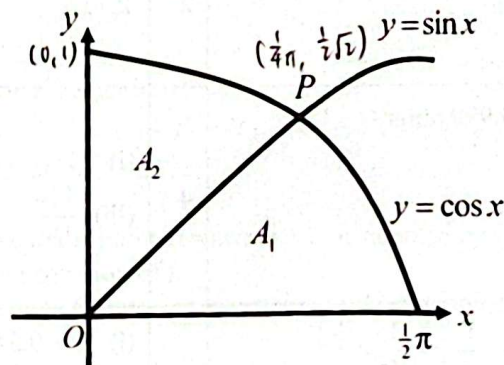
(ii) Hence evaluate $\int_0^\infty x^2 e^{-2x} dx$. [1]

[You may assume that ne^{-2n} and $n^2 e^{-2n} \rightarrow 0$ as $n \rightarrow \infty$.]

(b) The region bounded by the curve $y = \frac{4x}{x^2 + 1}$, the x -axis and the lines $x = 0$ and $x = 1$ is rotated through 2π radians about the x -axis. Use the substitution $x = \tan \theta$ to show that the volume of the solid obtained is given by $16\pi \int_0^{\frac{1}{4}\pi} \sin^2 \theta d\theta$, and evaluate this integral exactly. [6]

8 [9740/2015/P1/10]

Do not use a calculator in answering this question.



With origin O , the curves with equations $y = \sin x$ and $y = \cos x$, where $0 \leq x \leq \frac{1}{2}\pi$, meet at the point P with coordinates $(\frac{1}{4}\pi, \frac{1}{2}\sqrt{2})$. The area of the region bounded by the curves and the x -axis is A_1 and the area bounded by the curves and the y -axis is A_2 (see diagram).

(i) Show that $\frac{A_1}{A_2} = \sqrt{2}$. [4]

(ii) The region bounded by $y = \sin x$ between O and P , the line $y = \frac{1}{2}\sqrt{2}$ and the y -axis is rotated about the y -axis through 360° . Show that the volume of the solid formed is given by

$$\pi \int_0^{\frac{1}{2}\sqrt{2}} (\sin^{-1} y)^2 dy. \quad [2]$$

(iii) Show that the substitution $y = \sin u$ transforms the integral in part (ii) to $\pi \int_a^b u^2 \cos u du$, for limits a and b to be determined. Hence find the exact volume. [6]



9 [9740/2016/P2/2]

(a) (i) Find $\int x^2 \cos nx \, dx$, where n is a positive integer. [3]

(ii) Hence find $\int_{\pi}^{2\pi} x^2 \cos nx \, dx$, giving your answer in the form $a \frac{\pi}{n^2}$, where the possible values of a are to be determined. [2]

(b) The region bounded by the curve $y = \frac{x\sqrt{x}}{9-x^2}$, the x -axis and the lines $x=0$ and $x=2$ is rotated through 2π radians about the x -axis. Use the substitution $u = 9 - x^2$ to find the exact volume of the solid obtained, simplifying your answer. [5]

Answers

1	(i) $\frac{1}{2} \left(\frac{5\pi}{3} + \frac{\sqrt{3}}{4} \right); \frac{5\pi}{6} - \frac{\sqrt{3}}{8}$ (ii)(a) $\pi - 2$ (b) 5.391	2	2.73
3	(i) $2 \int_0^1 \sqrt{x} \sqrt{1-x} \, dx$; 0.999 units ² (ii) $\frac{4\pi}{15}$ units ³ (iii) $\frac{2}{3}$	4	(i) 11 (iii) $\frac{110}{3}$
5	(ii) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} e^{\frac{1}{2}} \right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} e^{\frac{1}{2}} \right)$ (iii) $\frac{1}{2} (1 - e^{-n^2}); \frac{1}{2}$ (iv) $1 - e^{-4}$ (v) 0.363	6	(i) $\beta = 0.347, \gamma = 1.532$ (ii) 0.781 units ² (iii) $\frac{9}{4}$ units ² (iv) $\{k : k \in \mathbb{R}, -1 < k < 3\}$
7	(a)(i) $\frac{1}{4} - \frac{1}{2} n^2 e^{-2n} - \frac{1}{2} n e^{-2n} - \frac{1}{4} e^{-2n}$ (ii) $\frac{1}{4}$ (b) $(2\pi^2 - 4\pi)$ units ³	8	(iii) $a = 0, b = \frac{\pi}{4}; \sqrt{2}\pi \left(\frac{\pi^2}{32} + \frac{\pi}{4} - 1 \right)$
9	(a)(i) $\frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx + C$ (ii) $\begin{cases} \frac{6\pi}{n^2}, & \text{for odd } n \\ \frac{2\pi}{n^2}, & \text{for even } n, \text{ including } n = 0 \end{cases}$ (b) $\frac{\pi}{2} \left(\frac{4}{5} + \ln \frac{5}{9} \right)$ units ³		



Parametric Equations

1 [9740/2007/P1/11]

A curve has parametric equations

$$x = \cos^2 t, \quad y = \sin^3 t, \quad \text{for } 0 \leq t \leq \frac{1}{2}\pi$$

- (i) Sketch the curve. [2]
- (ii) The tangent to the curve at point $(\cos^2 \theta, \sin^3 \theta)$, where $0 < \theta < \frac{1}{2}\pi$, meets the x - and y -axes at Q and R respectively. The origin is denoted by O . Show that the area of triangle OQR is

$$\frac{1}{12} \sin \theta (3 \cos^2 \theta + 2 \sin^2 \theta)^2. \quad [6]$$

- (iii) Show that the area under the curve for $0 \leq t \leq \frac{1}{2}\pi$ is $2 \int_0^{\frac{1}{2}\pi} \cos t \sin^4 t \, dt$, and use the substitution $\sin t = u$ to find this area. [5]

2 [9740/2009/P2/1]

The curve C has parametric equations

$$x = t^2 + 4t, \quad y = t^3 + t^2.$$

- (i) Sketch the curve for $-2 \leq t \leq 1$. [1]

The tangent to the curve at the point P where $t = 2$ is denoted by l .

- (ii) Find the cartesian equation of l . [3]
- (iii) The tangent l meets C again at the point Q . Use a non-calculator method to find the coordinates of Q . [4]

3 [9740/2010/P1/11]

A curve C has parametric equations

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}.$$

- (i) The point P on the curve has parameter p . Show that the equation of the tangent at P is $(p^2 + 1)x - (p^2 - 1)y = 4p$. [4]
- (ii) The tangent at P meets the line $y = x$ at the point A and the line $y = -x$ at the point B . Show that the area triangle OAB is independent of p , where O is the origin. [4]
- (iii) Find a cartesian equation of C . Sketch C , giving the coordinates of any points where C crosses the x - and y -axes and the equations of any asymptotes. [4]

4 [9740/2011/P1/3]

The parametric equations of a curve are

$$x = t^2, \quad y = \frac{2}{t}$$

- (i) Find the equation of the tangent to the curve at the point $\left(p^2, \frac{2}{p}\right)$, simplifying your answer. [2]
- (ii) Hence find the coordinates of the points Q and R where this tangent meets the x - and y -axes respectively. [2]
- (iii) Find a cartesian equation of the locus of the mid-point of QR as p varies. [3]



5 [9740/2012/P1/11]

A curve C has parametric equations

$$x = \theta - \sin \theta, \quad y = 1 - \cos \theta,$$

where $0 \leq \theta \leq 2\pi$.

