

Solutions and Comments for H2 Mathematics P1 2026

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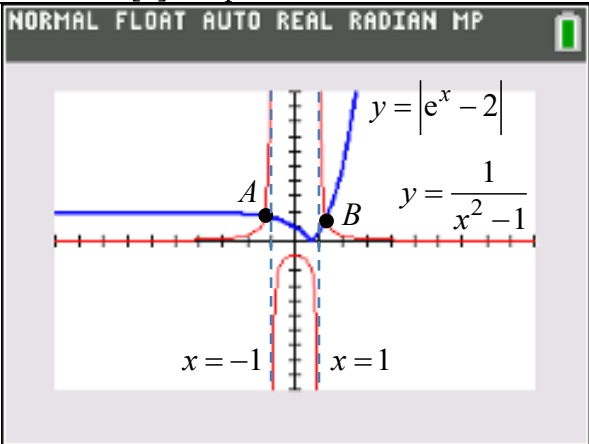
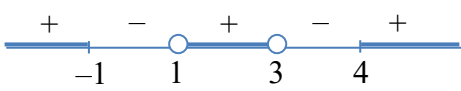
It is given that $y = \tan^{-1}(px)$, where p is a constant and $-\frac{\pi}{2} < \tan^{-1}(px) < \frac{\pi}{2}$.

(a) Show that $(1 + p^2 x^2) \frac{dy}{dx} = p$. [2]

(b) By further differentiation of the result in part (a), find the first two non-zero terms in the Maclaurin series for y . [4]

1	Solutions [6] Maclaurin Series.	
(a)	$y = \tan^{-1}(px)$ Differentiate with respect to x , $\frac{dy}{dx} = \frac{p}{1 + p^2 x^2}$ $(1 + p^2 x^2) \frac{dy}{dx} = p \quad (\text{Shown})$ <u>Alternative</u> $\tan y = px$ Differentiate with respect to x , $\sec^2 y \frac{dy}{dx} = p$ $(1 + \tan^2 y) \frac{dy}{dx} = p$ $(1 + p^2 x^2) \frac{dy}{dx} = p \quad (\text{Shown})$	Generally ok.
(b)	Differentiate with respect to x again, $(1 + p^2 x^2) \frac{d^2 y}{dx^2} + (2p^2 x) \frac{dy}{dx} = 0.$ Differentiate with respect to x again, $(1 + p^2 x^2) \frac{d^3 y}{dx^3} + (2p^2 x) \frac{d^2 y}{dx^2} + (2p^2 x) \frac{d^2 y}{dx^2} + (2p^2) \frac{dy}{dx} = 0$ At $x = 0$, $y = \tan^{-1}(0) = 0, \frac{dy}{dx} = p, \frac{d^2 y}{dx^2} = 0, \frac{d^3 y}{dx^3} = -2p^3.$ By Maclaurin's Theorem, $y = 0 + px + \frac{0}{2!} x^2 + \frac{-2p^3}{3!} x^3 + \dots$ $\approx px - \frac{p^3}{3} x^3$	Generally well done. Common errors involve mistakes in the implicit differentiation steps resulting in not having $\frac{d^2 y}{dx^2} = 0$ and thus the first 2 non-zero terms involve the terms in x and x^2 instead which is wrong. Presentation error: The first 2 non-zero terms should be px and $\frac{p^3}{3} x^3$, not p and $-\frac{p^3}{3}$.

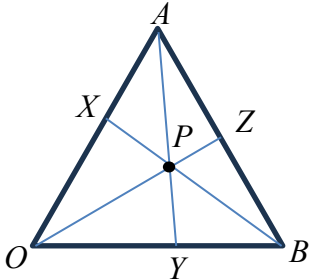
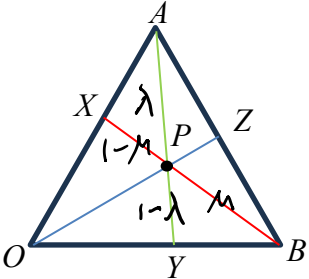
- 2 (a) Solve the inequality $|e^x - 2| > \frac{1}{x^2 - 1}$ using a graphical approach. [3]
- (b) Without the use of a graphing calculator, solve the inequality $\frac{x+2}{x-1} \geq \frac{2}{x-3}$. [3]

2	Solutions [6] Inequalities.	
(a)	 $A = (-1.2581, 1.7158)$ and $B = (1.2770, 1.5857)$ From the graphs, for $e^x - 2 > \frac{1}{x^2 - 1}$, $x < -1.26$ or $-1 < x < 1$ or $x > 1.28$ (3 s.f.)	Not well attempted as students failed to see the whole graph and hence missed out the answer, $-1 < x < 1$. Students need to understand the difference between the use of “or” and “and” in the final answer.
(b)	$y = \frac{1}{x^2 - 1}$ $\frac{x+2}{x-1} - \frac{2}{x-3} \geq 0$ $\frac{(x+2)(x-3) - 2(x-1)}{(x-1)(x-3)} \geq 0$ $\frac{x^2 - 3x - 4}{(x-1)(x-3)} \geq 0$ $\frac{(x+1)(x-4)}{(x-1)(x-3)} \geq 0$  $\therefore x \leq -1$ or $1 < x < 3$ or $x \geq 4$	Some students went ahead to do cross multiplication and were penalised. Students need to learn to read more carefully as there are several different combinations of the solution. Students need to understand the difference between the use “or” and “and” in the final answer.

