

Uppper Sec Emath

Revision Notes

Name: _____

MATHEMATICAL FORMULAE

Compound interest

$$\text{Total amount} = P \left(1 + \frac{r}{100} \right)^n$$

Mensuration

$$\text{Curved surface area of a cone} = \pi r l$$

$$\text{Surface area of a sphere} = 4\pi r^2$$

$$\text{Volume of a cone} = \frac{1}{3} \pi r^2 h$$

$$\text{Volume of a sphere} = \frac{4}{3} \pi r^3$$

$$\text{Area of triangle } ABC = \frac{1}{2} ab \sin C$$

$$\text{Arc length} = r\theta, \text{ where } \theta \text{ is in radians}$$

$$\text{Sector area} = \frac{1}{2} r^2 \theta, \text{ where } \theta \text{ is in radians}$$

Trigonometry

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Statistics

$$\text{Mean} = \frac{\sum fx}{\sum f}$$

$$\text{Standard deviation} = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2}$$

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1 Linear Equations

We'll start our journey in E-Math doing simple recaps on some of the important concepts we learnt in Lower Secondary Maths.

Solving a linear equation of 1 unknown

- Cross multiply / multiply both sides with the lowest common multiple of the denominator to remove the fraction.
- Expand all brackets.
- Apply opposite operations to both sides of the equation to shift the terms.
- All terms with the unknown should be on the left, all others should be on the right.

Worked Example 1.1

Solve the equation $7(x + 6) = 5(3x + 2)$.

[S2 BMSS P2/2015 MYE Q10]

[Solution]

$$7(x + 6) = 5(3x + 2)$$

$$7x + 42 = 15x + 10$$

$$-8x = -32$$

$$x = 4$$

Worked Example 1.2

Solve the equation $\frac{8(x-2)}{3} = \frac{7(5x-1)}{4}$.

[Solution]

$$\frac{8(x-2)}{3} = \frac{7(5x-1)}{4}$$

$$32(x-2) = 21(5x-1)$$

$$32x - 64 = 105x - 21$$

$$-73x = 43$$

$$x = -\frac{43}{73}$$

Worked Example 1.3

Chandra left home at 08 30 and headed for school. After cycling for x minutes at a speed of 30 km/h, he experienced a flat tyre. He spent 30 minutes to change the tyre and then continued the second part of his journey for $(x + 10)$ minutes at a speed of 24 km/h.

- (a) Find, in terms of x ,
- (i) the distance travelled in the first part of his journey.
 - (ii) the distance travelled in the second part of the journey.
- (b) Given that the distance from his home to school is 17.5 km, form an equation in x and solve it.
- (c) What time did Chandra finally arrive in school?

[Solution]

(a) (i) Distance travelled for the first part = $\frac{x}{60} \times 30$

$$= \frac{x}{2} \text{ km}$$

(ii) Distance travelled for the second part = $\frac{x+10}{60} \times 24$

$$= \frac{2x+20}{5} \text{ km}$$

(b) Since the total distance is 17.5 km,

$$\frac{x}{2} + \frac{2x+20}{5} = 17.5$$

$$\frac{5x+2(2x+20)}{10} = 17.5$$

$$5x + 4x + 40 = 175$$

$$9x = 135$$

$$x = 15$$

(c) Total time taken = 15 + 30 + 25

$$= 70 \text{ min}$$

$$= 1 \text{ hour } 10 \text{ min}$$

Time that Chandra finally arrives in school is 9.40 am.

Thinking Process

This is an easier variant of those kinds of questions that require students to form a quadratic equation.

Always use the information provided in the question to help form the needed equation.

Solving linear equations with unknown

There are 3 known methods to solving. Although we will learn all 3 methods, some methods are more recommended than others. method

1. Elimination [Not recommended unless obvious]

- Multiply or divide one / both linear equations by some constant so that they can contain an identical term.
- Eliminate the identical term by subtracting or adding the equations to obtain an equation with only one unknown.
- Use this equation to solve for the unknown.
- Substitute the result into any original equation to solve for the other unknown.

2. Substitution

- Using either equation, express one of the unknowns in terms of the other unknown.
- Substitute the transformed equation into the other linear equation to obtain an equation with only one unknown.
- Use this equation to solve for the unknown.
- Substitute the result into any original equation to solve for the other unknown.

3. Graphical Method [Question specific]

- The solution of the 2 linear equations is the intersection point of the two graphs. (linked to chapter on Coordinate Geometry!)

Worked Example 1.4

Solve the simultaneous equation

$$3x + y = 4$$

$$4x + 2y = 5$$

[S4 YHSS P2/2017 Prelim Q1(c)]

[Solution]

$$3x + y = 4 \dots \dots \dots (1)$$

$$4x + 2y = 5 \dots \dots \dots (2)$$

From Equation (1),

$$y = 4 - 3x \dots \dots \dots (3)$$

Substitute Equation (3) into Equation (2),

$$4x + 2(4 - 3x) = 5$$

$$4x + 8 - 6x = 5$$

$$2x = 3$$

$$x = \frac{3}{2}$$

Substitute $x = \frac{3}{2}$ into Equation (3),

$$\begin{aligned} y &= 4 - 3\left(\frac{3}{2}\right) \\ &= -\frac{1}{2} \end{aligned}$$

Tip!

This is a very standard kind of simultaneous equation question. Elimination can also be used here (multiply the first equation by 2, subtract the equations eliminate the y and to obtain an equation in x)

However, always first observe the equations before choosing which method can be used to solve the question. Usually, one is more obvious than the other.

Use your calculator to verify if the solutions are accurate!

Worked Example 1.5

A two-digit number is such that the sum of its digits is $\frac{1}{4}$ of the number. When the digits of the number are reversed and the number is subtracted from the original number, the result obtained is -27 . Find the original number.

[S4 NLSS P2/2014 MYE Q2(a)]

[Solution]

Let the tens digit of the original number be x and its ones digit be y .

The original number is $10x + y$.

The number obtained when the digits of the original number are reversed is $10y + x$.

$$\therefore x + y = \frac{1}{4}(10x + y)$$

$$4(x + y) = 10x + y$$

$$4x + 4y = 10x + y$$

$$3y = 6x$$

$$y = 2x \dots \dots \dots (1)$$

$$(10x + y) - (10y + x) = -27$$

$$9x - 9y = -27$$

$$y - x = 3 \dots \dots \dots (2)$$

Substitute Equation (1) into Equation (2),

$$2x - x = 3$$

$$x = 3$$

Substitute $x = 3$ into Equation (1),

$$y = 2(3)$$

$$= 6$$

Hence, the original number is 36.

Tips/Thinking Process

This is a variant of simultaneous equations. It may not be obvious at first glance that the techniques of simultaneous equations need to be used.

Always read the question in great detail. Always allow unknowns to be assumed to be x and y . This is a crucial technique that students need to get used to.

Usually students struggle with the concept of rotating the digits around, so be careful!

Worked Example 1.6

Given that $k > 0$, find the value of k given the following 2 equations

$$x^2 + y^2 = \frac{5}{2}xy \dots \dots (1)$$

$$\frac{x-y}{x+y} = k \dots \dots (2)$$

[S4 CCHS(M) P1/2017 Q9(a)]

[Solution]

$$x^2 + 2xy + y^2 - 2xy = \frac{5}{2}xy$$

$$(x+y)^2 = \frac{5}{2}xy$$

$$x+y = \sqrt{\frac{5}{2}xy}$$

$$x+y = \frac{\sqrt{5}}{\sqrt{2}}\sqrt{xy} \dots \dots (3)$$

From Equation (2),

$$x^2 - 2xy + y^2 + 2xy = \frac{5}{2}xy$$

$$(x-y)^2 = \frac{1}{2}xy$$

$$x-y = \sqrt{\frac{1}{2}xy}$$

$$x-y = \frac{1}{\sqrt{2}}\sqrt{xy} \dots \dots (4)$$

Substitute Equation (3) and (4) into (2),

$$\frac{\frac{1}{\sqrt{2}}\sqrt{xy}}{\frac{\sqrt{5}}{\sqrt{2}}\sqrt{xy}} = k$$

$$k = \frac{1}{\sqrt{5}}$$

Thinking Process

This is a more challenging variant of simultaneous equation questions. It is not obvious at first glance that the techniques of simultaneous equations are necessary here.

We get the hint that we need to factorise using the second equation. Since we are not solving for either x or y , it is not necessary for us to eliminate / use substitution.

We must then be able to conclude that factorising the first equation in two different ways using the quadratic identities, will allow us to get $x+y$ and $x-y$ for us to manipulate.

Solving Quadratic Equations

There are many methods to solve quadratic equations. Usually, questions in exam will be direct in specifying the method they want you to use, to test different skills. So unlike solving simultaneous equations, this time we need to know all the methods!

1. Factorisation

- We use this method when we are familiar with the quadratic equation and we can see that it is easily factorised.
- **Only when the right hand side of the equation is 0**, then we can divide by a constant to simplify the whole equation.
- In general, equations will be of the form

$$(x - p)(x - q) = 0$$

which we can solve to get $x = p$ or $x = q$. p and q are the known as the roots of the equation.

- Eg:

$$3x^2 + 6x - 9 = 0$$

Can be simplified to:

$$x^2 + 2x - 3 = 0$$

Factorising the equation gives us

$$(x + 3)(x - 1) = 0$$

And solving the equation gives us

$$x = -3 \quad \text{or} \quad x = 1$$

2. Quadratic Formula

- This is the most recommended method, since it works for any quadratic equation.
- The formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If the expression in the square-root is negative, we can even conclude that the equation has no solution.
- Eg:

$$3x^2 + 6x - 9 = 0$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(-9)}}{2(3)}$$

$$x = -3 \quad \text{or} \quad x = 1$$

3. Completing the Square

- This method is popular but is usually susceptible to many mistakes.
- It is very useful in solving questions on quadratic graphs, where completing the square helps us to find the turning point.
- The aim is to rearrange the quadratic equation $ax^2 + bx + c = 0$ into:

$$a(x - h)^2 + k = 0$$

- Make $(x - h)^2$ the subject of the equation:

$$(x - h)^2 = -\frac{k}{a}$$

- Take the square roots of both sides of the equation and solve for x

$$x = h + \sqrt{-\frac{k}{a}} \quad \text{or} \quad x = h - \sqrt{-\frac{k}{a}}$$

- From this we can see that
 - If $k < 0$, there are 2 solution for x ,
 - If $k = 0$, there is 1 solution for x ,
 - if $k > 0$, there are no solutions for x .
- In general,

$$ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \left[c - a\left(\frac{b}{2a}\right)^2\right]$$

$$= a(x - h)^2 + k$$

- Eg:

$$3x^2 + 6x - 9 = 0$$

$$3(x^2 + 2x) - 9 = 0$$

$$3((x + 1)^2 - 1^2) - 9 = 0$$

$$3(x + 1)^2 = 12$$

$$(x + 1)^2 = 4$$

$$x = \pm 2$$

$$x = 1 \quad \text{or} \quad x = -3$$

Worked Example 1.7

Factorise the following

$$6f^2 - 11f + 3$$

Hence, solve the following

$$6f^2 + 3 = 11f$$

[S4 SCGS P1/2018 Prelim Q1]

[Solution]

$$6f^2 - 11f + 3 = (2f - 3)(3f - 1)$$

$$6f^2 + 3 = 11f$$

$$6f^2 - 11f + 3 = 0$$

$$\therefore (2f - 3)(3f - 1) = 0 \text{ (from previous part)}$$

$$f = \frac{3}{2} \quad \text{or} \quad f = \frac{1}{3}$$

Worked Example 1.8

Solve the following equation $4x^2 - 8x - 5 = 0$ by completing the square.

[Solution]

Although we were previously taught the methods for completing the square, we can actually compare coefficients to obtain our final form.

Since we know the final outcome of “completing the Square” is $a(x - h)^2 + k = 0$

$$\begin{aligned}4x^2 - 8x - 5 &= a(x + h)^2 + h \\&= a(x^2 + 2hx + h^2) + h \\&= ax^2 + 2ahx + ah^2 + h\end{aligned}$$

Comparing coefficients,

$$a = 4 \dots \dots (1)$$

$$2ah = -8 \dots \dots (2)$$

$$ah^2 + k = -5 \dots \dots (3)$$

Substitute Equation (1) into Equation (2),

$$2(4)h = -8$$

$$h = -1$$

Substitute Equation (1) and $h = -1$ into Equation (3),

$$(4)(-1)^2 + k = -5$$

$$k = -9$$

$$\therefore 4(x - 1)^2 - 9 = 0$$

$$(x - 1)^2 = \pm \frac{9}{4}$$

$$x = \frac{1}{2} \quad \text{or} \quad x = \frac{5}{2}$$

Common Mistakes when Solving Equations

1. Unnecessary cancellations leading to loss of solutions

- Students are not allowed to cancel as how they please as it will result in the loss of essential solutions.