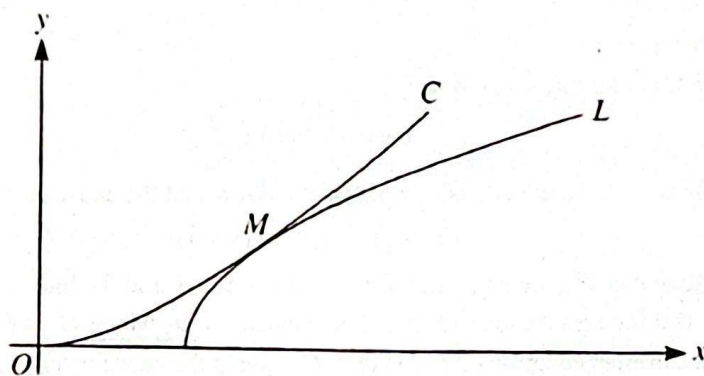
- (i) Show that $\frac{dy}{dx} = \cot \frac{1}{2}\theta$ and find the gradient of C at the point where $\theta = \pi$. What can be said about the tangents to C as $\theta \rightarrow 0$ and $\theta \rightarrow 2\pi$? [5]
- (ii) Sketch C , showing clearly the features of the curve at the points where $\theta = 0, \pi$ and 2π . [3]
- (iii) Without using a calculator, find the exact area of the region bounded by C and the x -axis [5]
- (iv) A point P on C has parameter p , where $0 < p < \frac{1}{2}\pi$. Show that the normal to C at P crosses the x -axis at the point with coordinates $(p, 0)$. [3]

6 [9740/2013/P1/11]

A curve C has parametric equations

$$x = 3t^2, \quad y = 2t^3.$$

- (i) Find the equation of the tangent to C at the point with parameter t . [3]
- (ii) The points P and Q on C have parameters p and q respectively. The tangent at P meets the tangent at Q at the point R . Show that the x -coordinate of R is $p^2 + pq + q^2$, and find the y -coordinate of R in terms of p and q . Given that $pq = -1$, show that R lies on the curve with equation $x = y^2 + 1$. [5]

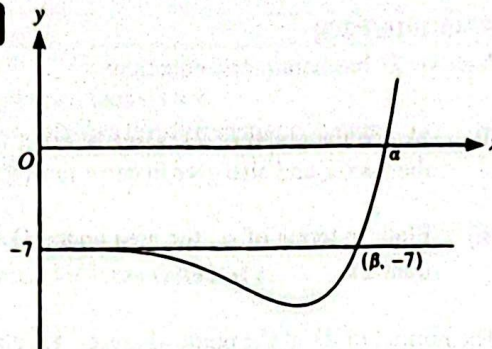


A curve L has equation $x = y^2 + 1$. The diagram shows the parts of C and L for which $y \geq 0$. The curves C and L touch at the point M .

- (iii) Show that $4t^6 - 3t^2 + 1 = 0$ at M . Hence, or otherwise, find the exact coordinates of M . [3]
- (iv) Find the exact value of the area of the shaded region bounded by C and L for which $y \geq 0$. [6]



7 [9740/2014/P1/7] Complex Numbers



It is given that $f(x) = x^6 - 3x^4 - 7$. The diagram shows the curve with equation $y = f(x)$ and the line with equation $y = -7$, for $x \geq 0$. The curve crosses the positive x -axis at $x = \alpha$, and the curve and the line meet where $x = 0$ and $x = \beta$.

- Find the value of α , giving your answer correct to 3 decimal places, and find the exact value of β . [2]
- Evaluate $\int_0^\alpha f(x) dx$, giving your answers correct to 3 decimal places. [2]
- Find, in terms of $\sqrt{3}$, the area of the finite region bounded by the curve and the line, for $x \geq 0$. [3]
- Show that $f(x) = f(-x)$. What can be said about the six roots of the equation $f(x) = 0$? [4]

8 [9740/2014/P2/1]

A curve C has parametric equations

$$x = 3t^2, \quad y = 6t.$$

- Find the value of t at the point on C where the tangent has gradient 0.4. [3]
- The tangent at the point $P(3p^2, 6p)$ on C meets the y -axis at the point D . Find the cartesian equation of the locus of the mid-point of PD as p varies. [4]

9 [9740/2015/P1/11]

A curve C has parametric equations

$$x = \sin^3 \theta, \quad y = 3 \sin^2 \theta \cos \theta, \quad \text{for } 0 \leq \theta \leq \frac{1}{2}\pi$$

- Show that $\frac{dy}{dx} = 2 \cot \theta - \tan \theta$. [3]
- Show that C has a turning point when $\tan \theta = \sqrt{k}$, where k is an integer to be determined. Find in non-trigonometric form, the exact coordinates of the turning point and explain why it is a maximum. [6]
- Show that the area of the region bounded by C and the x -axis is given by

$$\int_0^{\frac{1}{2}\pi} 9 \sin^4 \theta \cos^2 \theta d\theta.$$

Use your calculator to find the area, giving your answer correct to 3 decimal places. [3]

The line with equation $y = ax$, where a is a positive constant, meet C at the origin and at the point P .

- Show that $\tan \theta = \frac{3}{a}$ at P . Find the exact value of a such that the line passes through the maximum point of C . [3]



10 [9740/2016/P2/3]

A curve D has parametric equations

$$x = t - \cos t, \quad y = 1 - \cos t \quad \text{for } 0 \leq t \leq 2\pi$$

- (i) Sketch the graph of D . Give in exact form the coordinates of the points where D meet the x -axis, and also give in exact form the coordinates of the maximum point on the curve. [4]
- (ii) Find, in terms of a , the area under D for $0 \leq t \leq a$, where a is a positive constant less than 2π . [3]

The normal to D at the point where $t = \frac{1}{2}\pi$ cuts the x -axis at E and the y -axis at F .

- (iii) Find the exact area of triangle OEF , where O is the origin. [4]

Answers

1	(iii) $\frac{2}{5} \text{ units}^2$	2	(ii) $y = 2x - 12$ (iii) $Q(-3, -18)$
3	(ii) 4 (iii) $x^2 - y^2 = 4$	4	(i) $p^3y + x = 3p^2$ (ii) $Q(3p^2, 0), R\left(0, \frac{3}{p}\right)$ (iii) $x = \frac{27}{8y^2}$
5	(i) 0; tangents are parallel to y -axis (iii) $3\pi \text{ units}^2$	6	(i) $y = xt - t^3$ (ii) $pq(p + q)$ (iii) $\left(\frac{3}{2}, \frac{1}{\sqrt{2}}\right)$ (iv) $\frac{4}{15\sqrt{2}} \text{ units}^2$
7	(i) 1.885; $\sqrt{3}$ (ii) -0.597 (iii) $\frac{54}{35}\sqrt{3} \text{ units}^2$	8	(i) 2.5 (ii) $x = \frac{2}{27}y^2$
9	(ii) $k = 2; \left(\frac{2}{9}\sqrt{6}, \frac{2}{3}\sqrt{3}\right)$ (iii) 0.884 units^2 (iv) $a = \frac{3}{2}\sqrt{2}$	10	(i) x -intercepts: $(-1, 0), (2\pi - 1, 0)$; Max. $(\pi + 1, 2)$ (ii) $a - \cos a - \sin a + \frac{1}{4}\cos 2a + \frac{3}{4}$ (iii) $\frac{1}{4}(\pi + 1)^2 \text{ units}^2$



Differential Equations

1 [9740/2007/P1/4]

The current I in an electric circuit at time t satisfies the differential equation

$$4 \frac{dI}{dt} = 2 - 3I.$$

Find I in terms of t , given that $I = 2$ when $t = 0$.

[6]

State what happens to the current in this circuit for large values of t .

[1]

2 [9740/2008/P1/4(part of)(modified)]

(i) Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{3x}{x^2 + 1}.$$

[2]

(ii) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$.

[1]

(iii) What can you say about the gradient of every solution curve as $x \rightarrow \pm\infty$?

[1]

(iv) Sketch the graph of the solution found in part (ii).

[1]

3 [9740/2009/P2/4(modified)]

Two scientists are investigating the change of a certain population of size n thousand at time t years.

(i) One scientist suggests that n and t are related by the differential equation $\frac{d^2n}{dt^2} = 10 - 6t$.