- 3 A curve C has equation $y - 2\sqrt{3}x + 3e^{-y} = 0$, for $x > 0$. The normal to C at a point P has positive gradient and makes an angle of 30° with the x -axis. Find the exact equation of this normal. [6]

3	Solutions [6] Implicit Differentiation	
	Differentiate $y - 2\sqrt{3}x + 3e^{-y} = 0$ implicitly with respect to x, $\frac{dy}{dx} - 2\sqrt{3} + 3(e^{-y})\left(-\frac{dy}{dx}\right) = 0$ $(1 - 3e^{-y})\left(\frac{dy}{dx}\right) = 2\sqrt{3}$ $\frac{dy}{dx} = \frac{2\sqrt{3}}{1 - 3e^{-y}}$ Gradient of normal = $-\frac{1 - 3e^{-y}}{2\sqrt{3}}$. $-\frac{1 - 3e^{-y}}{2\sqrt{3}} = \tan 30^\circ$ $-1 + 3e^{-y} = (2\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)$ $e^{-y} = 1$ $y = 0$ When $y = 0$, $-2\sqrt{3}x + 3 = 0$ $x = \frac{\sqrt{3}}{2}$ $\therefore$ Coordinates of P is $\left(\frac{\sqrt{3}}{2}, 0\right)$ Exact equation of the normal to C at P: $y - 0 = \frac{1}{\sqrt{3}}\left(x - \frac{\sqrt{3}}{2}\right)$ $y = \frac{x}{\sqrt{3}} - \frac{1}{2}.$	Not done well. Most are able to do implicit differentiation with a handful unsuccessful in differentiating e^{-y} with respect to x. Some did not understand that $\tan 30^\circ$ gives the gradient of the normal and simply ignored the given info while many actually equated it to $\frac{dy}{dx}$ which is the gradient of the tangent instead. Some assumed $y = 0$ due to the information given as “made an angle with the x-axis”. While coincidentally point P is on the x-axis, this assumption cannot be given credit. Some still are not able to produce the correct equation of a normal, mainly still using the expression for $\frac{dy}{dx}$ for its gradient instead of the value of $\frac{dy}{dx}$ at point P.

- 4 With reference to the origin O , the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are both non-zero and non-parallel.
- The point X lies on OA such that $OX:XA=2:1$. The point Y lies on OB such that $OY:YB=3:2$. The lines AY and BX intersect at the point P .
- (a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [5]
- (b) Point Z lies on AB such that $4AZ = 3BZ$. Express $\overrightarrow{OZ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]
- (c) Show that O, P and Z are collinear. [2]

4	Solutions [8] Abstract Vectors	
(a)	$\overrightarrow{OX} = \frac{2}{3}\mathbf{a}$ and $\overrightarrow{OY} = \frac{3}{5}\mathbf{b}$ Method 1: $l_{AY}: \mathbf{r} = \mathbf{a} + \lambda \overrightarrow{AY}$ $= \mathbf{a} + \lambda \left(\frac{3}{5}\mathbf{b} - \mathbf{a} \right), \lambda \in \mathbb{R}$  $\overrightarrow{OP} = \mathbf{a} + \lambda \left(\frac{3}{5}\mathbf{b} - \mathbf{a} \right)$, for some $\lambda \in \mathbb{R}$ $= (1-\lambda)\mathbf{a} + \frac{3}{5}\lambda\mathbf{b}$ $l_{BX}: \mathbf{r} = \mathbf{b} + \mu \overrightarrow{BX}$ $= \mathbf{b} + \mu \left(\frac{2}{3}\mathbf{a} - \mathbf{b} \right), \mu \in \mathbb{R}$ $\overrightarrow{OP} = \mathbf{b} + \mu \left(\frac{2}{3}\mathbf{a} - \mathbf{b} \right)$, for some $\mu \in \mathbb{R}$ $= \frac{2}{3}\mu\mathbf{a} + (1-\mu)\mathbf{b}$	Majority was able to use the correct approach, but there was also many who were unable to start the question. The most commonly seen method was the intersection of lines. Some used the ratio theorem method which was slightly more complicated with $\lambda, 1-\lambda, \mu, 1-\mu$.  Other mistake: Equating $\lambda \overrightarrow{AY}$ and $\mu \overrightarrow{BX}$, instead of the vector equations of the 2 lines. $\lambda \overrightarrow{AY}$ and $\mu \overrightarrow{BX}$ are directional vectors (which are not localised), but the 2 lines are!
	Method 2: $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ $= \mathbf{a} + \lambda \overrightarrow{AY}$ $= \mathbf{a} + \lambda \left(\frac{3}{5}\mathbf{b} - \mathbf{a} \right)$ $= (1-\lambda)\mathbf{a} + \frac{3}{5}\lambda\mathbf{b}$	

	$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$ $= \mathbf{b} + \mu \overrightarrow{BX}$ $= \mathbf{b} + \mu \left(\frac{2}{3} \mathbf{a} - \mathbf{b} \right)$ $= \frac{2}{3} \mu \mathbf{a} + (1 - \mu) \mathbf{b}$		
	As $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and non-zero, comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$, $1 - \lambda = \frac{2}{3} \mu \quad \text{---(1)}$ $\frac{3}{5} \lambda = 1 - \mu \quad \text{---(2)}$ Solving (1) & (2) using GC, $\lambda = \frac{5}{9} \text{ and } \mu = \frac{2}{3}$ $\overrightarrow{OP} = (1 - \lambda) \mathbf{a} + \frac{3}{5} \lambda \mathbf{b}$ $= \left(1 - \frac{5}{9} \right) \mathbf{a} + \frac{3}{5} \left(\frac{5}{9} \right) \mathbf{b}$ $= \frac{4}{9} \mathbf{a} + \frac{1}{3} \mathbf{b}$	After obtaining a correct equation involving λ and μ, students naturally solved the unknowns by comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$. However, this comparison worked because $\mathbf{a}$ and $\mathbf{b}$ were non-parallel and non-zero. This condition must be stated! (Less than 5 students stated this!)	
(b)	$A \quad \overset{\text{3}}{\quad} \quad Z \quad \overset{\text{4}}{\quad} \quad B$ $4AZ = 3BZ$ $\overrightarrow{OZ} = \frac{4}{7} \overrightarrow{OA} + \frac{3}{7} \overrightarrow{OB}$ $= \frac{4}{7} \mathbf{a} + \frac{3}{7} \mathbf{b}$	Students who did not get this right made mistake with the ratio. $4AZ = 3BZ$ $\frac{AZ}{BZ} = \frac{3}{4} \text{ (not } \frac{4}{3} \text{)}$	
(c)	$\overrightarrow{OP} = \frac{4}{9} \mathbf{a} + \frac{1}{3} \mathbf{b}$ $= \frac{7}{9} \left(\frac{4}{7} \mathbf{a} + \frac{3}{7} \mathbf{b} \right)$ $= \frac{7}{9} \overrightarrow{OZ}$	$\overrightarrow{OZ} = \frac{4}{7} \mathbf{a} + \frac{3}{7} \mathbf{b}$ $= \frac{9}{7} \left(\frac{4}{9} \mathbf{a} + \frac{1}{3} \mathbf{b} \right)$ $= \frac{9}{7} \overrightarrow{OP}$	Students generally understood that collinearity required the parallel vectors condition, but few students did not state the common point condition. $\overrightarrow{PZ} = \dots$ $= \frac{8}{63} \mathbf{a} + \frac{2}{21} \mathbf{b}$ $= \frac{2}{7} \overrightarrow{OP}$ accepted
	$\overrightarrow{OP} \parallel \overrightarrow{OZ}$ Since O is the common point, O, P and Z are collinear.		