- Eg:

$$(x+1)(x-3) = (x-3)$$

$$(x+1) = 1$$

$$x = 0$$

- When cancelling like this, this will cause the solution of $(x-3)$ to disappear, resulting in the loss of an essential solution.

Correct solution:

$$(x+1)(x-3) = (x-3)$$

$$(x+1)(x-3) - (x-3) = 0$$

$$(x-3)(x+1-1) = 0$$

$$(x-3)(x) = 0$$

$$x = 3 \quad \text{or} \quad x = 0$$

2. Incorrect cancellations

- Some students will cancel incorrectly as well. Students are not allowed to cancel as they please as this will lead to the incorrect solution.

- Eg:

$$(x+1) + (x-3)(x+2) = (x-3)$$

$$(x+1) + (x+2) = 1$$

$$2x = -2$$

$$x = -1$$

- Cancelling like this is **INCORRECT!**

- This form of cancellation is incorrect as the $(x + 1)$ term does not contain the $(x - 3)$ term. Cancellations, in general, should be avoided in this topic as it will result to a loss of an essential solution.

Correct solution:

$$(x + 1) + (x - 3)(x + 2) = (x - 3)$$

$$x + 1 + x^2 - x - 6 - x + 3 = 0$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \quad \text{or} \quad x = 2$$

3. Intermediary steps

- Students **MUST** remember that all intermediary workings must be shown. Without these intermediary workings, students will not be issued the full credit of the question.

$$x^2 + 2x - 960 = 0$$

- Working 1:

$$(x - 30)(x + 32) = 0$$

x	$\times$	-30	$-30x$
x	$\times$	32	$32x$
x^2		-960	$2x$

- If the equation is factorisable using the “box-method”, show the relevant “box-method” working, and show the factorised form of the equation.
- Working 2:

$$x = -(2) \pm \frac{\sqrt{(2)^2 - 4(1)(-960)}}{2(1)}$$

$$x = \frac{(-2 \pm \sqrt{3844})}{2}$$

...

If the equation is not factorisable using the “box-method”, show the 2 steps. These 2 steps above are the most important to be shown as workings for the question.

Solving Fractional Equations

This part is similar to solving linear equations with one unknown. The only difference is that we will now get a quadratic instead of a linear equation.

Worked Example 1.9

Solve the equation

$$\frac{2x}{2x-3} + 1 = \frac{1}{2-3x}$$

[S4 YHSS P2/2017 Prelim Q1(d)]

[Solution]

$$\frac{2x}{2x-3} + 1 = \frac{1}{2-3x}$$

$$\frac{1}{2-3x} - \frac{2x}{2x-3} = 1$$

$$\frac{(2x-3)-2x(2-3x)}{(2-3x)(2x-3)} = 1$$

$$2x - 3 - 4x + 6x^2 = 4x - 6 - 6x^2 + 9x$$

$$12x^2 - 15x + 3 = 0$$

$$3(4x^2 - 5x + 1) = 0$$

$$3(4x - 1)(x - 1) = 0$$

$$x = \frac{1}{4} \quad \text{or} \quad x = 1$$

$4x$	-1	$-x$
x	-1	$-4x$
$4x^2$	1	$-5x$

Tip!

Always add brackets when combining fractions (especially subtracting). This is a very common source of mistakes among students.

Example of a common incorrect solution:

$$\frac{1170}{x-4} - \frac{1200}{x} = 5$$

$$\frac{1170(x) - 1200(x-4)}{x(x-4)} = 5$$

$$\frac{1170 - 1200x \ominus 4800}{x(x-4)} = 5$$


...

The additional brackets here are to serve as a reminder to all students that whenever students start manipulating the equation by finding the common denominator etc, when opening up the brackets, the sign of the terms in the brackets MUST change.

Correct solution:

$$\frac{1170}{x-4} - \left(\frac{1200}{x} \right) = 5$$

$$\frac{1170(x)}{(x-4)(x)} - \left[\frac{1200(x-4)}{x(x-4)} \right] = 5$$


$$\frac{1170x - [1200(x-4)]}{x(x-4)} = 5$$

$$\frac{1170x - 1200x \oplus 4800}{x(x-4)}$$


...

Common Mistake when Solving Fractional Equations

1. Inappropriate Cancellations

- The same concept is applied here as mentioned in the previous section, where cancellations will result in the loss of an essential solution.

- Eg 1:

$$\frac{2(x-5)(x+5) - x^2 + 10x}{(x-1)(x+5)}$$

- Performing a cancellation like this is unacceptable as the $(x+5)$ term only appears for the $2(x-5)(x+5)$ and $(x-1)(x+5)$ term. The $-x^2 + 10x$ expression does not contain $(x+5)$ term, hence we cannot cancel the $(x+5)$ term like this.

- Eg 2:

$$\frac{2x^3 - 2x^2 - \dots - \cancel{x^3} + \cancel{6x^2} + \cancel{5x} - 5x^2 - 40x - 25}{\cancel{x^3} + \cancel{6x^2} + \cancel{5x} - x^2 - 6x - 5}$$

- Performing a cancellation like this is unacceptable as cancelling common terms in the numerator and denominator like this is incorrect. Even though the terms x^3 , $6x^2$ and $5x$ appear both in the numerator and denominator, cancellation is only allowed if EVERY TERM contains the same expression.

2. Wrong usage of denominators

- Many students have issues choosing the correct denominators to be multiplied to the numerators to combine the fractions into a single fraction.
- Eg:

$$\frac{3}{4t} - \frac{5}{4t+8} = 1$$

$$\frac{3(8) - 5}{4t+8} = 1$$

...

- Students assumed that to make the denominators common, they will need to multiply 8 to the numerator to get the common denominator. 🤔
- The general rule of thumb is that if you are unsure of common denominator, just multiply the denominators together.
- Sometimes, this might create unnecessarily large denominators / numerators (eg: cubic expressions that we are unable to factorise). In these cases, take more effort to find the lowest common multiple.

Correct solution:

$$\frac{3}{4t} - \frac{5}{4t+8} = 1$$

$$\frac{3(4t+8) - 5(4t)}{4t(4t+8)} = 1$$

...

3. Multiplying denominators without thought

- Although the general rule of thumb is to multiply the denominators together when unsure (as mentioned earlier), there are certain scenarios where this will be disastrous.
- Eg:

$$\frac{5y-3}{2y^2+5y-3} - \frac{2}{1-2y} = 6$$

$$\frac{(5y-3)(1-2y) - 2(2y^2+5y-3)}{(2y^2+5y-3)(1-2y)} = 6$$

$$\frac{5y-10y-3+6y-4y^2-10y+6}{2y^2-4y^3+5y-10y^2-3+6y} = 6$$

$$\frac{-4y^2-14y+3}{-4y^3-8y^2+11y-3} = 6$$

- Most students will get stuck at this junction as they are unsure of how to simplify a cubic equation in the denominator. This is not taught in the syllabus so this should imply that the approach is completely incorrect.
- Students should first simplify all the quadratic expressions first as they will be factorisable. After factorisation, look out for common terms in the denominator then proceed with the least common denominator.

Correct solution

$$\frac{5y-3}{2y^2+5y-3} - \frac{2}{1-2y} = 6$$

$$\frac{5y-3}{(2y-1)(y+3)} - \frac{2}{1-2y} = 6$$

$$\frac{5y-3+2(y+3)}{(2y-1)(y+3)} = 6$$

$$\frac{7y+3}{(2y-1)(y+3)} = 6$$

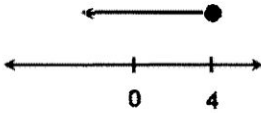
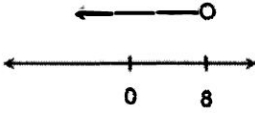
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2 Linear Inequalities

Symbols used in Linear Inequalities

Symbol	Name
$>$	Greater than
$<$	Less than
$\geq$	Greater than or equal to
$\leq$	Less than or equal to

Linear Inequalities on a Number Line

Symbols	Definition	Diagram Representation
●	Number under the dot is included in the solution	
○	Number under the circle is NOT included in the solution	

Worked Example 2.1

Solve the inequality

$$-\frac{17-8x}{4} < 2 - \frac{4-3x}{2} < 5\frac{1}{3}$$

Hence, state the smallest prime number that satisfies the inequality.

[S4 AHS P1/2018 Prelim Q4]

[Solution]

$$-\frac{17-8x}{4} < 2 - \frac{4-3x}{2} < 5\frac{1}{3}$$

$$-\frac{17-8x}{4} < 2 - \frac{4-3x}{2}$$

$$-\frac{17-8x}{4} < \frac{8-2(4-3x)}{4}$$

$$-\frac{17-8x}{4} < \frac{8-8+6x}{4}$$

$$-(17-8x) < 6x$$

$$2x < 17$$

$$x < 8\frac{1}{2}$$

$$2 - \frac{4-3x}{2} < 5\frac{1}{3}$$

$$\frac{4-(4-3x)}{2} < \frac{16}{3}$$

$$9x < 32$$

$$x < 3\frac{5}{9}$$

$$\therefore x < 3\frac{5}{9}$$

The smallest prime number is 2.

Tips

To solve such questions, always remember to split up the inequality as such. Group the first 2 expressions together as a set and solve, then group the last 2 expressions together as a set and solve.

Do recall algebraic fractional properties when performing subtraction of algebraic fractions.

$$2 - \frac{4-3x}{2}$$

$$\frac{4-(4-3x)}{2}$$

Do note that these included brackets are important as they remind students of the significance of the minus sign outside of the bracket.

Always check the solved inequalities at the end, if both inequalities are less than / greater than, choose the value that will work for both sides of the inequality.

$$x < 8\frac{1}{2}$$

This is an impossible solution for second inequality.

Worked Example 2.2

Solve the linear inequality and represent their solution on the number line provided.

$$\frac{4}{3}x - 14 \leq 3(x - 4) < 7.5$$



Hence, write down the smallest integer which satisfies the inequality above.

[S4 SMSS P2/2019 MYE Q20]

[Solution]

$$\frac{4}{3}x - 14 \leq 3(x - 4) < 7.5$$

$$x \geq -1\frac{1}{5}$$

$$\frac{4}{3}x - 14 \leq 3(x - 4)$$

$$3(x - 4) < 7.5$$

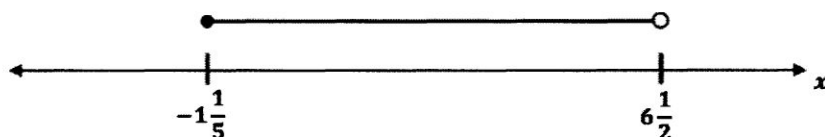
$$4x - 42 \leq 9x - 36$$

$$x - 4 < 2.5$$

$$5x \geq -6$$

$$x < 6\frac{1}{2}$$

$$-1\frac{1}{5} \leq x < 6\frac{1}{2}$$



The smallest integer which satisfies the inequality is -1 .

Thinking Process

$$-1\frac{1}{5} \leq x < 6\frac{1}{2}$$

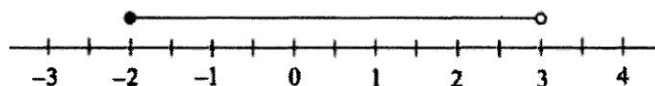
By convention, the left most number should be the smallest, and the right most number should be the biggest. This should be the proper presentation style and the solution needs to be logical.

It is not necessary to demarcate every single interval in the number line. Just state the minimum and maximum values.

Worked Example 2.3

The solution of the inequality, where c is a constant, is represented on the number line below.

$$-6\frac{1}{4} < \frac{x-4}{4} - 2x \leq c$$



Find the value of c .