Find the general solution of this differential equation.

[3]

(ii) The other scientist suggests that n and t are related by the differential equation

$\frac{dn}{dt} = 3 - 0.02n$. Find n in terms of t , given that $n = 100$ when $t = 0$. Explain in simple

terms what will eventually happen to the population using this model.

[7]

4 [9740/2010/P1/7]

A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant 20°C . As the liquid warms up, the rate of increase of its temperature $\theta^\circ\text{C}$ after time t minutes is proportional to the temperature difference $(20 - \theta)^\circ\text{C}$. Initially the temperature of the liquid is 10°C and the rate of increase of the temperature is 1°C per minute.

By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-\frac{1}{10}t}$.

[7]

Find the time it takes the liquid to reach a temperature of 15°C , and state what happens to θ for large values of t . Sketch a graph of θ against t .

[4]

5 [9740/2011/P1/8]

(i) Find $\int \frac{1}{100 - v^2} dv$.

[2]

(ii) A stone is dropped from a stationary balloon. It leaves the balloon with zero speed, and t seconds later its speed v metres per second satisfies the differential equation

$$\frac{dv}{dt} = 10 - 0.1v^2.$$

(a) Find t in terms of v . Hence find the exact time the stone takes to reach a speed of 5 metres per second.

[5]

(b) Find the speed of the stone after 1 second.

[3]

(c) What happens to the speed of the stone for large values of t ?

[2]



6 [9740/2012/P2/1]

- (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} = 16 - 9x^2,$$

giving your answer in the form $y = f(x)$.

[3]

- (b) Given that u and t are related by

$$\frac{du}{dt} = 16 - 9u^2,$$

and that $u = 1$ and when $t = 0$, find t in terms of u , simplifying your answer.

[5]

7 [9740/2013/P1/10(part of)]

The variables x , y and z are connected by the following differential equations.

$$\frac{dz}{dx} = 3 - 2z \quad (\text{A})$$

$$\frac{dy}{dx} = z \quad (\text{B})$$

- (i) Given that $z < \frac{3}{2}$, solve the equation (A) to find z in terms of x .

[4]

- (ii) Hence find y in terms of x .

[2]

- (iii) Use the result in part (ii) to show that

$$\frac{d^2y}{dx^2} = a \frac{dy}{dx} + b,$$

for constants a and b to be determined.

[3]

8 [9740/2014/P1/10]

The mass, x grams, of a certain substance present in a chemical reaction at time t minutes satisfies the differential equation

$$\frac{dx}{dt} = k(1 + x - x^2),$$

where $0 \leq x \leq \frac{1}{2}$ and k is a constant. It is given that $x = \frac{1}{2}$ and $\frac{dx}{dt} = -\frac{1}{4}$ when $t = 0$.

- (i) Show that $k = -\frac{1}{5}$.

[1]

- (ii) By first expressing $1 + x - x^2$ in completed square form, find t in terms of x .

[5]

- (iii) Hence find

- (a) the exact time taken for the mass of the substance present in the chemical reaction to become half its initial value,

[1]

- (b) the time taken for there to be none of the substance present in the chemical reaction, giving your answer correct to 3 decimal places.

[1]

- (iv) Express the solution of the differential equation in the form $x = f(t)$ and sketch the part of the curve with this equation which is relevant in this context.

[5]



9 [9740/2015/P2/1]

As a tree grows, the rate of increase of its height, h m, with respect to time, t years after planting, is modelled by the differential equation

$$\frac{dh}{dt} = \frac{1}{10} \sqrt{16 - \frac{1}{2}h}$$

The tree is planted as a seedling of negligible height, so that $h = 0$ when $t = 0$.

- (i) State the maximum height of the tree, according to this model. [1]
- (ii) Find an expression for t in terms of h , and hence find the time the tree takes to reach half its maximum height. [5]

10 [9740/2016/P1/9]

A stone is held on the surface of a pond and released. The stone falls vertically through the water and the distance, x metres, that the stone has fallen in time t seconds is measured. It is given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

- (i) The motion of the stone is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 10$$

- (a) By substituting $y = \frac{dx}{dt}$, show that the differential equation can be written as

$$\frac{dy}{dt} = 10 - 2y. \quad [1]$$

- (b) Find y in terms of t and hence find x in terms of t . [6]

- (ii) A second model for the motion of the stone is suggested, given by the differential equation

$$\frac{d^2x}{dt^2} = 10 - 5\sin \frac{1}{2}t.$$

Find x in terms of t for this model. [3]

- (iii) The pond is 5 metres deep. For each of these models, find the time the stone takes to reach the bottom of the pond, giving your answers to 2 decimal places. [2]



Answers

1	$I = \frac{2}{3} + \frac{4}{3}e^{-\frac{3}{4}t}; I \rightarrow \frac{2}{3}$	2	(i) $y = \frac{3}{2}\ln(x^2 + 1) + C$ (ii) $y = \frac{3}{2}\ln(x^2 + 1) + 2$ (iii) gradient tends to 0
3	(i) $5t^2 - t^3 + Ct + D$ (ii) population eventually reaches 150000	4	6.93min, $\theta \rightarrow 20^\circ C$
5	(i) $\frac{1}{20}\ln\left \frac{10+v}{10-v}\right + C$ (ii)(a) $t = \frac{1}{2}\ln\left \frac{10+v}{10-v}\right ; \frac{1}{2}\ln 3$ (b) 7.62 ms^{-1} (c) 10 ms^{-1}	6	(a) $y = 8x^2 - \frac{3}{4}x^4 + Cx + D$ (b) $t = \frac{1}{24}\ln\left \frac{4+3u}{7(4-3u)}\right $
7	(i) $z = \frac{3}{2} + Ae^{-2x}$ (ii) $\frac{3}{2} - \frac{A}{2}e^{-2x} + B$ (iii) $a = -2, b = 3$	8	(ii) $\frac{5}{4} - \left(x - \frac{1}{2}\right)^2; t = -\sqrt{5}\ln\left \frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}\right $ (iii)(a) $\sqrt{5}\ln\left(\frac{2\sqrt{5}+1}{2\sqrt{5}-1}\right)$ (b) 2.152 min (iv) $x = \frac{e^{\frac{t}{\sqrt{5}}}(\sqrt{5}+1)+1-\sqrt{5}}{2\left(1+e^{\frac{t}{\sqrt{5}}}\right)}$
9	(i) 32m (ii) $t = 160 - 40\sqrt{16 - \frac{1}{2}h}; 46.9 \text{ years}$	10	(i)(b) $x = 5t + \frac{5}{2}e^{-2t} - \frac{5}{2}$ (ii) $x = 5t^2 + 20\sin\frac{1}{2}t - 10t$ (iii) Model 1: 1.47s; Model 2: 1.05s



Module V Complex Numbers

Introduction to Complex Numbers

1 [9740/2008/P1/8]

A graphic calculator is **not** to be used in answering this question.

(i) It is given that $z_1 = 1 + (\sqrt{3})i$. Find the value of z_1^3 , showing clearly how you obtain your answer. [3]

(ii) Given that $1 + (\sqrt{3})i$ is a root of the equation

$$2z^3 + az^2 + bz + 4 = 0,$$

find the values of the real numbers a and b . [4]

(iii) For these values of a and b , solve the equation in part (ii), and show all the roots on an Argand diagram. [4]

2 [9740/2010/P2/1]

(i) Solve the equation $x^2 - 6x + 34 = 0$. [2]

(ii) One root of the equation $x^4 + 4x^3 + x^2 + ax + b = 0$, where a and b are real, is $x = -2 + i$. Find the values of a and b and the other roots. [5]

3 [9740/2011/P1/10(part of)]

Do not use a graphic calculator in answering this question.