- 5 (a) By writing $\cos 3x$ as $\cos(x+2x)$ and applying addition formula, show that

$$\cos 3x = \cos x(4\cos^2 x - 3). \quad [2]$$

- (b) By using integration by parts, or otherwise, find $\int \tan x \sin 3x \, dx$. [5]

5	Solutions [7] Integration Techniques (Trigo, by parts)	
(a)	$\begin{aligned} \cos 3x &= \cos(x+2x) \\ &= \cos x \cos 2x - \sin x \sin 2x \\ &= \cos x(2\cos^2 x - 1) - \sin x(2\sin x \cos x) \\ &= 2\cos^3 x - \cos x - 2\sin^2 x \cos x \\ &= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x \\ &= 4\cos^3 x - 3\cos x \\ &= \cos x(4\cos^2 x - 3). \quad (\text{Shown}) \end{aligned}$	Almost all could show the result.
(b)	$\begin{aligned} &\int \tan x \sin 3x \, dx \\ &= -\frac{1}{3} \cos 3x \tan x \\ &\quad - \int \sec^2 x \left(-\frac{1}{3} \cos 3x\right) dx \\ &= -\frac{1}{3} \cos 3x \tan x + \frac{1}{3} \int \frac{1}{\cos^2 x} [\cos x(4\cos^2 x - 3)] dx \\ &= -\frac{1}{3} \cos 3x \tan x + \frac{1}{3} \int 4\cos x - 3\sec x \, dx \\ &= -\frac{1}{3} \cos 3x \tan x + \frac{1}{3} (4\sin x - 3\ln \sec x + \tan x) + c \\ &= -\frac{1}{3} \cos 3x \tan x + \frac{4}{3} \sin x - \ln \sec x + \tan x + c. \end{aligned}$ $\begin{aligned} u &= \tan x, \quad \frac{dv}{dx} = \sin 3x \\ \frac{du}{dx} &= \sec^2 x, \quad v = -\frac{1}{3} \cos 3x \end{aligned}$	For integration by parts here, u must be $\tan x$. If $u = \sin 3x$, the integral will become too complicated to solve ($\int 3\ln \sec x \cos 3x \, dx$). As shown in the solution, when $u = \tan x$, $\cos 3x$ will show up in the second integral. This will allow part (a) result to be used. Some students took the wrong move to integrate by parts again to end up with a complicated integral. A common mistake by almost all students was leaving out the modulus sign in $\ln \sec x + \tan x$. In MF27, modulus sign is not required because there is a condition $x < \frac{\pi}{2}$. Other common mistakes: 1. $\int \sec x \, dx = \sec x \tan x$ 2. $du = \sec^2 x$ instead of $\frac{du}{dx} = \sec^2 x$.

- 6 A colony of bacteria is cultivated in a tank under experimental conditions. The amount of bacteria is y (in grams), at time t (in days) after the start of the experiment. The total growth rate of the bacteria is determined by the following two factors.

- The bacteria reproduce naturally and grow a rate directly proportional to the amount of bacteria present.
- An automated feeder continuously supplies a liquid nutrient solution to the tank. This nutrient solution stimulates an additional growth rate which is directly proportional to the time taken from the start of the experiment.

Initially, there is 3 grams of bacteria, and the total growth rate of the bacteria is 6 grams per day. After 1 day, the additional growth rate, due to the liquid nutrient, is 3 grams per day.

- (a) Show that $\frac{dy}{dt} = 2y + 3t$. [2]
- (b) By substituting $u = 2y + 3t$, show that the differential equation can be written as $\frac{du}{dt} = 2u + 3$. [2]
- (c) Find u in terms of t and hence y in terms of t . [5]

6	Solutions [9] Differential Equations.	
(a)	Let $\frac{dy_n}{dt}$ be the natural growth rate of the bacteria and let $\frac{dy_{add}}{dt}$ be the additional growth rate of the bacteria $\frac{dy_n}{dt} \propto y \Rightarrow \frac{dy_n}{dt} = ay$ and $\frac{dy_{add}}{dt} \propto t \Rightarrow \frac{dy_{add}}{dt} = bt$ where a, b are positive constants. $\frac{dy}{dt} = \frac{dy_n}{dt} + \frac{dy_{add}}{dt} = ay + bt$ When $t = 0, y = 3, \frac{dy}{dt} = 6 \Rightarrow a = 2$. When $t = 1, \frac{dy_{add}}{dt} = 3 \Rightarrow b = 3$. Therefore, $\frac{dy}{dt} = 2y + 3t$. (Shown)	Quite many students did not differentiate $\frac{dy}{dt}$ from $\frac{dy_n}{dt}$ and $\frac{dy_{add}}{dt}$. The results ' $2y + 3t$ ' are already provided in the question; there is a need to explain clearly how the '2' and '3' came about.
(b)	Given $u = 2y + 3t$, differentiate with respect to t, $\frac{du}{dt} = 2 \frac{dy}{dt} + 3$ $\frac{du}{dt} = 2(2y + 3t) + 3$ $\frac{du}{dt} = 2u + 3$. (Shown)	Generally well done. Though some cannot handle differentiation with 3 variables, e.g. when differentiating wrt y from $u = 2y + 3t$, they get instead of $\frac{du}{dy} = 2 + 3 \frac{dt}{dy}$.
(c)	Solving	Some students do not know how to apply the technique to 'separate the variables'.