[S4 SCGS P1/2019 Prelim Q8]

[Solution]

Since the constant is located where the inequality sign is $\leq$, we are only interested in the solution from the number line where $x \geq -2$.

$$\frac{x-4}{4} - 2x \leq c$$

By substituting $x = -2$, the limit of the inequality, we can find the value of c

$$\frac{((-2)-4)}{4} - 2(-2) = c$$

$$c = 2\frac{1}{2}$$

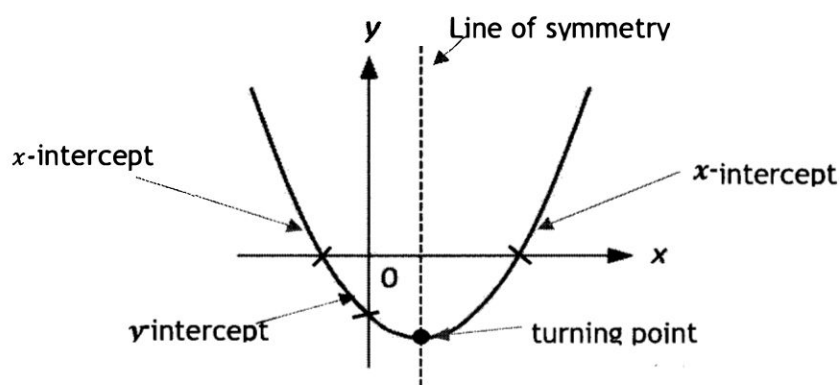
Tips

This is a rather unique way of testing the concepts of linear inequalities.

Students must be able to identify which value of x needs to be substituted into the inequality. By observing that the constant c is located at the “less than and equal to” sign, this should indicate to students to choose the value that corresponds to the solid dot on the number line.

3 Quadratic Graphs

Feature of quadratic graphs



$$y = ax^2 + bx + c \quad (a > 0)$$

Feature	Methodology	Explanation
x- intercepts	Substitute $y = 0$ into equation	The graph cuts the x -axis at those points where $y = 0$
y- intercept	Substitute $x = 0$ into equation	The graph cuts the y - axis at those points where $x = 0$
Turning point	x -coordinate: Midpoint of x -intercepts y -coordinate: Substitute x -coordinate	The turning point lies in the middle of the x -intercepts
Line of symmetry	$x =$ " x - coordinate of turning point"	The turning point lies on the line of symmetry of the graph

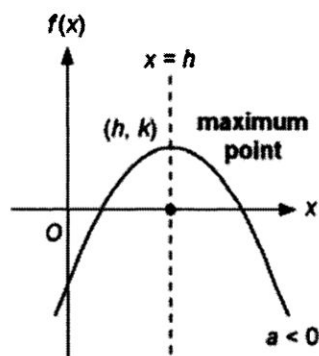
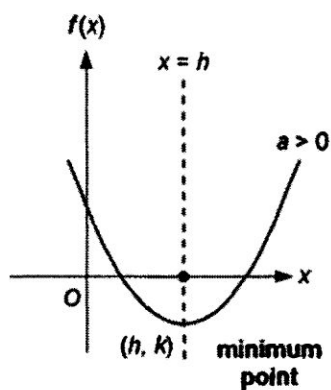
*Note that the features and methodologies are the same for "sad"-shape curves ($a < 0$)

Things to note

- Presentation for coordinates
 - If the question is asking for coordinates, brackets "()" must be used eg: (x, y)
- Presentation for "Line of Symmetry"
 - If the question is asking for the "line of symmetry", the question is asking for an equation. Since this is a vertical line, the standard equation should be $x = a$

Sketching of Quadratic Graph $y = ax^2 + bx + c$

1. Identify the shape of the quadratic graph based on the signature of the quadratic coefficient a .
2. y - intercepts of the graph can be obtained by solving for $x = 0$.
3. x - intercepts of the graph can be obtained by solving for $f(x) = 0$.
4. Determine the Maximum/Minimum values with the above mentioned 3 methods.



Curve	Assumptions	Feature
$y = \pm(x - h)^2 + k$	$h, k > 0$	(h, k) is the turning point of the graph
$y = \pm(x - p)(x - q)$	$p, q > 0$	$(p, 0)$ and $(q, 0)$ are the x -intercepts

Finding the roots of Quadratic Equations

1. Factorisation

- This method works when the equation is easily factorized.
- Factorise $f(x) = ax^2 + bx + c = 0$ to get the product of its linear factors $a(x - p)(x - q) = 0$ where p and q are known as the roots of the equation.
- Use the multiplication property that when $ab = 0 \Rightarrow a = 0$ or $b = 0$

2. Quadratic Formula

- The roots of the equation $y = ax^2 + bx + c = 0$ can be obtained with

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$$

3. Completing the Square

- Rearrange the quadratic equation $f(x) = ax^2 + bx + c = 0$ in the form of

$$f(x) = a(x - h)^2 + k = 0$$

- Make $(x - h)^2$ the subject of the equation

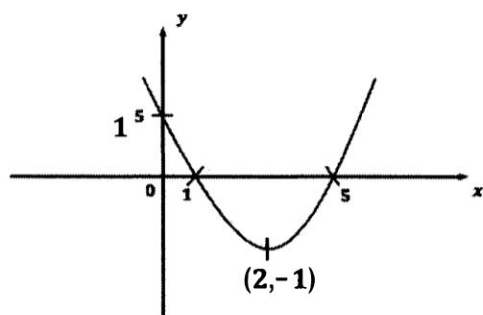
$$f(x) = (x - h)^2 = \frac{k}{a}$$

Take note that when square rooting on both sides, must take both the positive and negative of a square

- Take the square roots of both sides of the equation and solve for the values of x

$$x = h + \sqrt{\frac{k}{a}} \quad x = h - \sqrt{\frac{k}{a}}$$

Worked Example 3.1



The diagram shows a quadratic curve which can be expressed in the form

$$y = a(x + b)^2 + c$$

- (a) Find the values of a , b and c .
(b) Hence, find the x - intercepts, giving your answers correct to 2 decimal places.

[S4 TKSS P1/2015 PRELIM Q21]

[Solution]

(a) $y = a(x + b)^2 + c$

Since the minimum point is $(2, -1)$,

$$b = -2$$

$$c = -1$$

To solve for a , we substitute the y - intercept $(0, 1)$ into the equation

$$1 = a(0 - 2)^2 - 1$$

$$a = \frac{1}{2}$$

(b) At the x - intercepts, $y = 0$

$$0 = \frac{1}{2}(x - 2)^2 - 1$$

$$(x - 2)^2 = 2$$

$$x - 2 = \pm\sqrt{2}$$

$$x = 3.41421 \dots \quad \text{or} \quad x = 0.585786$$

$$x = 3.41 \text{ (2.d.p)} \quad \quad x = 0.59 \text{ (2.d.p)}$$

Worked Example 3.2

- (a) Express $y = x^2 + 2x - 8$ in the form $y = (x + p)(x + q)$.
(b) Hence, sketch $y = x^2 + 2x - 8$, labelling all intercepts and turning points appropriately.

[S4 CBSS P1/2016 PRELIM Q13(a) (Modified)]

[Solution]

$$\begin{aligned}\text{(a) } y &= x^2 + 2x - 8 \\ &= (x + 4)(x - 2)\end{aligned}$$

- (b) To find the minimum point, we need to find the midpoint of the x -intercepts.

$$\begin{aligned}\text{Midpoint} &= \left(\frac{-4+2}{2}, 0 \right) \\ &= (-1, 0)\end{aligned}$$

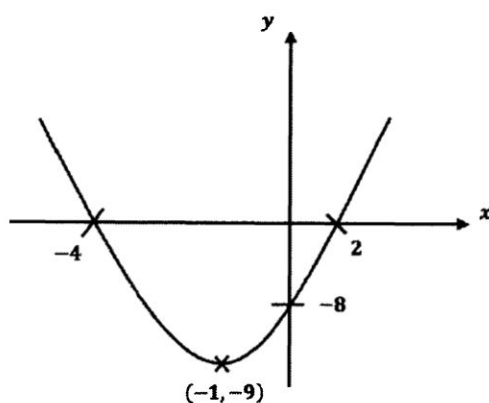
The mid-point and the minimum point have the same x -coordinate since they lie on the same line [line of symmetry] Substitute $x = -1$ into the equation.

$$\begin{aligned}y &= (-1)^2 + 2(-1) - 8 \\ &= -9\end{aligned}$$

Minimum point: $(-1, -9)$

To find the y -intercept, substitute $x = 0$ into the equation.

$$\begin{aligned}y &= (0)^2 + 2(0) - 8 \\ &= -8\end{aligned}$$



Worked Example 3.3

- (a) By completing the square, express $x^2 - 6x + 5$ in the form $(x - a)^2 - b$.
(b) Hence, solve the equation $x^2 - 6x + 5 = 0$.
(c) Sketch the graph of $y = x^2 - 6x + 5$

[S4 XMSS P1/2017 PRELIM Q24]

[Solution]

$$\begin{aligned} \text{(a)} \quad x^2 - 6x + 5 &= (x - 3)^2 - 9 + 5 \\ &= (x - 3)^2 - 4 \end{aligned}$$

$$\text{(b)} \quad x^2 - 6x + 5 = 0$$

$$(x - 3)^2 - 4 = 0$$

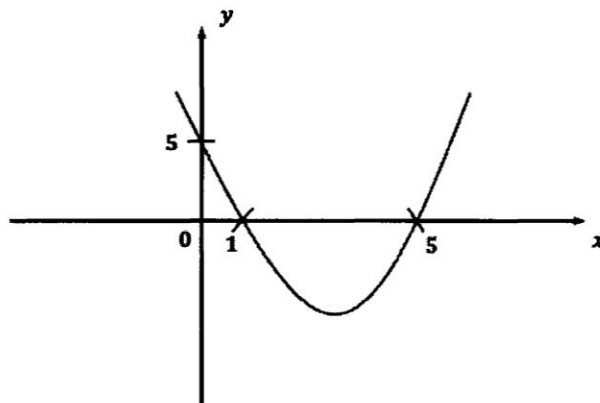
$$(x - 3)^2 = 4$$

$$(x - 3) = \pm 2$$

$$x - 3 = 2 \quad \text{or} \quad x - 3 = -2$$

$$x = 5 \quad \quad \quad x = 1$$

(c)



Tip!

- It is not necessary to draw the curve with a curve ruler, but the curve needs to be smooth and drawn with a pencil.
- While sketching the graphs, always label all the intercepts regardless of the question.

4 Indices

Laws of Indices

Property	Indices Law
Same Base	$a^m \times a^n = a^{m+n}$
	$\frac{a^m}{a^n} = a^{m-n}$
	$(a^m)^n = a^{mn}$
Same Power	$a^m \times b^m = (ab)^m$
	$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$
Radicals	$\sqrt[n]{a} = a^{\frac{1}{n}}$
	$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$
Others	$a^0 = 1$
	$a^{-n} = \frac{1}{a^n}$
	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Common Mistake

$$a^m + b^m = (a + b)^m$$

Many students get confused and use the above in their solutions! However, this is incorrect!

By substituting values of $a = 1$, $b = 2$ and $m = 3$

$$\text{LHS} = 1^3 + 2^3 = 9$$

$$\text{RHS} = (1 + 2)^3 = 27$$

$$\text{LHS} \neq \text{RHS}$$

Try to recall in quadratic identities, we have

$$a^2 + b^2 + 2ab = (a + b)^2$$

We can see very clearly that there is an extra term needed before we can factorise the expression into brackets.

Worked Example 4.1

Simplify the following, leaving your answer in positive index form

$$\frac{16a^3b^4}{7c^4} \div \frac{2(ab)^2}{21c^0} \times \frac{a^{n+1}}{4a^{n-2}}$$

[S4 MSHS P1/2017 PRELIM Q4]

[Solution]

Method 1:

$$\begin{aligned}\frac{16a^3b^4}{7c^4} \div \frac{2(ab)^2}{21c^0} \times \frac{a^{n+1}}{4a^{n-2}} &= \frac{16a^3b^4}{7c^4} \times \frac{21}{2a^2b^2} \times \frac{a^{(n+1)-(n-2)}}{4} \\ &= \frac{24ab^2}{c^4} \times \frac{a^{(n+1)-(n-2)}}{4} \\ &= \frac{6a^4b^2}{c^4}\end{aligned}$$

Method 2:

$$\begin{aligned}\frac{16a^3b^4}{7c^4} \div \frac{2(ab)^2}{21c^0} \times \frac{a^{n+1}}{4a^{n-2}} &= \frac{2^4a^3b^4}{7c^4} \times \frac{3 \times 7}{2a^2b^2} \times \frac{a^{n+1}}{2^2a^{n-2}} \\ &= 2^{4-1-2} \times 3 \times 7^{1-1} \times a^{3+n+1-2-(n-2)} b^{4-2} c^{0-4} \\ &= 2 \times 3 \times a^4 \times b^2 \times c^{-4} \\ &= \frac{6a^4b^2}{c^4}\end{aligned}$$

Tips/Thinking Process

This is a very common indices question, and we will present it in two ways to cater to 2 separate group of people.