(i) The roots of the equation $z^2 = -8i$ are z_1 and z_2 . Find z_1 and z_2 in cartesian form $x + iy$, showing your working. [4]

(ii) Hence, or otherwise, find in cartesian form the roots w_1 and w_2 of the equation

$$w^2 + 4w + (4 + 2i) = 0. [3]$$

4 [9740/2013/P1/4]

The complex number w is given by $1 + 2i$.

(i) Find w^3 in the form $x + iy$, showing your working. [2]

(ii) Given that w is a root of the equation $az^3 + 5z^2 + 17z + b = 0$, find the exact values of the real numbers a and b . [3]

(iii) Using these values of a and b , find the roots of this equation in exact form. [3]

5 [9740/2014/P1/5]

It is given that $z = 1 + 2i$.

(i) Without using a calculator, find the values of z^2 and $\frac{1}{z^3}$ in cartesian form $x + iy$, showing your working. [4]

(ii) The real numbers p and q are such that $pz^2 + \frac{q}{z^3}$ is real. Find, in terms of p , the values of q and the value of $pz^2 + \frac{q}{z^3}$. [3]



6 [9740/2016/P1/7]

Do not use a calculator in answering this question.

- (i) Verify that $-1+5i$ is a root of the equation $w^2 + (-1-8i)w + (-17+7i) = 0$. Hence, or otherwise, find the second root of the equation in cartesian form, $p+iq$, showing your working. [5]
- (ii) The equation $z^3 - 5z^2 + 16z + k = 0$, where k is a real constant, has a root $z = 1+ai$, where a is a positive real constant. Find the values of a and k , showing your working. [5]

Answers

1	(i) -8 (ii) $a = -3, b = 6$ (iii) $1+i\sqrt{3}$ or $1-i\sqrt{3}$ or $-\frac{1}{2}$	2	(i) $3+5i$ or $3-5i$ (ii) $a = -16, b = -20; x = 2-i$ or $x = 2$ or $x = -2$
3	(i) $2-2i, -2+2i$ (ii) $-3+i, -1-i$	4	(i) $-11-2i$ (ii) $a = 27, b = 295$ (iii) $1+2i, 1-2i, \frac{59}{27}$
5	(i) $z^2 = -3+4i; \frac{1}{z^3} = -\frac{11}{125} + \frac{2}{125}i$ (ii) $q = -250p; pz^2 + \frac{q}{z^3} = 19p$	6	(i) $2+3i$ (ii) $a = 3, k = -30$



Module VI Statistics

Permutations and Combinations

1 [9740/2007/P2/9]

A group of 12 people consists of 6 married couples.

- (i) The group stand in a line.
 - (a) Find the number of different possible orders, [1]
 - (b) Find the number of different possible orders in which each man stands next to his wife [3]
- (ii) The group stand in a circle.
 - (a) Find the number of different possible arrangements. [1]
 - (b) Find the number of different possible arrangements if men and women alternate. [2]
 - (c) Find the number of different possible arrangements if each man stands next to his wife and men and women alternate. [2]

2 [9740/2008/P2/10]

A group of diplomats is to be chosen to represent three islands, K , L and M . The group is to consist of 8 diplomats and is chosen from a set of 12 diplomats consisting of 3 from K , 4 from L and 5 from M . Find the number of ways in which the group can be chosen if it includes

- (i) 2 diplomats from K , 3 from L and 3 from M , [2]
- (ii) diplomats from L and M only, [2]
- (iii) at least 4 diplomats from M , [2]
- (iv) at least 1 diplomat from each island. [4]

3 [9740/2009/P2/8]

Find the number of ways in which the letters of words ELEVATED can be arranged if

- (i) there are no restrictions, [1]
- (ii) T and D must not be next to one another, [2]
- (iii) consonants (L, V, T, D) and vowels (E, A) must alternate, [3]
- (iv) between any two Es there must be at least 2 other letters. [3]

4 [9740/2014/P2/6]

A team in a particular sport consists of 1 goalkeeper, 4 defenders, 2 midfielders and 4 attackers. A certain club has 3 goalkeepers, 8 defenders, 5 midfielders and 6 attackers.

- (i) How many different teams can be formed by the club? [2]

One of the midfielders in the club is the brother of one of the attackers in the club.

- (ii) How many different teams can be formed which include exactly one of the two brothers. [3]

The two brothers leave the club. The club manager decides that one of the remaining midfielders can play as either a midfielder or as a defender.

- (iii) How many different teams can now be formed by the club? [3]



5 [9740/2015/P2/11]

This question is about arrangement of all eight letters in the word CABBAGES.

- (i) Find the number of arrangements of the eight letters that can be made. [2]
- (ii) Write down the number of these arrangements in which the letters are **not** in alphabetical order. [1]
- (iii) Find the number of different arrangements that can be made with both A's together and both B's together. [2]
- (iv) Find the number of arrangements that can be made with no two adjacent letters the same. [4]

Answers

1	(i)(a) 479001600 (b) 39916800 (ii)(a) 39916800 (b) 86400 (c) 240	2	(i) 120 (ii) 9 (iii) 210 (iv) 485
3	(i) 6720 (ii) 5040 (iii) 192 (iv) 480	4	(i) 31500 (ii) 16800 (iii) 8820
5	(i) 10080 (ii) 10079 (iii) 720 (iv) 5760		



Probability

1 [9740/2007/P2/10]

A player throws three darts at a target. The probability that he is successful in hitting the target with his first throw is $\frac{1}{8}$. For each of his second and third throws, the probability of success is

- twice the probability of success on the preceding throw if that throw was successful,
- the same as the probability of success on the preceding throw if that throw was unsuccessful.

Construct a probability tree showing this information.

[3]

Find

- (i) the probability that all three throws are successful, [2]
- (ii) the probability that at least two throws are successful, [2]
- (iii) the probability that the third throw is successful given that exactly two of the three throws are successful. [4]

2 [9740/2008/P2/7]

A computer game simulates a tennis match between two players, A and B . The match consists of at most three sets. Each set is won by either A or B , and the match is won by the first player to win two sets.

The simulation uses the following rules.

- The probability that A wins the first set is 0.6.
- For each set after the first, the conditional probability that A wins that set, given that A won the preceding set, is 0.7.
- For each set after the first, the conditional probability that B wins that set, given that B won the preceding set, is 0.8.

Calculate the probability that

- (i) A wins the second set, [2]
- (ii) A wins the match, [3]
- (iii) B won the first set, given that A wins the match. [3]

3 [9740/2009/P2/7]

A company buy $p\%$ of its electronic components from supplier A and the remaining $(100 - p)\%$ from supplier B . The probability of a randomly chosen component supplied by A is faulty is 0.05. The probability that a randomly chosen component supplied by B is faulty is 0.03.

- (i) Given that $p = 25$, find the probability that a randomly chosen component is faulty. [2]
- (ii) For a general value of p , the probability that a randomly chosen component that is faulty was supplied by A is denoted by $f(p)$. Show that $f(p) = \frac{0.05p}{0.02p + 3}$. Prove by differentiation that f is an increasing function for $0 \leq p \leq 100$, and explain what this statement means in the context of the question. [6]



4 [9740/2010/P2/7]

For events A and B it is given that $P(A) = 0.7$, $P(B) = 0.6$ and $P(A | B') = 0.8$. Find

- (i) $P(A \cap B')$, [2]
- (ii) $P(A \cup B)$, [2]
- (iii) $P(B' | A)$. [2]

For a third event C , it is given that $P(C) = 0.5$ and that A and C are independent.

- (iv) Find $P(A' \cap C)$. [2]
- (v) Hence state an inequality satisfied by $P(A' \cap B \cap C)$. [1]

5 [9740/2010/P2/8]

The digit 1, 2, 3, 4 and 5 are arranged randomly to form a five-digit number. No digit is repeated. Find the probability that

- (i) the number is greater than 30 000. [1]
- (ii) the last two digits are both even. [2]
- (iii) The number is greater than 30 000 and odd. [4]

6 [9740/2011/P2/9]

Camera lenses are made by two companies, A and B . 60% of all lenses are made by A and the remaining 40% by B . 5% of the lenses made by A are faulty. 7% of the lenses made by B are faulty.