$\frac{du}{dt} = 2u + 3$ $\int \frac{1}{2u+3} du = \int 1 dt$ $\frac{1}{2} \ln 2u+3 = t + c$ $\ln 2u+3 = 2t + d \text{ where } d = 2c$ $ 2u+3 = e^{2t+d}$ $2u+3 = \pm e^{2t+d}$ $2u+3 = Ae^{2t} \text{ where } A = \pm e^d$ $2u = Ae^{2t} - 3$ $u = \frac{1}{2}(Ae^{2t} - 3)$ $2y + 3t = \frac{1}{2}(Ae^{2t} - 3)$ When $y = 3, t = 0$. $6 + 0 = \frac{1}{2}(Ae^0 - 3) \Rightarrow A = 15.$ Therefore, $u = \frac{1}{2}(15e^{2t} - 3)$ and $y = \frac{1}{2}\left(\frac{1}{2}(15e^{2t} - 3) - 3t\right) = \frac{15}{4}e^{2t} - \frac{3}{2}t - \frac{3}{4}.$	A few did not include modulus (nor did not explain why modulus is not necessary). Some stop at getting the general solution but fail to see that we can apply the initial conditions to further obtain the particular solutions (with unknown constants A solved). Some did not express u in terms of t to answer the question.
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- 7 Let a and b be fixed positive real numbers. A variable line C passing through (a, b) with varying gradient m has equation $y - b = m(x - a)$, where $m < 0$. The line C cuts the x -axis and y -axis at the points $P(p, 0)$ and $Q(0, q)$ respectively.

(a) (i) Show that A , the area of triangle OPQ , where O is the origin, is given by

$$A = \frac{1}{2} \left(2ab - a^2m - \frac{b^2}{m} \right) \text{ units}^2. \quad [2]$$

(ii) Using differentiation, find the value of m that gives the minimum A . [3]

- (b) As m varies, the line C rotates about point (a, b) causing the points P and Q to move along the x - and y - axes, respectively. At the instant when $m = -1$, point P is moving to the right, where the rate of change of p is 1 unit per second and point Q is moving downwards. At this instant, find the rate of change of q , in units per second, leaving your answer in terms of a and b . [4]

7	Solutions [9] Applications of differentiation, Min-Max, Rates of change.	
(a)(i)	At point P, $-b = m(x - a)$ $p = a - \frac{b}{m}$ $A = \frac{1}{2}(OP)(OQ)$ $= \frac{1}{2} \left(a - \frac{b}{m} \right) (b - am)$ $= \frac{1}{2} \left(ab - a^2m - \frac{b^2}{m} + ab \right)$ $= \frac{1}{2} \left(2ab - a^2m - \frac{b^2}{m} \right) \text{ units}^2.$ At point Q, $q = b - am$	Mostly well done. Though some did wrong substitutions, e.g. for point $P(p, 0)$, 'p' and '0' are substituted into a and b rather than x and y. 😊
(a)(ii)	Differentiate A with respect to m, $\frac{dA}{dm} = -\frac{a^2}{2} + \frac{b^2}{2m^2}$ Let $\frac{dA}{dm} = 0$, $-\frac{a^2}{2} + \frac{b^2}{2m^2} = 0$ $m^2 = \frac{b^2}{a^2}$ $m = \pm \frac{b}{a}$ $m = -\frac{b}{a} \quad \left(\text{reject } \frac{b}{a} \text{ because } m < 0, \frac{b}{a} > 0 \right)$	Majority have idea how to go about answering the question but either made mistakes in differentiation or did not use the condition given in question to identify the correct value of m. One serious mistake made by quite a few students is that instead of finding the value of m which gives stationary A value and then showing that A is a minimum, the students use

	$\frac{d^2 A}{dm^2} = -\frac{b^2}{m^3} > 0 \text{ (as } m < 0, b^2 > 0 \text{)}$ By second derivative test, A attains a minimum at $m = -\frac{b}{a}$.	second derivative test to decide which is the correct value of m .
(b)	At Q on C, At P on C, $q = b - am \quad \text{and} \quad p = a - \frac{b}{m}$ $\frac{dq}{dm} = -a \quad \quad \quad \frac{dp}{dm} = \frac{b}{m^2}$ As $\frac{dp}{dt} = 1$ and $m = -1$, by chain rule, $\frac{dp}{dm} \frac{dm}{dt} = \frac{dp}{dt}$ $\left(\frac{b}{m^2}\right) \left(\frac{dm}{dt}\right) = 1$ $\frac{dm}{dt} = \frac{(-1)^2}{b} = \frac{1}{b}$ $\frac{dq}{dm} \frac{dm}{dt} = \frac{dq}{dt}$ $\frac{dq}{dt} = (-a) \left(\frac{1}{b}\right) = -\frac{a}{b} \text{ units per second.}$	Badly done. There are quite a few challenges here: - The need to link various results together. - Recognising which of a, b and m are constants and which are variables when performing differentiation. - Dealing with p and q rather than x and y.
	<u>Alternative</u> Implicitly differentiate with respect to t, $p = a - \frac{b}{m} \quad \quad \quad q = b - am$ $\frac{dp}{dt} = \left(\frac{b}{m^2}\right) \frac{dm}{dt} \quad \text{and} \quad \frac{dq}{dt} = -a \frac{dm}{dt}$ As $\frac{dp}{dt} = 1$ and $m = -1$, $1 = \left(\frac{b}{1^2}\right) \frac{dm}{dt}$ $\frac{dm}{dt} = \frac{1}{b}$ $\frac{dq}{dt} = -a \left(\frac{1}{b}\right) = -\frac{a}{b} \text{ units per second.}$	

- 8 (a)** A sequence $u_1, u_2, u_3, \dots$ is such that $u_1 = 5$ and $u_{n+1} = 3u_n + 2n$.