Method 1 makes use of normal cancellations of indices and numbers. This method, when used well, can allow us to solve the questions quicker. However, this method is prone to error, as there are many factors involved (different variables, positive and negative powers, constants, etc)

Method 2 is a generic method that can be used for any kind of indices question. We prime factorise all the constants, and use the laws of indices to write the whole expression in one line.

Questions that use the phrase “leaving your answer in positive index form” require all the powers to be positive, so in this question we move c^{-4} to the denominator.

Worked Example 4.2

Find the value of k given that

$$\frac{\sqrt[4]{x^{-3}} \times x^{\frac{1}{2}}}{x^{-2}} = x^{\frac{2}{3}k}$$

[S4 AHS P1/2016 PRELIM Q2]

[Solution]

$$\begin{aligned}\text{LHS} &= \frac{\sqrt[4]{x^{-3}} \times x^{\frac{1}{2}}}{x^{-2}} \\ &= x^{-\frac{3}{4}} \times x^{\frac{1}{2}} \times x^2 \\ &= x^{-\frac{3}{4} + \frac{1}{2} + 2} \\ &= x^{\frac{7}{4}}\end{aligned}$$

By comparing powers, $\frac{7}{4} = \frac{2}{3}k$

$$\therefore k = \frac{21}{8}$$

Worked Example 4.3

Solve the equation

$$27^{2x-1} = 3^2 (3^x)$$

[S4 TKGS P1/2011 PRELIM Q10(a)]

[Solution]

$$27^{2x-1} = 3^2 (3^x)$$

$$3^{3(2x-1)} = 3^{2+x}$$

By comparing powers,

$$\therefore 3(2x - 1) = 2 + x$$

$$6x - 3 = 2 + x$$

$$5x = 5$$

$$x = 1$$

Thinking Process

These questions are common in exams and require us to solve equations in one variable. The main objective is to simplify the expression on both sides of the equality sign, and compare powers.

Treat each expression on each side as its own 'mini question'. The comparing of powers is just a small extension of that 😊

Common mistakes

Ensure that both sides of the equal sign only contain 1 element before comparing the powers.

Combine all elements together using indices laws first before comparing.

$$\begin{array}{c} \textcircled{27}^{2x-1} = 3^2 \textcircled{3^x} \\ 2x - 1 = 2x \end{array} \quad \mathbf{X}$$

Mastering the laws of indices is key to getting these type of questions right! There are two key mistakes here.

1. Not ensuring that both sides of the equation are of the same base. You cannot compare powers when the bases are the same! (27 and 3)
2. Wrong laws of indices. When we multiply two terms of the same base, we add the powers, not multiply them. Hence the power on the right should be $2 + x$, not $2x$.

Worked Example 4.4

Solve the following equation

$$3^{2x+1} - 2(3^{x+1}) = 9$$

[Solution]

$$3^{2x+1} - 2(3^{x+1}) = 9$$

$$3(3^{2x}) - 2(3^x)(3) = 9$$

$$\text{Let } 3^x = y$$

$$3y^2 - 6y - 9 = 0$$

$$y^2 - 2y - 3 = 0$$

$$(y - 3)(y + 1) = 0$$

$$y = 3 \quad \text{or} \quad y = -1$$

$$\therefore 3^x = 3 \quad 3^x = -1 \text{ (N.A)}$$

$$x = 1$$

Thinking Process

This is another common variant of the indices branch of questions. Although this looks like the earlier questions, it is very different, as it is not possible to compare powers on both sides of the equality from the start.

To solve these kind of questions, we need to think of it as a **quadratic equation**. Notice that we are splitting the powers such that we get 3^{2x} and 3^x , which we can do substitution to get y^2 and y .

Common Mistake

Many students end up stopping at $y = 3$ or $y = -1$. Remember that the variable in the initial equation is x , so we have to end up solving for the unknown variable x , not y .

Worked Example 4.5

Show that the following expression is divisible by 35 for all real positive integers of x

$$3^{2x+3} + \left(\frac{1}{9}\right)^{-1-x} - \sqrt{81^x}$$

[S4 MSHS P1/2017 PRELIM Q11(a)]

[Solution]

$$\begin{aligned} 3^{2x+3} + \left(\frac{1}{9}\right)^{-1-x} - \sqrt{81^x} &= 3^{2x+3} + (3^{-2})^{-1-x} - (3^{4x})^{\frac{1}{2}} \\ &= 3^{2x+3} + 3^{2+2x} - 3^{2x} \\ &= 3^3(3^{2x}) + 3^2(3^{2x}) - (3^{2x}) \\ &= 3^{2x}(3^3 + 3^2 - 1) \\ &= 3^{2x}(35) \end{aligned}$$

From the calculations above, it shows that the expression has a factor of 35. Hence it is divisible by 35.

Worked Example 4.6

Evaluate the following

$$\frac{2^{1004} - 15(2^{1000}) + 1}{2^{1001} + 2}$$

[S4 VS P1/2017 PRELIM Q8]

[Solution]

$$\begin{aligned} \frac{2^{1004} - 15(2^{1000}) + 1}{2^{1001} + 2} &= \frac{2^4(2^{1000}) - 15(2^{1000}) + 1}{2(2^{1000}) + 2} \\ &= \frac{2^{1000} + 1}{2(2^{1000} + 1)} \\ &= \frac{1}{2} \end{aligned}$$

Tips

There might be some unique indices questions that appear in the exams, most of them require some form of factorization or identifying of common terms. In Q4.5 we identified 3^{2x} as the common term, while in Q4.6 we identified 2^{1000} as the common term (to add like terms together), before identifying $2^{1000} + 1$ as the common term (to cancel).

The best way to get used to these kind of questions is to understand the thinking behind.

Simple and Compound Interest

Simple Interest

$$I = \frac{Prt}{100}$$

$$\text{Total amount} = P + \frac{Prt}{100}$$

- P is the principal amount
- r is the rate of interest
- t is the time period (in years)

Compound Interest

$$\text{Total amount} = P \left(1 + \frac{r}{100}\right)^n$$

$$I = P \left(1 + \frac{r}{100}\right)^n - P$$

- P is the principal amount
- r is the rate of interest
- n is the number of periods of the loan

Worked Example 4.7

The cash price of a waffle maker is \$149. Rose wants to start a waffle shop business and buys 5 waffle makers on hire purchase. She pays a deposit of 15% of the cash price followed by 24 equal monthly instalments with interest charged at a flat rate of 1.5% per annum. Calculate the amount of the monthly instalments, correct to the nearest cent.

[S4 SST P2/2018 Prelim Q7(a)]

[Solution]

$$\text{Amount paid using instalment} = (\$149 \times 5) \times 85\%$$

$$= \$633.25$$

$$\text{Interest incurred} = \$633.25 \times 1.5\% \times \frac{24}{12}$$

$$= \$18.9975$$

$$\text{Amount of monthly instalment} = \frac{633.25 + 18.9975}{24}$$

$$= \$27.17697 \dots$$

$$= \$27.18 \text{ (nearest cent)}$$

Worked Example 4.8

Darby invested \$5 400 in a bank that pays $m\%$ interest per annum which is compounded half yearly. If she received \$5 847.89 after 2.5 years, calculate the value of m .

[S4 HIHS P1 /2015 Prelim Q10]

[Solution]

$$\$5\,847.89 = \$5400 \left(1 + \frac{\left(\frac{m}{2}\right)}{100} \right)^5$$

$$\left(1 + \frac{m}{200} \right)^5 = \frac{5\,847.89}{5400}$$

$$1 + \frac{m}{200} = \sqrt[5]{\frac{5\,847.89}{5400}}$$

$$m = 200 \left(\sqrt[5]{\frac{5\,847.89}{5400}} - 1 \right)$$

$$= 3.212810$$

$$= 3.21(3.s.f)$$

Tips/Thinking Process

This is a rather simple question that requires the usage of the formula. Do note that this formula is provided in the formula sheet.

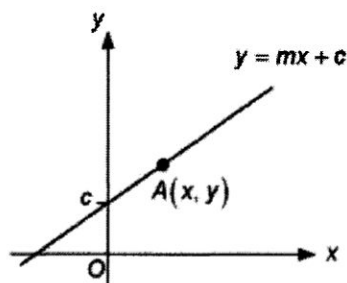
$$= \$5400 \left(1 + \frac{\left(\frac{m}{2}\right)}{100} \right)^5$$

Why $n = 5$? Since the interest is compounded half yearly, this means that every year, the interest will be taken 2 times. Since it is 2.5 years, the loan period will be 5.

Why $\frac{m}{2}$? The interest is calculated based on a per annum bases, but the interest is compounded half yearly. Hence, every time within the year is half of the interest.

5 Coordinate Geometry

General Properties



Term	Name	Definition
c	y -intercept	Represents the y -value where the line cuts the y -axis (when $x = 0$)
m	gradient	Represents the change in the y -value arising from a per unit change in x
$-\frac{c}{m}$	x -intercept	Represents the x -value where the line cuts the x -axis (when $y = 0$)

Common Mistake

We have learnt that the equation of straight line is $y = mx + c$.

Consider the equation:

$$2y = 4x + 8$$

Many students will forget that the coefficient of y has to be 1, and hence, conclude that the gradient is 4.

We must remember to do the appropriate manipulation. In this case, we should get

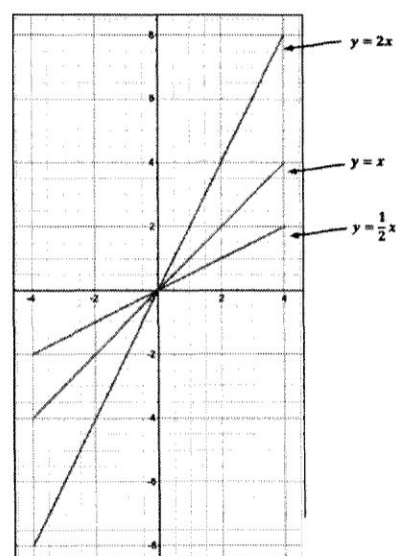
$$y = 2x + 4$$

Interpretation of the value of m

Do note that from the value of the gradient, we can tell how steep a line is.

The smaller the gradient value, the shallower the gradient is going to be.

The greater the value, the steeper the gradient is going to be.



Gradient formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

m value	Behaviour
$m > 0$	Positive gradient, upwards sloping
$m < 0$	Negative gradient, downwards sloping
$m = 0$	Parallel to the x -axis, horizontal line
m undefined	Parallel to the y -axis, vertical line
$m_1 = m_2$	Two lines are parallel

We have learnt the basic way of using $\frac{\text{rise}}{\text{run}}$ to calculate gradient in Lower Secondary.

In E-Math, we can use the difference in coordinates to calculate the rise and run.

Rise: change in y -coordinates

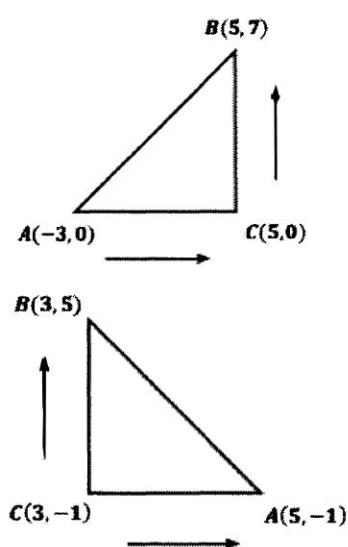
Run: change in x -coordinates

Using inspection to find length

Inspection is a fast way to find lengths of lines in the cartesian coordinate system. These lengths can be read off based on the coordinates. Draw a right triangle and label the coordinates respectively.

For both triangles, coordinate C is vertically below coordinate B , and is horizontally across from coordinate A . We can count the horizontal and vertical change to find the changes in x and y coordinates from A to B .

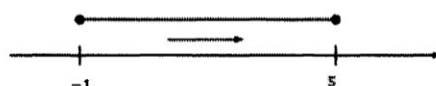
Using the second triangle as an example, the change is:



Horizontal change = $3 + 5 = 8$ units

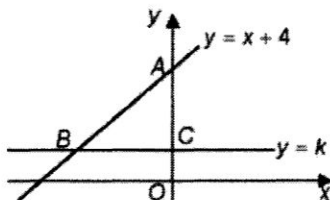


Vertical change = $1 + 5 = 6$ units



Worked Example 5.1

The graphs of $y = x + 4$ and $y = k$ are shown in the diagram below.



It is given that $y = x + 4$ cuts the y -axis at A , and cuts the line of $y = k$ at B .

If the length of line segment AB is $3\sqrt{2}$, find

- (a) the coordinates of A ,
- (b) the value of k ,
- (c) the perpendicular distance from C to the line AB , where C is the point of intersection of $y = k$ with the y -axis.