- (i) One lens is selected at random. Find the probability that
 - (a) it is faulty, [2]
 - (b) it is made by A , given that its faulty. [1]
- (ii) Two lenses are selected at random. Find the probability that
 - (d) exactly one of them is faulty [2]
 - (e) both were made by A , given that exactly one is faulty. [3]

7 [9740/2011/P2/11]

A committee of 10 people is chosen at random from a group of 18 women and 12 men. The number of women on the committee is denoted by R .

- (i) Find the probability that $R = 4$. [3]
- (ii) The most probable number of women on the committee is denoted by r . By using the fact that $P(R = r) > P(R = r + 1)$, show that r satisfies the inequality
$$(r + 1)!(17 - r)!(9 - r)!(r + 3)! > r!(18 - r)!(10 - r)!(r + 2)!$$
and use this inequality to find the value of r . [5]

8 [9740/2012/P2/5]

The probability that a hospital patient has a particular disease is 0.001. A test for the disease has probability p of giving a positive result when the patient has the disease, and equal probability p of giving a negative result when the patient does not have the disease. A patient is given the test.

- (i) Given that $p = 0.995$, find the probability that
 - (a) the result of the test is positive, [2]
 - (b) the patient has the disease given that the result of the test is positive. [2]
- (ii) It is given that instead that there is a probability of 0.75 that the patient has the disease given that the result of the test is positive. Find the value of p , giving your answer correct to 6 decimal places. [3]



9 [9740/2012/P2/7]

A group of fifteen people consists of one pair of sisters, one set of three brothers and ten other people. The fifteen people are arranged randomly in a line.

- (i) Find the probability that the sisters are next to each other. [2]
- (ii) Find the probability that the brothers are *not* all next to one another. [2]
- (iii) Find the probability that the sisters are next to each other and the brothers *are* all next to one another. [2]
- (iv) Find the probability that *either* the sisters are next to each other *or* the brothers are all next to one another *or* both. [2]

Instead the fifteen people are arranged in a circle.

- (v) Find the probability that the sisters are next to each other. [1]
- (i) Find the probability that the customer has insufficient money to pay for the watermelons if she has only \$20. [4]

A random sample of n red watermelons is chosen. Find the smallest value of n such that the probability that more than one of the red watermelons weigh more than 3kg is more than 0.5.

[3]

10 [9740/2013/P2/8]

For events A and B it is given that $P(A) = 0.7$, $P(B|A') = 0.8$ and $P(A|B') = 0.88$. Find

- (i) $P(B \cap A')$, [1]
- (ii) $P(A' \cap B')$, [2]
- (iii) $P(A \cap B)$. [3]

11 [9740/2013/P2/11]

A machine is used to generate codes consisting of three letters followed by two digits. Each of the three letters generated is equally likely to be any of the twenty-six letters of the alphabet A-Z. Each of the two digits generated is equally likely to be any of the nine digits 1-9. The digit 0 is not used.

Find the probability that a randomly generated chosen code has

- (i) three different letters and two different digits, [2]
- (ii) the second digit higher than the first digit, [2]
- (iii) exactly two letters the same or two digits the same, but not both, [4]
- (iv) exactly one vowel (A, E, I, O or U) and exactly one even digit. [4]

12 [9740/2014/P2/10]

A game has three sets of ten symbols, and one symbol from each set is randomly chosen to be displayed on each turn. The symbols are as follows.

Set 1	+	+	+	+	x	x	x	o	o	*
Set 2	+	+	+	x	o	o	o	o	*	*
Set 3	+	+	x	x	x	x	o	o	o	*

For example, if a + is chosen from set 1, a O symbol is chosen from set 2 and a * symbol is chosen from set 3, the display would be + o *.

- (i) Find the probability that, on one turn,
 - (a) * * * is displayed, [1]
 - (b) at least one * is displayed, [2]
 - (c) two x symbols and one + are displayed, in any order. [3]
- (ii) Given that exactly one of the symbols displayed is *, find the probability that the other two symbols are + and o. [4]



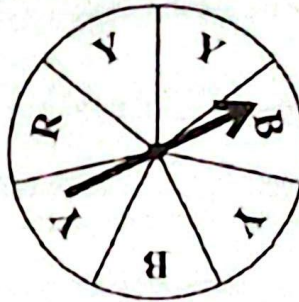
13 [9740/2015/P2/9]

For events A , B and C it is given that $P(A) = 0.45$, $P(B) = 0.4$, $P(C) = 0.3$ and $P(A \cap B \cap C) = 0.1$. It is also given that events A and B are independent, and that events A and C are independent.

- (i) Find $P(B|A)$. [1]
- (ii) Given also that events B and C are independent, find $P(A' \cap B' \cap C')$. [3]
- (iii) Given instead that events B and C are not independent, find the greatest and least possible values of $P(A' \cap B' \cap C')$. [4]

14 [9740/2016/P2/5]

In a game of chance, a player has to spin a fair spinner. The spinner has 7 sections and an arrow which has an equal chance of coming to rest over any of the 7 sections. The spinner has 1 section labelled R, 2 sections labelled B and 4 sections labelled Y (see diagram).



The player then has to throw one of three fair six-sided die, coloured red, blue or yellow. If the spinner comes to rest over R the red die is thrown, if the spinner comes to rest over B the blue die is thrown and if the spinner comes to rest over Y the yellow die is thrown. The yellow die has one face with * on it, the blue die has two faces with * on it and the red die has three faces with * on it. The player wins the game if the die thrown comes to rest with a face showing * uppermost.

- (i) Find the probability that a player wins a game. [2]
- (ii) Given that a player wins a game, find the probability that the spinner came to rest over B. [1]
- (iii) Find the probability that a player wins 3 consecutive games, each time throwing a die of a different colour. [2]

15 [9740/2016/P2/7]

Permutations and Combinations

The management board of a company consists of 6 men and 4 women. A chairperson, a secretary and a treasurer are chosen from the 10 members of the board. Find the number of ways the chairperson, the secretary and the treasurer can be chosen so that

- (i) they are all women, [1]
- (ii) at least one is a woman and at least one is a man. [3]

The 10 members of the board sit at random around a table. Find the probability that

- (iii) the chairperson, the secretary and the treasurer sit in three adjacent places, [3]
- (iv) the chairperson, the secretary and the treasurer are all separated from each other by at least one other person. [3]



Answers

1	(i) $\frac{1}{64}$ (ii) $\frac{21}{256}$ (iii) $\frac{13}{17}$	2	(i) 0.5 (ii) 0.512 (iii) 0.109
3	(i) $\frac{7}{200}$	4	(i) 0.32 (ii) 0.92 (iii) $\frac{16}{35}$ (iv) 0.15 (v) $P(A' \cap B \cap C) \leq 0.15$
5	(i) $\frac{3}{5}$ (ii) $\frac{1}{10}$ (iii) $\frac{7}{20}$	6	(i)(a) 0.058 (b) $\frac{15}{29}$ (ii)(a) 0.109272 (b) 0.313
7	(i) 0.0941 (ii) $r = 6$	8	(i)(a) 0.00599 (b) $\frac{199}{1198}$ (ii) 0.999666
9	(i) $\frac{2}{15}$ (ii) $\frac{34}{35}$ (iii) $\frac{2}{455}$ (iv) $\frac{43}{273}$ (v) $\frac{1}{7}$	10	(i) 0.24 (ii) 0.06 (iii) 0.26
11	(i) $\frac{400}{507}$ (ii) $\frac{4}{9}$ (iii) $\frac{1201}{6084}$ (iv) $\frac{1225}{6591}$	12	(i)(a) $\frac{1}{500}$ (b) $\frac{44}{125}$ (c) $\frac{29}{500}$ (ii) $\frac{71}{306}$
13	(i) 0.4 (ii) 0.185 (iii) $0.165 \leq P(A' \cap B' \cap C') \leq 0.33$	14	(i) $\frac{11}{42}$ (ii) $\frac{4}{11}$ (iii) $\frac{4}{1029}$
15	(i) 24 (ii) 576 (iii) $\frac{1}{12}$ (iv) $\frac{5}{12}$		



Binomial Distribution

1 [9740/2009/P2/11(part of)]

A fixed number, n , of cars is observed and the number of those cars that are red is denoted by R .