It is known that the n th term of this sequence is given by

$$u_n = a(3^n) + bn + c,$$

where a, b and c are constants.

Find the constants a, b and c .

[4]

- (b) (i)** State the value of k such that $\sum_{r=1}^{2n} \frac{1}{r} = \sum_{r=1}^n \left(\frac{1}{2r+k} + \frac{1}{2r} \right)$. [1]

- (ii)** Hence, show that $\sum_{r=n+1}^{2n} \frac{1}{r} = \sum_{r=1}^n \left(\frac{1}{2r-1} - \frac{1}{2r} \right)$. [2]

- (iii)** Write down the value of $\lim_{n \rightarrow \infty} \sum_{r=n+1}^{2n} \frac{1}{r}$. [1]

8	Solutions [8] Recurrence Relations, Summation.	
(a)	Using recurrence relation with $u_1 = 5$, $u_2 = 3u_1 + 2(1) = 17$ $u_3 = 3u_2 + 2(2) = 55$ Using the given nth term expression, $u_1 = a(3^1) + b(1) + c = 3a + b + c = 5 \text{ --- (1)}$ $u_2 = a(3^2) + b(2) + c = 9a + 2b + c = 17 \text{ --- (2)}$ $u_3 = a(3^3) + b(3) + c = 27a + 3b + c = 55 \text{ --- (3)}$ Solving the system of equations using GC, $a = \frac{13}{6}, b = -1, c = -\frac{1}{2}$	Many got u_2 and u_3 wrong due to confusion in substituting the value of n giving results like $u_{1+1} = 3u_1 + 2(\text{X}) = 19$, subsequently equations 2 and 3 are wrong. Some tried to find the general term through repeated substitution but are all unsuccessful in their attempts.
(b)(i)	$k = -1$ Because, $\sum_{r=1}^n \left(\frac{1}{2r-1} + \frac{1}{2r} \right) = \left(\frac{1}{1} + \frac{1}{2} \right) + \left(\frac{1}{3} + \frac{1}{4} \right) + \dots + \left(\frac{1}{2n-1} + \frac{1}{2n} \right)$ $= \sum_{r=1}^{2n} \frac{1}{r}.$	Very limited successful attempts here with most responses giving no response at all. There are some good guesses as to the value of k.

(b)(ii)	$\sum_{r=n+1}^{2n} \frac{1}{r} = \sum_{r=1}^{2n} \frac{1}{r} - \sum_{r=1}^n \frac{1}{r}$ $= \sum_{r=1}^n \left(\frac{1}{2r-1} + \frac{1}{2r} \right) - \sum_{r=1}^n \frac{1}{r} \quad (\text{from (b)})$ $= \sum_{r=1}^n \left(\frac{1}{2r-1} + \frac{1}{2r} - \frac{1}{r} \right)$ $= \sum_{r=1}^n \left(\frac{1}{2r-1} - \frac{1}{2r} \right) \quad (\text{shown})$	Many non-responses here as well. Many of those who attempted managed to get the first line of the solution. Some tried to apply the result from part (b) but are unsuccessful due to not knowing the value of k. Some responses tried using substitution but were all unsuccessful with this method.
(b)(iii)	$\lim_{n \rightarrow \infty} \sum_{r=n+1}^{2n} \frac{1}{r} = \sum_{r=1}^{\infty} \left(\frac{1}{2r-1} - \frac{1}{2r} \right)$ $= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{r+1}}{r} + \dots$ $= \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r}$ $= \ln(1+1) \quad (\text{by MF27, Maclaurin Series})$ $= \ln 2.$	Most responses tried arguing that the sum goes to 0 as the individual terms goes to 0 while n tends to infinity. However, a simple check on the sum will conclude that it is not a possible answer as we are adding non-zero terms. Only a handful realises that the sum is a Maclaurin series for $\ln(1+x)$ where $x = 1$. Some responses include using the GC to sum to a very large number to achieve enough accuracy for 3 significant figures ending up as 0.693.

9 (a) The function f is defined by $f : x \mapsto 4 + e^{-x}$, where $0 < x \leq 4$.

(i) Show that f^{-1} exists. [1]

(ii) Find $f^{-1}(x)$ and the domain of f^{-1} . [3]

(b) The function g is defined by

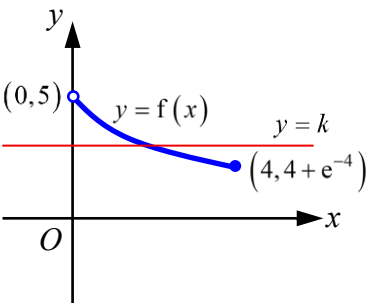
$$g : x \mapsto f(x), \text{ for } 0 < x \leq 4,$$

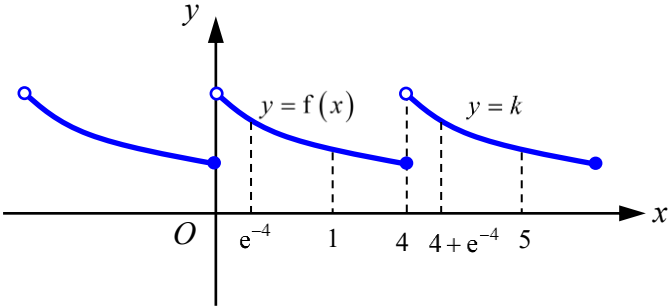
and that $g(x) = g(x+4)$ for all real values of x .