[Solution]

- (a) At A , $x = 0$

$$\therefore y = (0) + 4$$

$$= 4$$

$$\therefore A(0, 4)$$

- (b) Considering coordinates of B , when $y = k$,

$$k = x + 4 \Rightarrow x = k - 4$$

$$\therefore B(k - 4, k)$$

$$C(0, k)$$

$$\text{By Pythagoras Theorem, } AB = \sqrt{(k - 4)^2 + (4 - k)^2}$$

$$\sqrt{(k - 4)^2 + (4 - k)^2} = 3\sqrt{2}$$

$$2(k - 4)^2 = (3\sqrt{2})^2$$

$$2(k - 4)^2 = 18$$

$$k - 4 = \pm 3$$

$$\therefore k = 1 \text{ or } k = 7 \text{ (reject } k = 7 \because k < 4)$$

$$(c) \text{ Area of } \triangle ABC = \frac{1}{2}(3)(4 - 1) = \frac{9}{2} \text{ units}^2$$

Let h be the perpendicular distance from AB to C .

$$\text{Area of } \triangle ABC = \frac{1}{2}(AB)(h) = \frac{3\sqrt{2}}{2}h$$

$$\therefore \frac{3\sqrt{2}}{2}h = \frac{9}{2}$$

$$3\sqrt{2}h = 9$$

$$h = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \text{ units}$$

Tips

Q5.1(b) is a common question, where we are required to use find the coordinates of an unknown point, using the equation of the lines that pass through it.

At coordinate B , we can assume that the coordinates are of the form (x, y) .

Since $y = k$ passes through B , we now get the form (x, k) .

Since $y = x + 4$ also passes through it, and we know the y -coordinate is k , we can substitute into the equation to get $k = x + 4$, giving us the coordinate of B as $(k - 4, k)$.

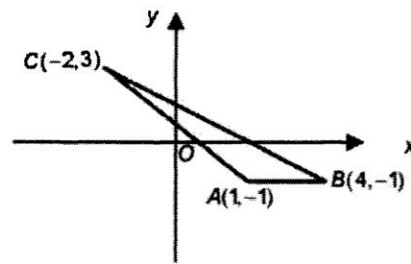
Q5.1(c) is also a common question, where we are required to find the perpendicular distance from a point to a line. This is more common in trigonometry, although it appears occasionally in coordinate geometry.

To solve this, we use two ways of finding the area of a triangle.

The first way $\frac{1}{2}(\text{base})(\text{height})$ helps us to find the area of the triangle.

The second way $\frac{1}{2}(\text{new base})(\text{new perpendicular height})$ allows us to find the perpendicular distance, since we know the length of the new base (usually a diagonal).

Worked Example 5.2



The points $A(1, -1)$, $B(4, -1)$ and $C(-2, 3)$ as shown in the diagram. Find

- (a) the equation of the line AC ,
- (b) the value of $\sin \angle BAC$.

[S4 AHS P1/2018 Prelim Q8(Modified)]

[Solution]

- (a) To find the equation of the line, we need 1 point and 1 gradient

$$\begin{aligned}\text{Gradient of line } AC &= \frac{3 - (-1)}{-2 - 1} \\ &= -\frac{4}{3}\end{aligned}$$

$$\therefore y = -\frac{4}{3}x + c$$

Using the point $A(1, -1)$,

$$-1 = -\frac{4}{3}(1) + c$$

$$c = \frac{1}{3}$$

$$\therefore y = -\frac{4}{3}x + \frac{1}{3}$$

- (b) Let D be the coordinate vertically below C and horizontally to the right of A .

$$\therefore D(-2, -1)$$

Length of $CD = 4$ units

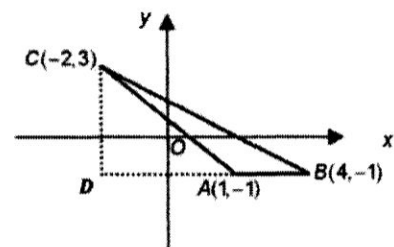
Length of $AD = 3$ units

$$\begin{aligned}\text{Length of } AC &= \sqrt{3^2 + 4^2} \\ &= 5 \text{ units}\end{aligned}$$

$$\therefore \sin \angle BAC = \sin \angle CAD$$

$$= \frac{CD}{CA}$$

$$= \frac{4}{5}$$



Worked Example 5.3

Given that the line L has equation $3x + 4y = 12$. Find

- (a) the coordinates of the point at which it cuts the y -axis.
- (b) the equation of the lines which is the image of the line L under a reflection about the line $x = 4$.

[S4 NGHS P1/2012 Prelim Q7]

[Solution]

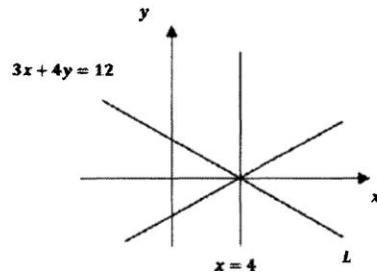
- (a) At the y -axis, $x = 0$

$$\therefore 3(0) + 4y = 12$$

$$y = 3$$

Hence, the coordinate is $(0, 3)$.

- (b)



Since we are reflected about the line $x = 4$, the gradient of the image of L are negatives of each other.

$$3x + 4y = 12$$

$$y = 3 - \frac{3}{4}x$$

Gradient of line L is $-\frac{3}{4}$.

Since we know that the line passes through the point where $x = 4$, we can find the common coordinate between the line L and the image.

$$y = 3 - \frac{3}{4}(4) = 0$$

Common coordinate = $(4, 0)$

$\therefore$ Gradient of the image line is $\frac{3}{4}$

$$\therefore y = \frac{3}{4}x + c$$

Using the common coordinate $(4, 0)$,

$$0 = \frac{3}{4}(4) + c \Rightarrow c = -3$$

$$\therefore y = \frac{3}{4}x - 3$$

Tips/Thinking Process

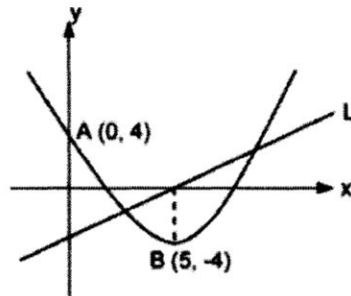
The concept of reflections may be new for some students that are not exposed to such questions.

Do note the following with regards to reflection:

- Reflection about a vertical line
 - The reflected line (known as the image)'s gradient is the negative of the original line's gradient.
 - Both the original line and the image have a common point, on that reflected line. To find that coordinate, since that line is vertical, the x –coordinate is already known. Substitute the x –coordinate into the original equation to get the y –coordinate.
- Reflection about a horizontal line
 - Concepts are the same as above, except that to find the common point, the y –coordinate is already known.

Worked Example 5.4

The diagram below shows a quadratic curve C of equation $y = x^2 + bx + a$, where a and b are constants to be determined. $A(0,4)$ and $B(5,-4)$ are two points that lie on C . A straight line L is drawn across the curve C , intersecting it at points $(3,-1)$ and $(8,2)$.



Find,

- (i) the values of a and b .
- (ii) the distance between A and B .
- (iii) the gradient of the line L .
- (iv) the equation of another line parallel to the line L passing through a point $(5, 12)$.

[S4 TSS P1/2009 Prelim Q19]

[Solution]

- (i) Since both A and B lie on the curve, substitute both points to find 2 equations

At $A(0, 4)$,

$$4 = (0)^2 + b(0) + a$$

$$a = 4$$

At $B(5, -4)$,

$$-4 = (5)^2 + b(5) + 4$$

$$5b = -33$$

$$b = 6\frac{3}{5}$$

- (ii) Distance between A and $B = \sqrt{(5-0)^2 + (-4-4)^2}$
 $= 9.43398 \dots$
 $= 9.43 \text{ units (3s.f.)}$

- (iii) To find the gradient,

$$\text{Gradient} = \frac{2 - (-1)}{8 - 3}$$

$$= \frac{3}{5}$$

- (iv) Since the 2 lines are parallel, they have the same gradient. Hence, substitute the point into the equation of the line.

$$y = \frac{3}{5}x + c$$

$$12 = \frac{3}{5}(5) + c$$

$$c = 9$$

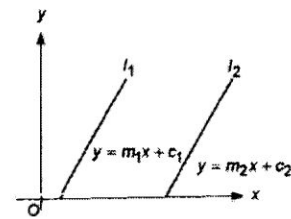
$$\therefore y = \frac{3}{5}x + 9$$

Revision on Parallel Lines

Remember for 2 lines to be parallel, the gradients of the first line is equal to the gradient of the second. In other words,

$$m_1 = m_2$$

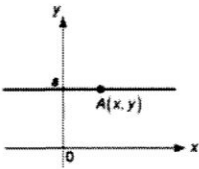
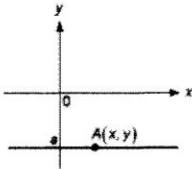
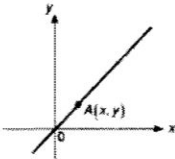
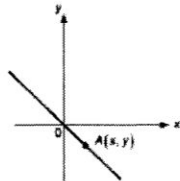
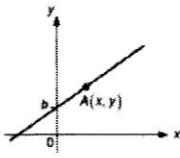
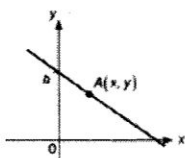
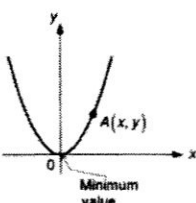
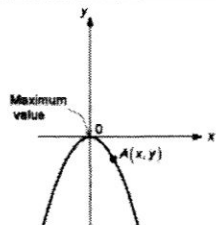
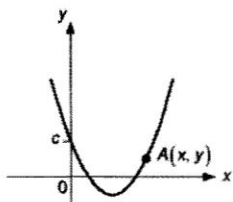
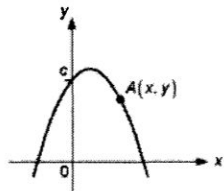
These parallel lines will not intersect each other at all.



6 Functions & Graphs

Let's first go through the various types of graphs available in the syllabus.

Graph Recap

Equation	Graph when $a > 0$	Graph when $a < 0$
$y = a$		
$y = ax$		
$y = ax + b$ ($b \neq 0$)		
$y = ax^2$		
$y = ax^2 + bx + c$		

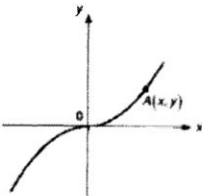
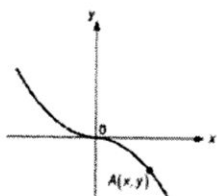
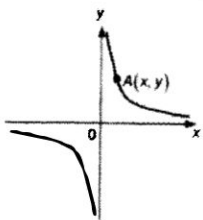
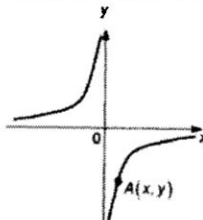
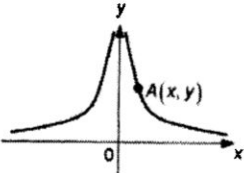
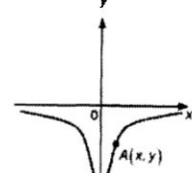
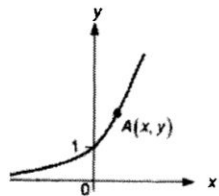
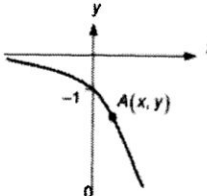
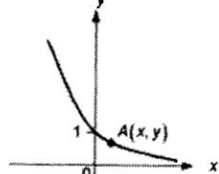
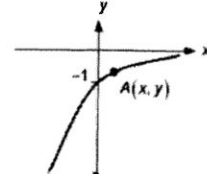
Take Note

How to tell the shape of the quadratic curve?

It depends on the signum of a .

Signum of a	Type of curve
a positive ($a > 0$)	"happy"-face curve
a negative ($a < 0$)	"sad"-face curve

New Graphs

Equation	Graph when $a > 0$	Graph when $a < 0$
$y = ax^3$		
$y = \frac{a}{x}$		
$y = \frac{a}{x^2}$		
$y = a(2)^x$		
$y = a\left(\frac{1}{2}\right)^x$		

Some interesting new ideas on graphs

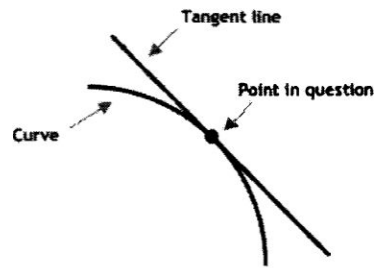
1. Vertical Translation

- We learnt this before in coordinate geometry, where the graph of $y = x + c$ where $c > 0$, is parallel to the line $y = x$, just ' c ' units higher
- This shift upwards is known as a vertical translation
- Similarly, if $c < 0$, the graph shifts lower
- This concept of shifting upwards and downwards can be applied to all graphs!