- (i) State, in context, two assumptions needed for R to be well modelled by a binomial distribution. [2]

Assume now that R has the distribution $B(n, p)$.

- (ii) Given that $n = 20$ and $p = 0.15$, find $P(4 \leq R < 8)$. [2]
(iii) Given that $n = 20$ and $P(R = 0 \text{ or } 1) = 0.2$, write down an equation for the value of p , and find this value numerically. [2]

Answers

- | | |
|---|--|
| 1 | (ii) 0.346
(iii) $(1-p)^{20} + 20p(1-p)^{19} = 0.2$; $p = 0.142$ |
|---|--|



Normal Distribution

1 [9740/2007/P2/8]

Chickens and Turkeys are sold by weight. The masses, in kg, of chickens and turkeys are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean Mass	Standard Deviation
Chickens	2.2	0.5
Turkeys	10.5	2.1

Chickens are sold at \$3 per kg and turkeys at \$5 per kg.

- (i) Find the probability that a randomly chosen chicken has a selling price exceeding \$7. [2]
- (ii) Find the probability of the event that both a randomly chosen chicken has a selling price exceeding \$7 and a randomly chosen turkey has a selling price exceeding \$55. [3]
- (iii) Find the probability that the total selling price of a randomly chosen chicken and a randomly chosen turkey is more than \$62. [4]
- (iv) Explain why the answer to part (iii) is greater than the answer to part (ii). [1]

2 [9740/2008/P2/11]

The random variable X has the distribution $N(50, 8^2)$. Given that $X_1 + X_2$ are two independent observations of X , find

- (i) $P(X_1 + X_2 > 120)$, [2]
- (ii) $P(X_1 > X_2 + 15)$. [3]

The random variable Y is related to X by the formula $Y = aX + b$, where a and b are constants with $a > 0$. Given that $P(Y < 74) = P(Y > 146) = 0.0668$, find the values of $E(Y)$ and $\text{Var}(Y)$, and hence find the values of a and b . [7]

3 [9740/2009/P2/9]

The thickness in cm of a mechanics textbook is a random variable with the distribution $N(2.5, 0.1^2)$.

- (a) The mean thickness of n randomly chosen mechanics textbooks is denoted by $\bar{M}$ cm. Given that $P(\bar{M} > 2.53) = 0.0668$, find the value of n . [3]

The thickness in cm of a statistics textbook is a random variable with the distribution $N(2.0, 0.08^2)$.

- (b) Calculate the probability that 21 mechanics textbooks and 24 statistics textbooks will fit onto a bookshelf of length 1m. State clearly the mean and variance of any normal distribution you use in your calculation. [3]
- (c) Calculate the probability that the total thickness of 4 statistics textbooks is less than three times the thickness of 1 mechanics textbook. State clearly the mean and variance of any normal distribution you use in your calculation. [3]
- (d) State an assumption needed for your calculation in parts (ii) and (iii). [1]



4 [9740/2010/P2/9]

In this question you should state clearly the values of the parameters of any normal distribution you use.

Over a three-month period Ken makes X minutes of peak-rate telephone calls and Y minutes of cheap-rate calls. X and Y are independent random variables with the distributions $N(180, 30^2)$ and $N(400, 60^2)$ respectively.

- (i) Find the probability that, over a three-month period, the number of minutes of cheap-rate calls made by Ken is more than twice the number of minutes of peak-rate calls. [4]

Peak-rate calls cost \$0.12 per minute and cheap-rate calls cost \$0.05 per minute.

- (ii) Find the probability that, over a three-month period, the total cost of Ken's calls is greater than \$45. [3]
(iii) Find the probability that the total cost of Ken's peak-rate calls over two independent three-month periods is greater than \$45. [3]

5 [9740/2011/P2/5]

The continuous random variable X has the distribution $N(\mu, \sigma^2)$. It is known that $P(X < 40.0) = 0.05$ and $P(X < 70.0) = 0.975$. Calculate the values of μ and σ . [4]

6 [9740/2013/P2/6]

The continuous random variable Y has the distribution $N(\mu, \sigma^2)$. It is known that $P(Y < 2a) = 0.95$ and $P(Y < a) = 0.25$. Express μ in the form of ka , where k is a constant to be determined. [4]

7 [9740/2015/P2/12]

In this question you should state clearly the values of the parameters of any normal distribution you use.

The masses in grams of apples have the distribution $N(300, 20^2)$ and the masses in grams of pears have the distribution $N(200, 15^2)$. A certain recipe requires 5 apples and 8 pears.

- (i) Find the probability that the total mass of 5 randomly chosen apples is more than 1600 grams. [2]
(ii) Find the probability that the total mass of 5 randomly chosen apples is more than the total mass of 8 randomly chosen pears. [3]

The recipe requires the apples and pears to be prepared by peeling them and removing the cores. This process reduces the mass of each apple by 15% and the mass of each pear by 10%.

- (iii) Find the probability that the total mass, after preparation, of 5 randomly chosen apples and 8 randomly chosen pears is less than 2750 grams. [4]

8 [9740/2016/P2/9(part of)]

Binomial Distribution

- (a) The random variable X has distribution $N(15, a^2)$ and $P(10 < X < 20) = 0.5$. Find the value of a . [2]
(b) The random variable Y has distribution $B(4, p)$ and $P(Y = 1) + P(Y = 2) = 0.5$. Show that $4p^4 - 12p^2 + 8p = 1$ and find the possible values of p . [4]



Answers

1	(i) 0.395 (ii) 0.160 (iii) 0.392	2	(i) 0.0385 (ii) 0.0924 ; $E(Y) = 110$, $\text{Var}(Y) = 576$; $a = 3.00$, $b = -40.0$
3	(a) 25 (b) 0.203 (c) 0.0707	4	(i) 0.681 (ii) 0.234 (iii) 0.362
5	$\mu = 53.7$, $\sigma = 8.32$	6	$\mu = 1.29a$
7	(i) 0.0127 (ii) 0.0524 (iii) 0.742	8	(a) $a = 7.41$ (b) $p = 0.599$ or 0.166



Hypothesis Testing

1 [9740/2007/P2/7]

A large number of students in a college have completed a geography project. The time, x hours, taken by a student to complete the project is noted for a random sample of 150 students. The results are summarised by

$$\sum x = 4626, \sum x^2 = 147691.$$

Find unbiased estimates for the population mean and variance. [2]

Test, at 5% significance level, whether the population mean time for student to complete the project exceeds 30 hours. [4]

State, giving a valid reason, whether any assumption about the population are needed in order for the test to be valid. [1]

2 [9740/2011/P2/10]

In a factory, the time in minutes for an employee to install an electronics component is normally distributed continuous random variable T . The standard deviation of T is 5.0 and under ordinary conditions the expected value of T is 38.0. After background music is introduced into the factory, a sample of n components is taken and the mean time taken for randomly chosen employees to install them is found to be $\bar{t}$ minutes. A test is carried out, at the 5% significance level, to determine whether the mean time taken to install a component has been reduced.