(i) Explain why the composite function g^2 exists. [1]

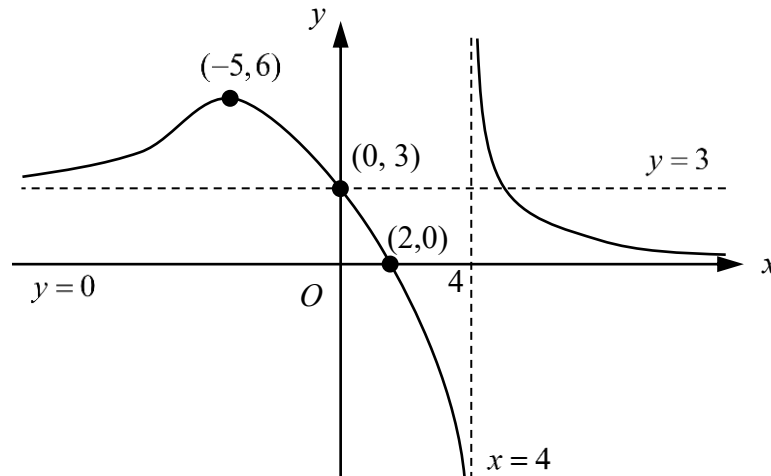
(ii) Find the exact value of $g^2(2)$. [2]

(iii) State the range of the composite function g^2 . [2]

9	Solutions [9] Functions.	
(a)(i)	 Any horizontal line $y = k$ cuts the graph of $y = f(x)$ at most once. Hence, f is one-one and f^{-1} exists. (shown)	Not well done though a standard question. How students argue often suggests that horizontal line test merely require one line to not cut the graph of $f(x)$ more than once it that's not true. Students need to be careful in how they phrase their argument to bring across the correct meaning. Many also did not show a fully correct graph for function f.
(a)(ii)	Let $y = f(x)$ $y = 4 + e^{-x}$ $e^{-x} = y - 4 \quad \text{--- (*)}$ $x = -\ln(y - 4)$ $\therefore f^{-1}(x) = -\ln(x - 4)$ and $D_{f^{-1}} = [4 + e^{-4}, 5)$	Generally well done. Some students made mistakes in applying law of logarithm. Some included modulus for 'y - 4' which is not necessary (see (*)). There are also errors with inclusion/exclusion of end-points for the domain.

(b)(i)	$R_g = [4 + e^{-4}, 5) \subset \mathbb{R} = D_g$, thus g^2 exists.	Many students did not realise that $D_g = \mathbb{R}$.
(b)(ii)	$ \begin{aligned} g^2(2) &= gg(2) \\ &= g(4 + e^{-2}) \\ &= g(e^{-2}) \\ &= 4 + e^{-e^{-2}}. \end{aligned} $	Many have idea how to go about the mapping of values for composite function but did not know the need to reduce $4 + e^{-2}$ to e^{-2} before performing the second round of mapping.
(b)(iii)	$ \begin{aligned} D_g = \mathbb{R} &\xrightarrow{g} R_g = [4 + e^{-4}, 5) \xrightarrow{g} \left(4 + e^{-(5-4)}, 4 + e^{-(4+e^{-4}-4)}\right) \\ &= (4 + e^{-1}, 4 + e^{-e^{-4}}] \\ \therefore R_{g^2} &= (4 + e^{-1}, 4 + e^{-e^{-4}}] \text{ or } (4.37, 4.98] \text{ (3 s.f.)} \end{aligned} $ 	Same problem mainly with the mapping as in (b)(ii)

- 10 (a) The diagram shows the graph of $y = f(x)$. The graph intersects the axes at $(2,0)$ and $(0,3)$. It has a turning point at $(-5, 6)$ and asymptotes with the equations $x = 4$, $y = 0$ and $y = 3$.



On separate diagrams, sketch each of the following graphs. For each graph, state the equation of any asymptotes, the coordinates of any points where the curve crosses the axes and the coordinates of any turning points.

(i) $y = f(|x|)$, [2]

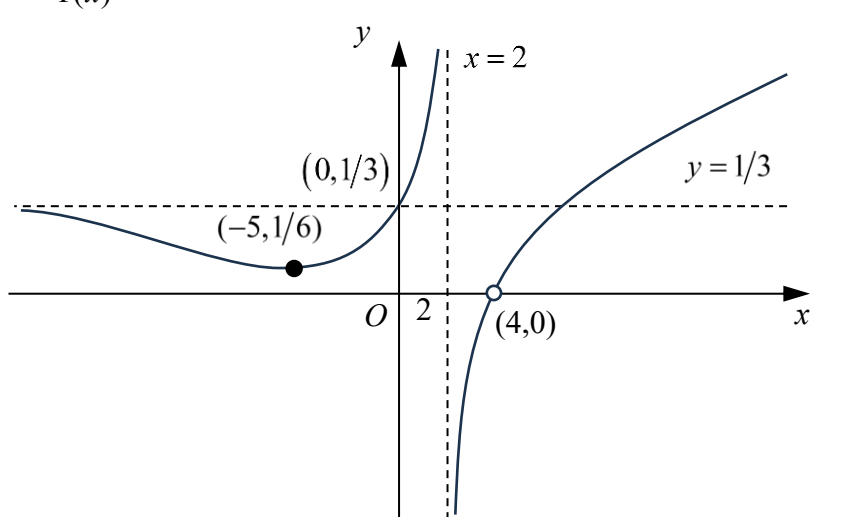
(ii) $y = \frac{1}{f(x)}$. [3]

- (b) Express $\frac{9x+5}{3x+2}$ in the form $a - \frac{b}{3x+2}$ where a and b are positive constants.

Hence, describe a sequence of transformations which transform the graph of $y = \frac{9x+5}{3x+2}$

to the graph of $y = \frac{1}{x}$. [5]

10	Solutions [10] Graphing Transformations.	
(a)(i)	$y = f(x)$	Generally ok. Some students did not ensure that for the answer, the point (3, 0) should be 'sharp' and not 'rounded' (like shape of maximum point). Some students mistakenly showed the curve bending towards the line $y = 3$ instead of the x-axis. There should be proper labelling of all x, y-intercepts, vertical and horizontal asymptotes.