2. Asymptotes

- An asymptote is a line that the graph approaches very near to, but never touches.
- Notice that the horizontal asymptote for the graphs $y = \frac{a}{x}$, $y = \frac{a}{x^2}$ and e^x are all the x -axis, which is $y = 0$.

Estimating gradient of a point

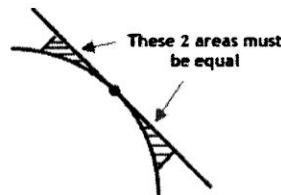


1. Identify the point on the curve where the gradient is to be found.
2. Draw a straight line **tangential** to the curve.
3. Read off the coordinates from 2 convenient points (points A and B) on the straight line.
4. Use the following formula below to calculate the gradient of the line.

$$m = \frac{y_B - y_A}{x_B - x_A}$$

Tip!

1. Check if the area on the left-hand side of the line to the curve is the same as the area on the right-hand side of the line to the curve.



2. Ensure that your **working lines** are shown, when the question is phrase like:
“use your graph to find the value of y when $x = 2$ ”
3. The solution of two simultaneous equations is the intersection point of the two graphs.
 - Always remember to fish out the original equation from the given equation (using manipulation) to get the additional line.
 - Eg: Assume that we are given the graph $y = x^2 + 3x - 3$, and want to solve the equation $x^2 + 4x = 4$ using the graph.

$$x^2 + 4x = 4$$

$$x^2 + 3x - 3 - 1 + x = 0$$

$$\underbrace{x^2 + 3x - 3}_{\text{Original Equation}} = \underbrace{-x + 1}_{\text{New Equation (to be drawn)}}$$

Original Equation New Equation (to be drawn)

Worked Example 6.1

Answer the whole of this question on a sheet of graph paper.

The table gives some values of x and the corresponding values of y where

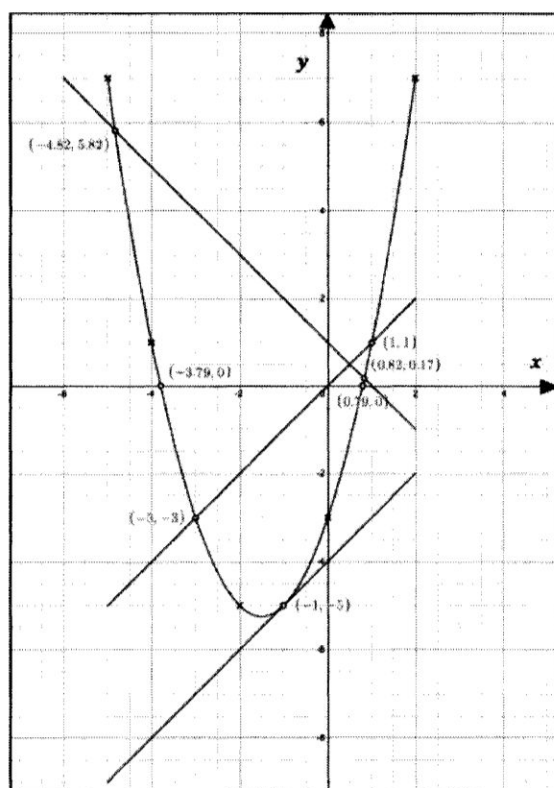
$$y = x^2 + 3x - 3$$

x	-5	-4	-3	-2	-1	0	1	2
y	7	1	-3	-5	-5	-3	1	7

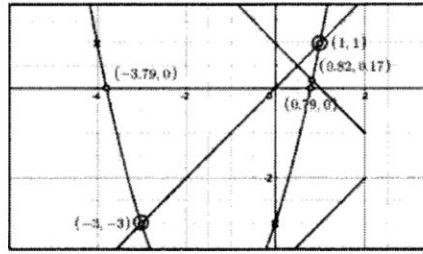
- (a) Using a scale of 2cm to represent 1 unit on the x -axis and 2cm to represent 2 units on the y -axis, draw a graph of $y = x^2 + 3x - 3$ for $-5 \leq x \leq 2$.
- (b) Use your graph to find
- The range of values of x for which $x^2 + 3x - 3 \leq x$.
 - The solutions to the equation $x^2 + 4x = 4$
- (c) By drawing a suitable tangent to the curve, find the coordinates of the point at which the gradient of the tangent is equal to 1.
- (d) Use the graph to solve $x^2 + 3x - 3 = 0$.

[Solution]

(a)



(b) (i)

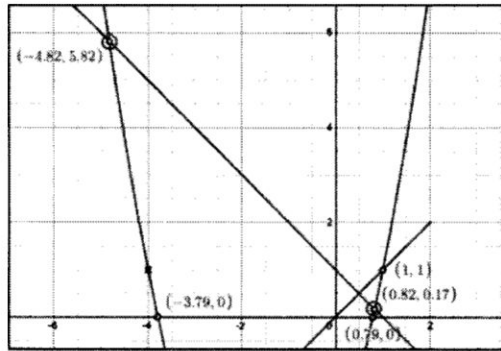


To solve the inequality, we need to sketch the line $y = x$.

From the graph, the intersection points are $x = -3$ and $x = 1$.

Hence, $-3 \leq x \leq 1$ as curve lies below the straight line $y = x$.

(ii)



$$x^2 + 4x = 4$$

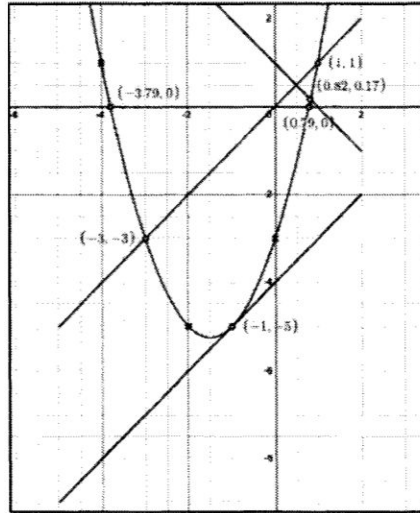
$$x^2 + 3x - 3 - 1 + x = 0$$

$$x^2 + 3x - 3 = -x + 1$$

Hence, to find the solutions, we need to sketch the line $y = -x + 1$.

From the graph, the solutions to the equation are $x = -4.82$ and $x = 0.82$.

(c)

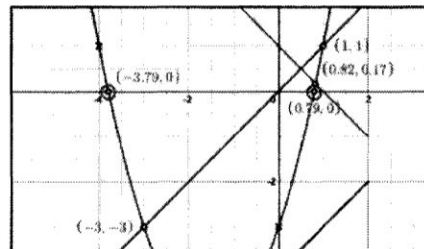


This part is rather unique as the question asks you to find the line where the gradient is 1. Note that the earlier part (b)(i) requires students to draw the line $y = x$, where the gradient is also 1.

Hence, to find the point, draw a line that is parallel to the line $y = x$.

From the graph, tangent is 1 at the point $(-1, -5)$.

(d)



To solve the equation $x^2 + 3x - 3 = 0$, we need to find the intersections of the curve with the x -axis

From the graph, $x = -3.79$ and $x = 0.79$.

Worked Example 6.2

Answer the whole of this question on a sheet of graph paper.

The variables x and y are connected by the equation

$$y = \frac{1}{20} \left(225 - 2x^2 - \frac{72}{x} \right)$$

Some of the corresponding values of x and y are given in the table below

x	0.5	1	2	3	4	6	8	10	12
y	4.0	7.6	9.1	a	8.8	b	4.4	0.9	-3.5

- (a) Find the value of a and b , correct to 1 decimal place.
- (b) Using a scale of 1 cm to represent 1 unit on both axes, plot the graph of the given equation above for the range $0.5 \leq x \leq 12$.
- (c) Use the graph to estimate the value of
- (i) y when $x = 5$
 - (ii) x when $y = 5$
- (d) By drawing a tangent, find the gradient of the graph at $x = 8$.
- (e) By adding another line to your graph, find the solutions of the equation

$$10 + 25x = \left(225 - 2x^2 - \frac{72}{x} \right)$$

[Solution]

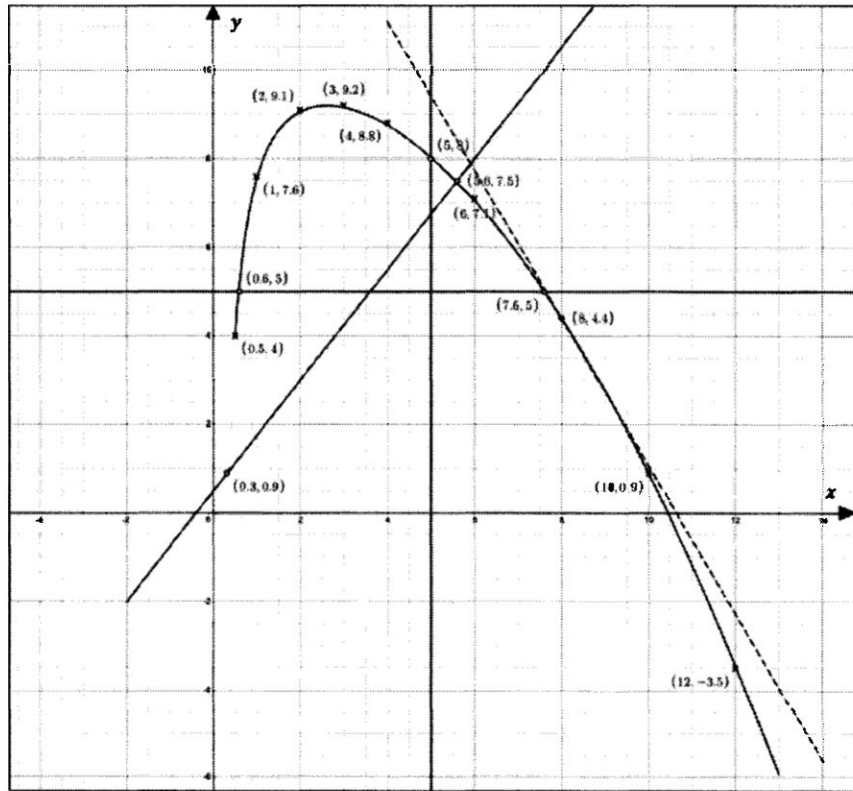
- (a) When $x = 3$,

$$\begin{aligned} a &= \frac{1}{20} \left(225 - 2(3)^2 - \frac{72}{(3)} \right) \\ &= 9.15 \\ &= 9.2 \end{aligned}$$

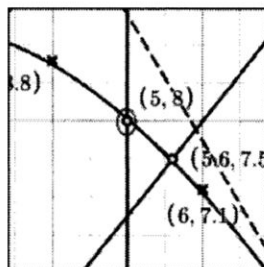
When $x = 6$,

$$\begin{aligned} b &= \frac{1}{20} \left(225 - 2(6)^2 - \frac{72}{(6)} \right) \\ &= 7.05 \\ &= 7.1 \end{aligned}$$

(b)



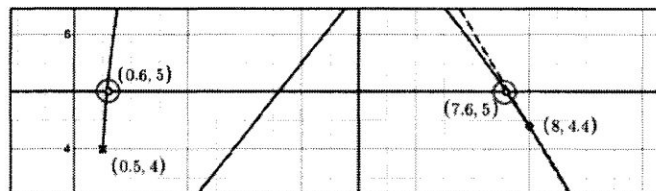
(c) (i)



When $x = 5, y = 8$

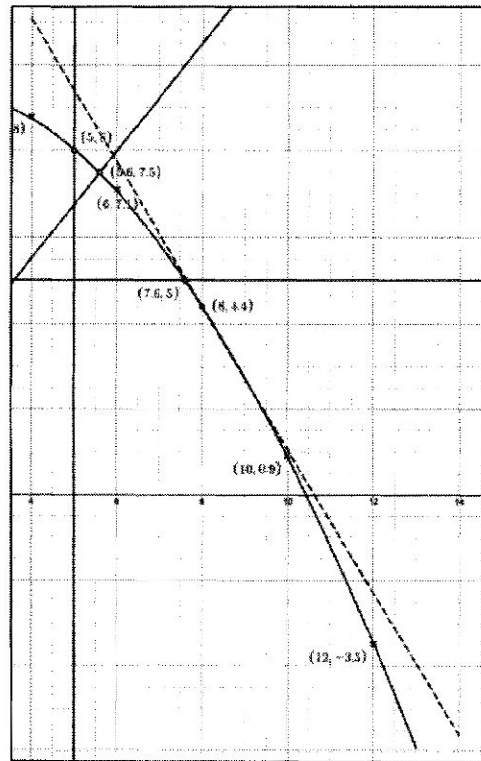
For such questions, always show the working in the graph. Always sketch out the needed line and show the intersection points.