(i) State appropriate hypotheses for the test, defining and symbols you use. [2]

(ii) Given that $n = 50$, state the set of values of $\bar{t}$ for which the result of the test would be to reject the null hypothesis. [3]

(iii) It is given that instead that $\bar{t} = 37.1$ and the result of the test is that the null hypothesis is not rejected. Obtain an inequality involving n , and hence find the set of values that n can take. [4]

3 [9740/2012/P2/6]

On a remote island a zoologist measures the tail lengths of a random sample of 20 squirrels. In a species of squirrel known to her, the tail lengths have mean 14.0 cm. She carries out a test, at 5% significance level, of whether squirrels on the island have the same mean tail length as the species known to her. She assumes that the tail lengths of squirrels on the island are normally distributed with standard deviation 3.8 cm.

(i) State appropriate hypotheses for the test. [1]

The sample mean tail length is denoted by $\bar{x}$ cm.

(ii) Use an algebraic method to calculate the set of values of $\bar{x}$ for which the null hypothesis would not be rejected. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [3]

(iii) State the conclusion of the test in the case where $\bar{x} = 15.8$. [2]



4 [9740/2016/P2/6(part of)]

The Managing Director of a company knows that, some years ago, the mean age of employees was 37 years. He believes that the mean age of employees now is less than 37 years. The Company Secretary obtains a suitable sample of 80 employees in order to carry out a hypothesis test of the Managing Director's belief that the mean age of employees now is less than 37 years. You are given that the population variance of the ages is 140 years².

- (i) Write down appropriate hypotheses to test the Managing Director's belief. You are given that the result of the test, using a 5% significance level, is that the Managing Director's belief should be accepted. Determine the set of possible values of the mean age of the sample of employees. [4]
- (ii) You are given instead that the mean age of the sample of employees is 35.2 years, and that the result of a test at the $\alpha\%$ significance level is that the Managing Director's belief should not be accepted. Find the set of possible values of α . [3]

Answers

1	30.8, 33.7 ; Reject H_0 as there is sufficient evidence at 5% significance level; No assumption needed as n is large, CLT can be applied.	2	$H_0 : \mu = 38.0$ (i) $H_1 : \mu < 38.0$ (ii) $\{\bar{t} : \bar{t} \in \mathbb{R}, \bar{t} \leq 36.8\}$ (iii) $\{n : n \in \mathbb{Z}^+, n \leq 83\}$
3	(i) $H_0 : \mu = 14.0$ $H_1 : \mu \neq 14.0$ (ii) $\{\bar{x} : \bar{x} \in \mathbb{R}, 12.4 < \bar{x} < 15.6\}$ (iii) Reject H_0 as there is sufficient evidence 5% significance level	4	(i) $H_0 : \mu = 37$ $H_1 : \mu < 37$; $\{\bar{x} : \bar{x} \in \mathbb{R}, \bar{x} \leq 34.8\}$ (ii) $\{\alpha : \alpha \in \mathbb{R}, 0 < \alpha < 8.68\}$



Correlation and Linear Regression

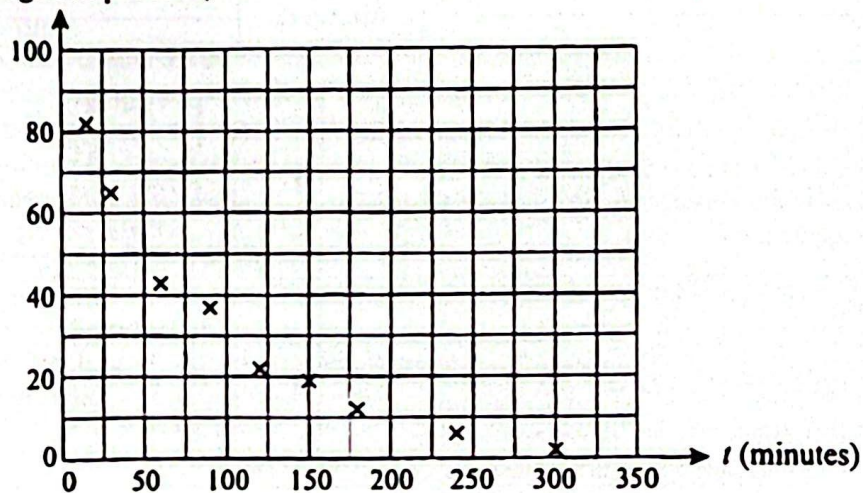
1 [9740/2007/P2/11]

Research is being carried out into how the concentration of a drug in the bloodstream varies with time, measured from when the drug is given. Observations at successive times give the data shown in the following table.

Time (t minutes)	15	30	60	90	120	150	180	240	300
Concentration (x microgram per litre)	82	65	43	37	22	19	12	6	2

It is given that the value of the product moment correlation coefficient for this data is -0.912 , correct to 3 decimal places. The scatter diagram for the data is shown below.

x (micrograms per litre)



Calculate the equation of the regression line x on t . [2]

Calculate the corresponding estimated value of x when $t = 300$, and comment on the suitability of the linear model. [2]

The variable y is defined by $y = \ln x$. For variables y and t ,

(i) calculate the product moment correlation coefficient and comment on its value, [2]

(ii) calculate the equation of the appropriate regression line. [3]

Use a regression line to give the best estimate that you can of the time when the drug concentration is 15 microgram per litre. [2]



2 [9740/2008/P2/8]

A certain metal discolours when exposed to air. To protect the metal against discolouring, it is treated with a chemical. In an experiment, different quantities, x ml, of the chemical were applied to standard samples of the metal, and the times, t hours, for the metal to discolour were measured. The results are given in the table.

x	1.2	2.0	2.7	3.8	4.8	5.6	6.9
t	2.2	4.5	5.8	7.3	7.6	9.0	9.9

- (i) Calculate the product moment correlation coefficient between x and t , and explain whether your answer suggests that a linear model is appropriate. [3]
(ii) Draw a scatter diagram for the data. [1]

One of the values of t appears to be incorrect.

- (iii) Indicate the corresponding point on your diagram by labelling it P , and explain why the scatter diagram for the remaining points may be consistent with a model of the form $t = a + b \ln x$. [2]
(iv) Omitting P , calculate least squares estimates of a and b for the model $t = a + b \ln x$. [2]
(v) Assume that the value of x at P is correct. Estimate the value of t for this value of x . [1]
(vi) Comment on the use of the model in part (iv) in predicting the value of t when $x = 8.0$. [1]

3 [9740/2009/P2/6]

The table gives the world record time, in seconds above 3 minutes 30 seconds, for running 1 mile as at 1st January in various years.

Year, x	1930	1940	1950	1960	1970	1980	1990	2000
Time, t	40.4	36.4	31.3	24.5	21.1	19.0	16.3	13.1

- (i) Draw a scatter diagram to illustrate the data. [2]
(ii) Comment on whether a linear model would be appropriate, referring both to the scatter diagram and the context of the question. [2]
(iii) Explain why in this context a quadratic model would probably not be appropriate for long term predictions. [1]
(iv) Fit a model of the form $\ln t = a + bx$ to the data, and use it to predict the world record time as at 1st January 2010. Comment on the reliability of your prediction. [3]



4 [9740/2010/P2/10(modified)]

A car is placed in a wind tunnel and the drag force F for different wind speeds v , in appropriate units, is recorded. The results are shown in the table.

v	0	4	8	12	16	20	24	28	32	36
F	0	2.5	5.1	8.8	11.2	13.6	17.6	22.0	27.8	33.9

- (i) Draw the scatter diagram for these values, labeling the axes clearly. [2]

It is thought that the drag force F can be modeled by one of the formulae

$$F = a + bv \quad \text{or} \quad F = c + dv^2$$

where a , b , c and d are constants.

- (ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between
(a) v and F ,
(b) v^2 and F . [2]
- (iii) Use your answers to parts (i) and (ii) to explain which of $F = a + bv$ or $F = c + dv^2$ is the better model. [1]
- (iv) It is required to estimate the value of v for which $F = 26.0$. Find the equation of a suitable regression line, and use it to find the required estimate. Explain why neither the regression line of v on F nor the regression line of v^2 on F should be used. [3]

5 [9740/2011/P2/8]

- (i) Sketch a scatter diagram that might be expected for the case where x and y are related approximately by $y = a + bx^2$, where a is positive b is negative. Your diagram should include 5 points, approximately equally spaces with respect to x , and with all x - and y -values positive. [2]

The table gives the values of seven observations of bivariate data, x and y .