(a)(ii)	$y = \frac{1}{f(x)}$ 	Common mistakes for this part include: • Not labelling the minimum point, axial intercepts, vertical and horizontal asymptotes correctly • Wrongly indicating the graph tending towards the line $y = \frac{1}{3}$ as x tends towards positive infinity • Not showing the correct behaviour for the curve from $x = 0$ to $x = 4$.
(b)	$\frac{9x+5}{3x+2} = 3 - \frac{1}{3x+2}$ $y = 3 - \frac{1}{3x+2} \xrightarrow{y \rightarrow y+3} y = -\frac{1}{3x+2}$ $\xrightarrow{y \rightarrow -y} y = \frac{1}{3x+2}$ $\xrightarrow{x \rightarrow \frac{x}{3}} y = \frac{1}{x+2}$ $\xrightarrow{x \rightarrow x-2} y = \frac{1}{x}$ Sequence of transformations: 1. Translate 3 units in the negative y-direction 2. Reflect about the x-axis 3. Scale by factor 3 parallel to x-axis 4. Translate 2 units in the positive x-direction	Students were able to show good skills in determining the necessary sequence of transformations as needed. However, care needs to be taken for the correct description of all the transformation as needed: • Translation (not shift or left/right move) of a units in positive/negative x or y-axis direction • Scaling of factor a (not a unit) parallel to x or y-axis • Reflection (not flip or mirror) about x or y-axis.
	<u>Alternatively</u> $y = 3 - \frac{1}{3x+2} \xrightarrow{y \rightarrow -y} y = -3 + \frac{1}{3x+2}$ $\xrightarrow{y \rightarrow y-3} y = \frac{1}{3x+2}$ $\xrightarrow{x \rightarrow x - \frac{2}{3}} y = \frac{1}{3x}$ $\xrightarrow{x \rightarrow \frac{x}{3}} y = \frac{1}{x}$ Sequence of transformations: 1. Reflect about the x-axis 2. Translate 3 units in the positive y-direction 3. Translate $\frac{2}{3}$ units in the positive x-direction 4. Scale by factor 3 parallel to x-axis.	Some students mistakenly transform the graph of $y = \frac{1}{x}$ to $y = 3 - \frac{1}{3x+2}$ instead. To reduce the curve from $y = \frac{1}{3x+2}$ to $y = \frac{1}{3x}$, the correct transformation should involve replacing 'x' with '$x - \frac{2}{3}$', and not '$x - 2$' and thus it should be a translation of $\frac{2}{3}$ (not 2) units in positive x-direction.

11 Do not use a calculator in answering this question.**(a)** The function f is defined by $f(z) = (2-i)z^2 + 10z + 20 + 10i$.**(i)** Find the roots of $f(z) = 0$. [3]The function p is defined by $p(z) = z^4 + az^3 + bz^2 + cz + d$, where a, b, c and d are real numbers. It is given that $f(z)$ is a factor of $p(z)$.**(ii)** Without any further calculations, state all the roots of $p(z) = 0$, explaining your answer clearly. [2]**(b)** Solve the simultaneous equations, giving z and w in the form $a + bi$, where a and b are real numbers.

$$z * w = 8 - 20i,$$

$$w - 2z = 10.$$

[5]

11	Solutions [10] Complex Numbers.	
(a)(i)	$(2-i)z^2 + 10z + 20 + 10i = 0$ In the above equation, $a = 2-i$, $b = 10$, $c = 20 + 10i$ $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-10 \pm \sqrt{(10)^2 - 4(2-i)(20+10i)}}{2(2-i)}$ $= \frac{-10 \pm \sqrt{100 - 4(40 + 20i - 20i + 10)}}{2(2-i)}$ $= \frac{-10 \pm \sqrt{100 - 200}}{2(2-i)}$ $= \frac{-10 \pm \sqrt{-100}}{2(2-i)}$ $= \frac{-10 \pm 10i}{2(2-i)}$ $= \frac{-5 \pm 5i}{2-i}$ $z = \frac{-5+5i}{2-i} \times \frac{2+i}{2+i} \quad \text{or} \quad z = \frac{-5-5i}{2-i} \times \frac{2+i}{2+i}$ $= \frac{-10-5i+10i-5}{2^2+1} \quad = \frac{-10-5i-10i+5}{2^2+1}$ $= \frac{-15+5i}{5} \quad = \frac{-5-15i}{5}$ $z = -3+i \quad \text{or} \quad z = -1-3i$ The roots of $f(z) = 0$ are $z = -3+i$ and $z = -1-3i$.	Badly attempted. Some students substituted wrong values into the equation. Some students used the GC to determine the answer. Since question stated no use of calculator, all workings need to be shown clearly including the multiplication of the conjugate of the denominator.

(a) (ii)	Since all the coefficients of $p(z)=0$ are real, and since $f(z)$ is a factor of $p(z)$, it means that $z=-3+i$ and $z=-1-3i$ are roots of $p(z)=0$, it means that $z=-3-i$ and $z=-1+3i$ are also roots. Hence, the roots of $p(z)=0$ are $z=-3+i, -3-i, -1+3i$ and $-1-3i$.	Students failed to share that all coefficients of $p(z)=0$ are real and hence were penalised.
(b)	$z * w = 8 - 20i \quad \text{---(1)}$ $w - 2z = 10 \quad \text{---(2)}$ From (2): $w = 2z + 10 \quad \text{---(3)}$ Substitute (3) into (1): $z * w = 8 - 20i$ $z * (10 + 2z) = 8 - 20i$ Let $z = x + iy$, $x, y \in \mathbb{R}$ Then $(x - iy)(10 + 2x + 2iy) = 8 - 20i$ $10x + 2x^2 + 2ixy - 10iy - 2ixy + 2y^2 = 8 - 20i$ $10x + 2x^2 + 2y^2 - 10iy = 8 - 20i$ Compare imaginary terms: $-10y = -20$ $y = 2$ Compare real terms: $10x + 2x^2 + 2(2)^2 = 8$ $2x^2 + 10x = 0$ $2x(x + 5) = 0$ $2x = 0$ or $x + 5 = 0$ $x = 0$ or $x = -5$ When $x = 0, y = 2,$ $z = 2i, w = 10 + 4i$ When $x = -5, y = 2,$ $z = -5 + 2i, w = 4i$	Badly attempted with slips in the calculation. Quite a number of students rejected one of the 2 answers with various reasons and hence did not get the full marks for this question.