(ii)



When $y = 5, x = 0.6$ or $x = 7.6$

(d) To find the gradient, we sketch a tangent at the point (8,4.4)



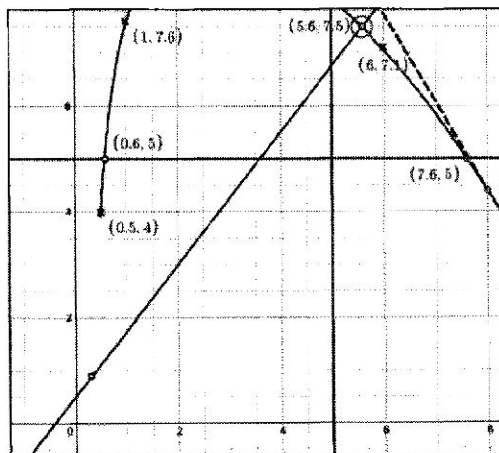
$$\begin{aligned}\text{Gradient} &= \frac{9.41 - (-2.28)}{5 - 12} \\ &= -1.67\end{aligned}$$

(e) To find the other line, multiply $\frac{1}{20}$ to both sides of the given equation.

$$\frac{10 + 25x}{20} = \frac{1}{20} \left(225 - 2x^2 - \frac{72}{x} \right)$$

$$\frac{1}{2} + \frac{5}{4}x = \frac{1}{20} \left(225 - 2x^2 - \frac{72}{x} \right)$$

Sketch the line: $y = \frac{1}{2} + \frac{5}{4}x$

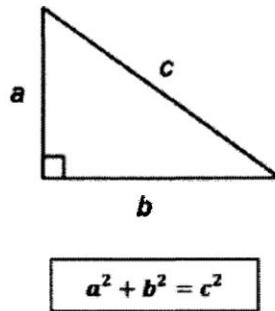


From the graph, $x = 5.6$

7 Trigonometry

Pythagoras Theorem (recap!)

Pythagoras Theorem states that for a right-angled triangle with sides a , b and hypotenuse c , the sum of the squares of the sides is equal to the square of the hypotenuse.



Note!

Pythagoras Theorem can only be used in RIGHT-ANGLED triangles and the hypotenuse is always the longest side of the triangle.

Converse of Pythagoras Theorem (recap!)

When we are presented with a triangle, we cannot immediately assume that it is right-angled. To prove it is right-angled, calculate the sum of the squares of the two sides, and the square of the hypotenuse separately.

Example

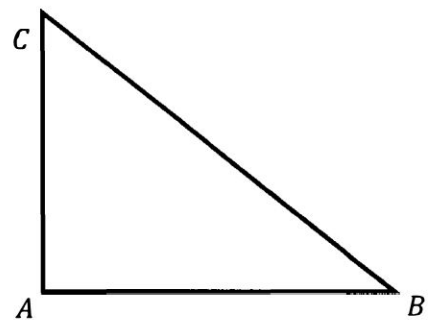
Given that $AB = 12$ cm, $AC = 5$ cm and $BC = 13$ cm, prove that triangle ABC is a right-angled triangle.

Step 1: $AB^2 + AC^2 = 12^2 + 5^2 = 169$ cm

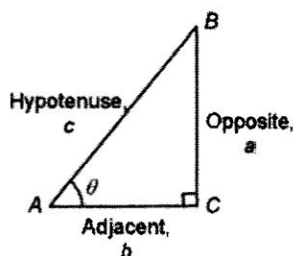
Step 2: $BC^2 = 13^2 = 169$ cm

Since $AB^2 + AC^2 = BC^2$,

By Converse of Pythagoras Theorem,
Triangle ABC is a right-angled triangle.



Trigonometric Ratios



To help with memorization of these ratios, you can use

1. TOA: $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$
2. CAH: $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$
3. SOH: $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$

Note!

Similar to Pythagoras Theorem, you can only use TOA CAH SOH in right-angled triangles.

Good to know: Special Angles

θ	0°	30°	45°	60°	90°
	0 rad	$\frac{1}{6}\pi$ rad	$\frac{1}{4}\pi$ rad	$\frac{1}{3}\pi$ rad	$\frac{1}{2}\pi$ rad
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

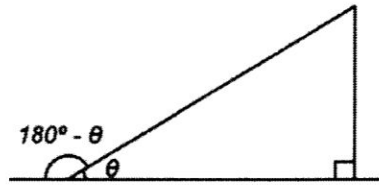
Tip!

You don't have to memorise these values.

1. Ensure your calculator is in degree mode and key the values into the calculator.
2. If the results show a decimal, square the value and you will derive a fraction.
3. Your exact value will therefore be the square root of the fraction on the calculator.

Trigonometric Ratios of Obtuse Angles

- Acute angle: $0^\circ < x < 90^\circ$
- Obtuse angle: $90^\circ < x < 180^\circ$



Ratio	Formula
sine	$\sin \theta = \sin(180^\circ - \theta)$
cosine	$\cos \theta = -\cos(180^\circ - \theta)$
tangent	$\tan \theta = -\tan(180^\circ - \theta)$

Eg:

$$\sin 150^\circ = \sin 30^\circ$$

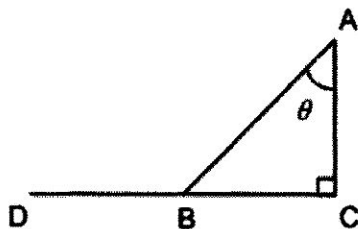
$$\cos 140^\circ = -\cos 40^\circ$$

$$\tan 120^\circ = -\tan 60^\circ$$

Worked Example 7.1

Triangle ABC is a right-angled triangle where $AB = 17$ cm, $BC = 8$ cm and $\angle ACB = 90^\circ$.

Given that D is a point on AC such that $\frac{CD}{AD} = \frac{2}{3}$,



Find, leaving your answer as a fraction in its simplest form, the value of

- a) $\tan \angle CBD$,
- b) $\cos \angle ADB$

[Solution]

- (a) $AC^2 = AB^2 - BC^2$ (By Pythagoras Theorem)

$$AC^2 = 17^2 - 8^2$$

$$AC = \sqrt{17^2 - 8^2}$$

$$= 15 \text{ (reject } -15)$$

$$AD = \frac{3}{3+2} \times 15 = 9 \text{ cm}$$

$$CD = \frac{2}{3+2} \times 15 = 6 \text{ cm}$$

$$\tan \angle CBD = \frac{DC}{BC} = \frac{6}{8} = \frac{3}{4}$$

- (b) $BD^2 = BC^2 + DC^2$ (By Pythagoras Theorem)

$$BD^2 = 8^2 + 6^2$$

$$BD = \sqrt{100} = 10 \text{ cm (Reject } -10)$$

$$\cos \angle ADB = -\cos(180 - \angle ADB)$$

$$= -\cos \angle BDC$$

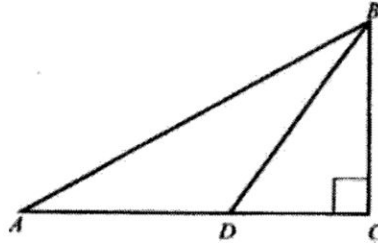
$$= -\frac{DC}{BD}$$

$$= -\frac{6}{10}$$

$$= -\frac{3}{5}$$

Worked Example 7.2

In the diagram, triangle ABC is a right-angle triangle and DBC is a straight line.



Given that $\cos \theta = \frac{5}{13}$, find the value of

- a) $\sin \theta$,
- b) $\tan(90^\circ - \theta)$,
- c) $\cos \angle DBA$.

[Solution]

(a) $BC^2 = AB^2 - AC^2$ (By Pythagoras Theorem)

$$BC^2 = 13^2 - 5^2$$

$$BC = \sqrt{13^2 - 5^2}$$

$$= 12 \text{ (reject } -12)$$

$$\sin \theta = \frac{BC}{AB} = \frac{12}{13}$$

(b) $\tan(90^\circ - \theta) = \tan \angle ABC$

$$= \frac{AC}{BC}$$

$$= \frac{5}{12}$$

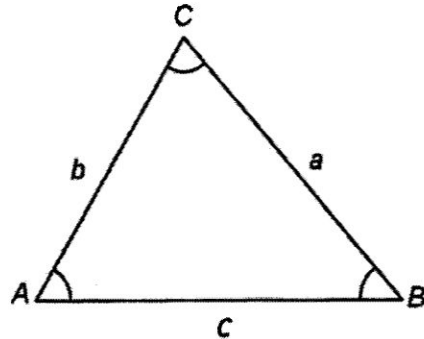
(c) $\cos \angle DBA = -\cos \angle ABC$

$$= -\frac{BC}{AB}$$

$$= -\frac{12}{13}$$

Sine and Cosine Rules

Triangle ABC is shown in the diagram below, where the side of lengths a, b and c are opposite the vertices A, B and C ,



Important Formula: Sine Rule

$$\frac{\sin \angle CAB}{a} = \frac{\sin \angle ACB}{b} = \frac{\sin \angle ABC}{c}$$

$$\frac{a}{\sin \angle CAB} = \frac{b}{\sin \angle ACB} = \frac{c}{\sin \angle ABC}$$

These formulas will be provided in Formula Sheet!

Important Formula: Cosine Rule

$$c^2 = a^2 + b^2 - 2(a)(b) \cos \angle ACB$$

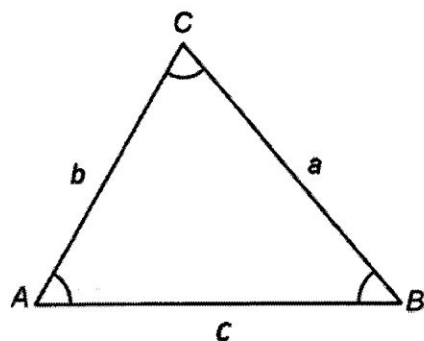
$$\cos \angle ACB = \frac{c^2 - a^2 - b^2}{-2(a)(b)}$$

- You use sine and cosine rule when you are working with a non-right-angled triangles.
- Sine and cosine rules also work on non-right angled triangles, although It would be easier to use Trigonometric Ratios (TOA CAH SOH)

Tip!

- The size of the angle always corresponds to the length opposite, therefore for the largest angle, the length opposite of the angle will also be the largest.
- For Sine rule, always leave the unknown in the numerator, i.e. if you are solving for length, the length should be in the numerator and angle in the denominator, vice versa.

Area of Triangle (Right-Angled and Non-Right-Angled Triangle)



In lower secondary we learnt the formula of Area of triangle = $\frac{1}{2}(\text{base})(\text{height})$. However, this only works for right-angled triangles. The following is a generic formula that will allow us to find the area of any triangle.

$$\text{Area of triangle} = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$$

We can see that the angle we use is always between the 2 lengths we use.

Note!

The angle chosen in the formula above is always between the two corresponding lengths!

Shortest distance from a point to a line

For the shortest distance from a point to a line, it will always be the **perpendicular distance** from the point to the line. We covered this briefly in Coordinate Geometry earlier.

To solve such questions:

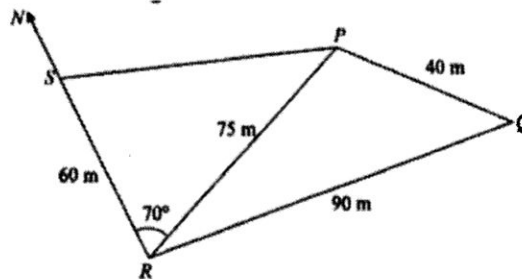
1. Draw a perpendicular line in a diagram to identify the triangle and the new base.
2. Use the formulas for area of triangle.

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}ab \sin C$$

The perpendicular distance it is equivalent to the height, therefore all you must do is to identify the base and find the area of triangle!

Worked Example 7.3

The diagram below shows a plot of land $PQRS$. The lengths of PR , RS , RQ , PQ and length of $\angle PRS$ is labelled clearly in the diagram.



Find,

- (a) the area of land PRS ,
- (b) the length of PS ,
- (c) $\angle PRQ$,
- (d) the shortest distance from R to PQ .