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y	18.8	16.9	14.5	11.7	8.6	4.9	0.8

- (ii) Calculate the value of the product moment correlation coefficient, and explain why its value does not necessarily mean that the best model for the relationship between x and y is $y = c + dx$. [2]
- (iii) Explain how to use the values obtained by calculating the product moment correlation coefficient to decide, for this data, whether $y = a + bx^2$ or $y = c + dx$ is the better model. [1]
- (iv) It is desired to use the data in the table to estimate the values of y for which $x = 3.2$. Find the equation of the least-squares regression line of y on x^2 . Use your equation to calculate the desired estimate. [3]



6 [9740/2012/P2/8]

Amy is revising for a mathematics examination and takes a different practice paper each week. Her marks, $y\%$ in week x , are as follows.

Week x	1	2	3	4	5	6
Percentage mark y	38	63	67	75	71	82

- Draw a scatter diagram showing these marks. [1]
- Suggest a possible reason why one of the marks does not seem, to follow the trend. [1]
- It is desired to predict Amy's marks on future papers. Explain why, in this context, neither a linear nor a quadratic model is likely to be appropriate. [2]

It is decided to fit a model of the form $\ln(L - y) = a + bx$, where L is a suitable constant. The product moment correlation coefficient between x and $\ln(L - y)$ is denoted by r . The following table gives values of r for some possible values of L .

L	91	92	93
r		-0.929944	-0.929918

- Calculate the value of r for $L = 91$, giving your answer correct to 6 decimal places. [1]
- Use the table and your answer to part (iv) to suggest with a reason which of 91, 92, or 93 is the most appropriate value for L . [1]
- Using this value for L , calculate the values of a and b , and use them to predict the week in which Amy will obtain her first mark of at least 90%. [4]
- Give an interpretation, in context, of the value of L . [1]

7 [9740/2013/P2/10]

- Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B), and (C) below. In each case your diagram should include 6 points, approximately equally spaced with respect to x and with all x - and y -values positive. The letters a, b, c, d, e and f represents constants.

- $y = a + bx^2$, where a is positive and b is negative,
- $y = c + d \ln x$, where c is positive and d is negative,
- $y = e + \frac{f}{x}$, where e is positive and f is negative. [3]

A motoring website gives the following information about the distance travelled, y km, by a certain type of car at different speed, $x \text{ km h}^{-1}$, on a fixed amount of fuel.

Speed, x	88	96	104	112	120	128
Distance, y	148	147	144	138	126	107

- Draw a scatter diagram for these values, labelling the axes. [1]
- Explain which of the three cases in part (i) is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. [2]
- It is required to estimate the distance travelled at a speed of 110 km h^{-1} . Use the case that you identified in (iii) to find the equation of a suitable regression line, and use your equation to find the required estimate. [3]



8 [9740/2014/P2/8]

- (a) Sketch a scatter diagram that might be expected when x and y are related approximately by $y = px^2 + t$ in each of the cases (i) and (ii) below. In each case your diagram should include 6 points, approximately equally spaced with respect to x , and with all x -values positive.
- (i) p and t are both positive.
- (ii) p is negative and t is positive. [2]
- (b) The ages in months (m) and prices in dollars (P) of a random sample of ten used cars of a certain model are given in the table.

m	11	20	28	36	40	47	58	62	68	75
P	112800	102600	76500	72000	72000	69000	65800	57000	50600	47600

It is thought that the price after m months can be modelled by one of the formulae $P = am + b$, $P = c \ln m + d$, where a, b, c and d are constants.

- (i) Find, corrected to 4 decimal places, the value of the product moment correlation coefficient between
- (A) m and P ,
- (B) $\ln m$ and P . [2]
- (ii) Explain which of $P = am + b$ and $P = c \ln m + d$ is a better model and find the equation of a suitable regression line for this model. [3]
- (iii) Use the equation of your regression line to estimate the price of a car that is 50 months old. [1]

9 [9740/2015/P2/10]

In an experiment the following information was gathered about air pressure P , measured in inches mercury, at different heights above sea-level h , measured in feet.

h	2000	5000	10 000	15 000	20 000	25 000	30 000	35 000	40 000	45 000
P	27.8	24.9	20.6	16.9	13.8	11.1	8.89	7.07	5.52	4.28

- (i) Draw a scatter diagram for these values, labelling the axes. [1]
- (ii) Find, correct to 4 decimal places, the product moment correlation coefficient between
- (a) h and P ,
- (b) $\ln h$ and P ,
- (c) $\sqrt{h}$ and P . [3]
- (iii) Using the most appropriate case from part (ii), find the equation which best models air pressure at different heights. [3]
- (iv) Given that 1 metre = 3.28 feet, rewrite the equation from part (iii) so that it can be used to estimate the air pressure when the height is given in metres. [2]



10 [9740/2016/P2/8]

A website about electric motors gives information about the percentage efficiency of motors depending on their power, measured in horsepower. Xian has copied the following table for a particular type of electric motor, but he has copied one of the efficiency values wrongly.

Power, x	1	1.5	2	3	5	7.5	10	20	30	40	50
Efficiency, $y\%$	72.5	82.5	84.0	87.4	87.5	88.5	89.5	90.2	91.0	91.7	92.4

- (i) Plot a scatter diagram on graph paper for these values, labelling the axes, using a scale of 2 cm to represent 10% efficiency on the y -axis and an appropriate scale for the x -axis. On your diagram circle the point that Xian has copied wrongly. [2]

For parts (ii), (iii) and (iv) of this question you should exclude the point for which Xian has copied the efficiency wrongly.

- (ii) Explain from your scatter diagram why the relationship between x and y should not be modelled by an equation of the form $y = ax + b$. [1]
- (iii) Suppose that the relationship between x and y is modelled by an equation of the form $y = \frac{c}{x} + d$, where c and d are constants. State with a reason whether each of c and d is positive or negative. [2]
- (iv) Find the product moment correlation coefficient and the constants c and d for the model in part (iii). [3]
- (v) Use the model $y = \frac{c}{x} + d$, with the values of c and d found in (iv), to estimate the efficiency value (y) that Xian has copied wrongly. Give two reasons why you would expect this estimate to be reliable. [3]

Answers

1	$x = -0.260t + 66.2$; $x = -11.7$ (i) $r = -0.994$ (ii) $\ln x = -0.0123t + 4.62$; 155 min	2	(i) $r = 0.970$ (iv) $a = 1.42$, $b = 4.40$ (v) $t = 8.32$ hours
3	(iv) $\ln t = -0.0161x + 34.9$; 3 min 41.4s	4	(ii)(a) $r = 0.9860$ (b) $r = 0.9907$ (iv) $F = 3.20 + 0.0242v^2$; $v = 30.7$
5	(ii) $r = -0.992$ (iv) $y = 22.2 - 0.856x^2$; $y = 13.5$	6	(iv) $r = -0.929744$ (v) $L = 92$ (vi) $a = 4.10$, $b = -0.280$; Week 13
7	(iii) Case A: $r = -0.939$ (iv) $y = -0.00462x^2 + 190$; $y = 134$ km	8	(b)(i)(A) $r = -0.9470$ (B) $r = -0.9749$ (ii) $p = (-33700)\ln m + 196000$ (iii) \$64000
9	(ii)(a) $r = -0.9807$ (b) $r = -0.9748$ (c) $r = -0.9986$ (iii) $p = -0.147\sqrt{h} + 34.8$ (iv) $p = -0.266\sqrt{h} + 34.8$	10	(iii) c : negative, d : positive (iv) $r = -0.980$; $c = -17.5$, $d = 91.8$ (v) $y = 85.9$