- 12** Underwater sonar buoys are deployed, in the ocean, to detect and monitor threats in the ocean and on the ocean surface. Two underwater sonar buoys, buoy A and buoy B, are located at points A and B respectively. The underwater sonar buoys are in the water body.

Points are defined relative to an origin $(0,0,0)$ on the ocean bed, with units in metres.

The ocean surface is modelled as the plane with equation $3y + 4z = 240$. A submarine moves along a straight path with equation $\frac{x}{2} = y - 20 = \frac{-z + 13}{2}$. A and B have coordinates $(-10, 31, 5)$ and $(4, 17, 19)$ respectively.

- Find a cartesian equation of the plane containing the path of the submarine and buoy B. [3]
- Determine the acute angle between the path of the submarine and the ocean surface. [2]
- An unidentified boat on the ocean surface is detected by buoys A and B. The boat is found to be equidistant from both buoys A and B. Describe geometrically the set of all possible positions of the location of the unidentified boat. [5]
- At low tide, the ocean surface falls below the current ocean surface at a perpendicular distance of 0.8 metres. The ocean surface at low tide is also modelled as a plane. Find the cartesian equation of this plane. [2]

12	Solutions [12] 3D Vectors.	
(a)	$\mathbf{r} = \begin{pmatrix} 0 \\ 20 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$ $\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{d}_2 = \begin{pmatrix} 4 \\ 17 \\ 19 \end{pmatrix} - \begin{pmatrix} 0 \\ 20 \\ 13 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 6 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 - 6 \\ -(12 + 8) \\ -6 - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -20 \\ -10 \end{pmatrix} = -10 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 53$ Cartesian equation: $2y + z = 53$.	Generally ok. Many unsuccessful solutions tried crossing vector $\overrightarrow{OB}$ with the direction vector of the line. This is wrong as the origin is not known to be on the plane. Some did not convert the equation of the line to parametric form properly. Some responses did not answer to the requirements of the question which is to find the Cartesian Equation of the plane.
(b)	Let θ be the acute angle between the submarine's path of travel and the ocean surface.	Generally ok as well. Some responses did not use the direction vector of the line and the normal of the plane, both of which are given by the question.

	$\sin \theta = \frac{\left \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \right }{\sqrt{2^2 + 1^2 + (-2)^2} \sqrt{0^2 + 3^2 + 4^2}}$ $\theta = \sin^{-1} \left(\frac{ -5 }{3(5)} \right)$ $\theta = \sin^{-1} \left(\frac{1}{3} \right)$ $\theta = 19.5^\circ \text{ (1 d.p.)}$	
(c)	Method 1 (Find plane containing all points equidistant to A and B) Let C be (x, y, z). $ \overrightarrow{AC} = \overrightarrow{BC} $ $(x+10)^2 + (y-31)^2 + (z-5)^2 = (x-4)^2 + (y-17)^2 + (z-19)^2$ $x^2 + 20x + 10^2 + y^2 - 62y + 31^2 = x^2 - 8x + 16 + y^2 - 34y + 17^2$ $+ z^2 - 10z + 5^2 = + z^2 - 38z + 19^2$ $20x - 62y - 10z + 1086 = -8x - 34y - 38z + 666$ $28x - 28y + 28z = -420$ $x - y + z = 15 \quad \text{---(1)}$	Most are blank responses. Among those who attempted, very few are successful in solving this part. [Method 1] Due to the algebraic load of this method very few among those who attempted this method got the equation of the plane.
	Method 2 (Find plane $\perp$ to $\overline{AB}$ and contains midpt of A and B) Midpoint of AB, $M = \left(\frac{-10+4}{2}, \frac{31+17}{2}, \frac{5+19}{2} \right) = (-3, 24, 12)$ $\overrightarrow{AM} = \begin{pmatrix} -10 \\ 31 \\ 5 \end{pmatrix} - \begin{pmatrix} -3 \\ 24 \\ 12 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ -7 \end{pmatrix} = -7 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ Equation of plane // to $\overrightarrow{AM}$ and contains M: $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 24 \\ 12 \end{pmatrix} = -15$ $x - y + z = -15 \quad \text{---(1)}$	[Method 2] While most who attempted this method are able to get the midpoint of AB, most are unable to follow up with the algorithm of this method to get the plane perpendicular to line segment AB passing through the midpoint. Most tried to find the foot of perpendicular of the midpoint onto the sea surface which is an unreliable method as they need vector $\overline{AB}$ to be parallel to the sea surface to succeed. Any deviation from this condition will cause the foot of perpendicular point to not

		be equidistant from points A and B . On the contrary, The foot of perpendicular from the locus of the boat to line segment AB is the midpoint but the method is not as efficient.
	Equation of ocean surface is $3y + 4z = 240$ ---(2) As the boat is on the ocean surface and equidistant from A and B, Solving (1) & (2), by GC, $x = 65 - \frac{7}{3}t,$ $y = 80 - \frac{4}{3}t,$ $z = t$ $\mathbf{r} = \begin{pmatrix} 65 \\ 80 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}, \lambda \in \mathbb{R}$ The set of all possible positions where the unidentified boat is located is on a line which passes through the point $(65, 80, 0)$ and is parallel to $\begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$.	Very few responses among all those who attempted realise the path of the boat is an intersection of 2 planes. Most simply stopping at the end of method 1 or 2.
(d)	The current ocean surface, $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = 240$ $\mathbf{r} \cdot \frac{1}{5} \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \frac{240}{5}$ At low tide,	Many blank responses as well here. Most who attempted tried the distance method to varying degree of success. Most are unfamiliar with the method itself and therefore unable to solve this part properly.

	$\mathbf{r} \cdot \frac{1}{5} \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \frac{240}{5} - \frac{4}{5}$ $\mathbf{r} \cdot \frac{1}{5} \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = \frac{236}{5}$ $\mathbf{r} \cdot \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = 236$ $3y + 4z = 236.$	
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