[Solution]

(a) Area of $PRS = \frac{1}{2} \times 60 \times 75 \times \sin 70^\circ$
 $= 2114.3 \text{ m}^2$ (5 s. f.)
 $= 2110 \text{ m}^2$ (3 s. f.)

(b) Using Cosine Rule,

$$PS^2 = 60^2 + 75^2 - 2(60)(75) \cos 70^\circ$$
$$PS = \sqrt{60^2 + 75^2 - 2(60)(75) \cos 70^\circ}$$
$$= 78.402 \text{ m (5 s. f.)}$$
$$= 78.4 \text{ m (3 s. f.)}$$

(c) Using Cosine Rule,

$$PQ^2 = PR^2 + RQ^2 - 2(PR)(RQ) \cos \angle PRQ$$
$$40^2 = 75^2 + 90^2 - 2(75)(90) \cos \angle PRQ$$
$$\cos \angle PRQ = \frac{40^2 - 75^2 - 90^2}{-2(75)(90)}$$
$$= \frac{97}{108}$$
$$\angle PRQ = \cos^{-1} \frac{97}{108}$$
$$= 26.084^\circ \text{ (3 d. p.)}$$
$$= 26.1^\circ \text{ (1 d. p.)}$$

(d) To find the shortest distance from P to RQ , it is equivalent to finding the perpendicular distance.

Step 1: Find the area of triangle PRQ

$$\begin{aligned}\text{Area of triangle } PRQ &= \frac{1}{2} \times 75 \times 90 \times \sin 26.084^\circ \\ &= 1484.0 \text{ m}^2 \text{ (5 s.f.)}\end{aligned}$$

Step 2: Equate the area of triangle with the formula for area of right angled triangle

$$1484.0 = \frac{1}{2} \times RQ \times PX$$

Where X is the foot of perpendicular from P to RQ .

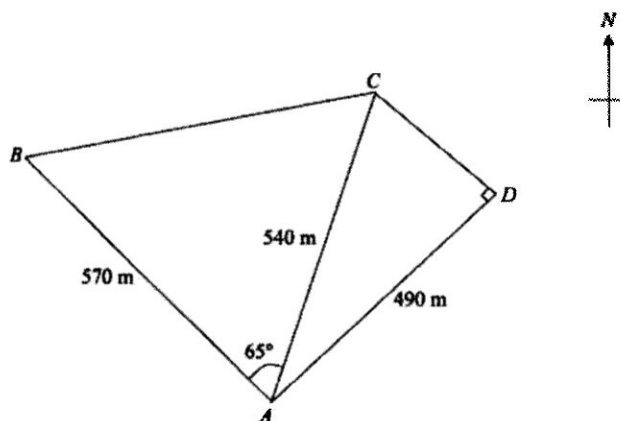
$$\therefore PX = 1484.0 \div \frac{1}{2} \div 40 = 74.2 \text{ m (3sf)}$$

Tip!

Try to always leave your workings in 5sf instead of 3sf. This is to ensure more accurate calculations in future workings. It may seem troublesome, but the difference in accuracy (5sf more accurate than 3sf), might cause the values to change very drastically. We can always present it in 5sf first before then rounding them off into 3sf.

Worked Example 7.4

The diagram below shows a field $ABCD$ on a horizontal ground, crossed by a path AC .



It is given that $AB = 570$ m, $AC = 540$ m, $AD = 490$ m and $\angle BAC = 65^\circ$. Find,

- (a) the length of BC ,
- (b) the value of $\sin \angle BCA$.

[Solution]

- (a) Using Cosine Rule,

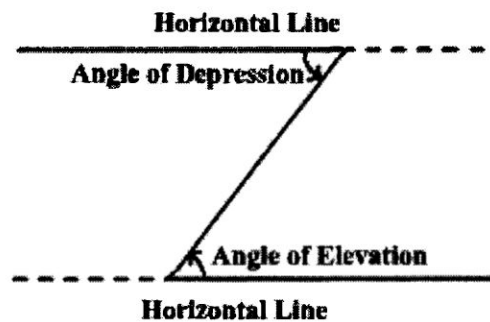
$$\begin{aligned}BC^2 &= 570^2 + 540^2 - 2(570)(540) \cos 65^\circ \\BC &= \sqrt{570^2 + 540^2 - 2(570)(540) \cos 65^\circ} \\&= 596.94 \text{ m (5 s.f.)} \\&= 597 \text{ m (3 s.f.)}\end{aligned}$$

- (b) Using Sine Rule,

$$\begin{aligned}\frac{\sin \angle BCA}{AB} &= \frac{\sin \angle BAC}{BC} \\ \frac{\sin \angle BCA}{570} &= \frac{\sin 65^\circ}{596.94} \\ \sin \angle BCA &= 570 \times \frac{\sin 65^\circ}{596.94} \\ \angle BCA &= \sin^{-1}\left(570 \times \frac{\sin 65^\circ}{596.94}\right) \\ &= 59.929^\circ \text{ (3 d.p.)} \\ &= 59.9^\circ \text{ (1 d.p.)}\end{aligned}$$

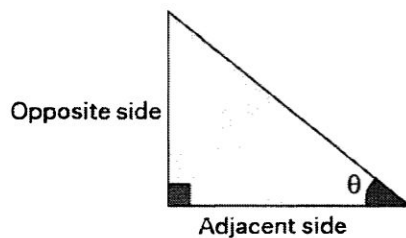
Angle of Depression and Elevation

- It is the angle subtended between the horizontal plane and the line joining the point.



Note!

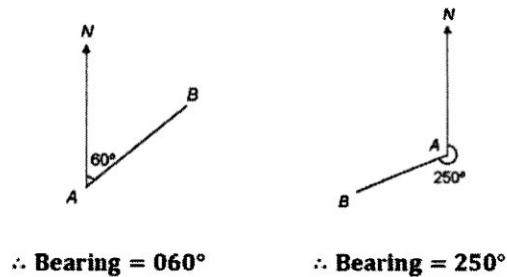
- 1) Angle of Depression and Angle of Elevation are actually ALTERNATE angles.
- 2) Questions testing on Angle of Depression or Elevation will always be phrased in this way: Angle of Depression or Elevation of A from B, where B is your starting point and A is your ending point.
- 3) You will always make use of trigonometric ratio **$\tan \theta$** in this topic to solve questions.



$$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Bearings

A bearing is an angle, measured in degrees, between the north line (north being 0° or 360°) and the line linking two points (the line of sight or travel direction). In other words, we start facing north, and “turn” clockwise until we face the desired object.



Take note of the following ways of phrasing:

- Bearing of B from A = Standing at A , looking at B
- Bearing of A from B = Standing at B , looking at A .

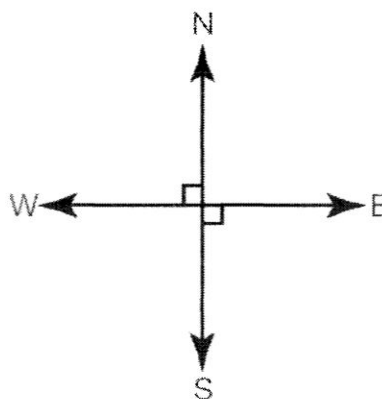
Common Mistake

Presentation of bearings when they are lesser than 100°

- All presentations of bearings must be in 3 digits, even if it is lesser than 100° .
- Incorrect presentation: 60° .
- Correct presentation: 060° .

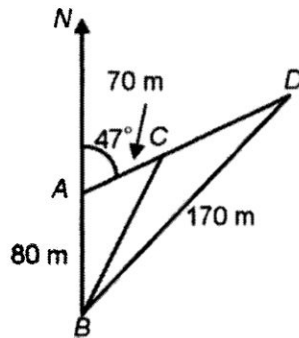
Tip!

It is be very helpful to understand the four directions on the map, North, South, East and West, as you are working with angles in this topic.



Worked Example 7.5

The diagram shows 4 points, A , B , C and D on the ground with D at a bearing of 047° from A . Point B is 80 m due south of A . Point C is on the line AD such that it is 70 m from A . BD is 170 m.



Find,

- (i) the distance BC ,
- (ii) $\angle ADB$,
- (iii) the bearing of D from B ,

There is a tree at point A . A boy measures the angle of elevation of the top of the tree from point B and finds that it is 23° .

- (iv) Find the height of the tree.

[Solution]

- (a) $\angle BAD = 180^\circ - 47^\circ = 133^\circ$ (Adjacent angles on a straight line)

By Cosine Rule,

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC) \cos \angle BAD$$

$$BC^2 = 80^2 + 70^2 - 2(80)(70) \cos 133^\circ$$

$$BC = \sqrt{11300 - 11200 \cos 133^\circ}$$

$$= 137.62 \text{ m (5 s.f.)}$$

$$= 138 \text{ m (3 s.f.)}$$

- (b) By Sine Rule,

$$\frac{\sin \angle ADB}{AB} = \frac{\sin \angle BAD}{BC}$$

$$\frac{\sin \angle ADB}{80} = \frac{\sin 133^\circ}{170}$$

$$\sin \angle ADB = 80 \times \frac{\sin 133^\circ}{170}$$

$$\angle ADB = \sin^{-1}\left(80 \times \frac{\sin 133^\circ}{170}\right)$$

$$= 20.131^\circ \text{ (3 d.p.)}$$

$$= 20.1^\circ (1 \text{ d.p.})$$

(c) Bearing of D from B = $\angle ABD$

$$= 180^\circ - \angle ADB - \angle BAD$$

$$= 180^\circ - 20.131^\circ - 133^\circ$$

$$= 26.869^\circ (3 \text{ d.p.})$$

$$= 26.9^\circ (1 \text{ d.p.})$$

(d) Let point E be the top of the tree.

The angle of elevation of the top of the tree, point E, from point B is between the horizontal line AB and the line connecting point B and point E, $\angle ABE$.

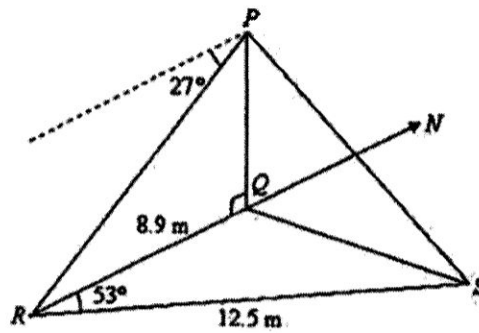
$$\tan \angle ABE = \frac{AE}{AB}$$

$$\tan 23^\circ = \frac{AE}{80}$$

$$\text{Height of the tree} = AE = 80 \tan 23^\circ$$

$$= 34.0 \text{ m } (3 \text{ s.f.})$$

Worked Example



In the diagram above, Q , R and S are three points on horizontal ground. $RQ = 8.9$ m and $SR = 12.5$ m. A vertical flagpole PQ stands at Q and the angle of depression of R from P is 27° . R is due south of Q and the bearing of S from R is 053°

(a) Find

- (i) the length of QS
- (ii) the angle QSR
- (iii) the bearing of S from Q
- (iv) the area of triangle QRS
- (v) the height of the flagpole PQ

(b) A man walks from R to S . Find the greatest angle of elevation of P from a point on RS

[S4 DSS P2/2017 MYE Qn 9]

Solution

(a) (i) To find QS , we shall use cosine rule

$$\begin{aligned} QS^2 &= QR^2 + RS^2 - 2(QR)(RS) \cos \angle QRS \\ &= (8.9)^2 + (12.5)^2 - 2(8.9)(12.5) \cos 53^\circ \\ &= 235.46 - 222.5 \cos 53^\circ \\ \therefore QS &= \sqrt{235.46 - 222.5 \cos 53^\circ} \\ &= 10.07750 \dots \\ &= 10.1 \text{ m (3s.f)} \end{aligned}$$

(ii) To find $\angle QSR$, we shall use sine rule

$$\begin{aligned} \frac{\sin \angle QSR}{QR} &= \frac{\sin \angle QRS}{QS} \\ \frac{\sin \angle QSR}{8.9} &= \frac{\sin 53^\circ}{\sqrt{235.46 - 222.5 \cos 53^\circ}} \\ \angle QSR &= \sin^{-1} \left(\frac{8.9 \sin 53^\circ}{\sqrt{235.46 - 222.5 \cos 53^\circ}} \right) \\ &= 44.855309 \dots \\ &= 44.9^\circ \text{ (1d.p)} \end{aligned}$$

- (iii) To find the bearing, we shall use the property: exterior angle is equal to the sum of interior opposite angles of a triangle

$$\text{Bearing of } S \text{ from } Q = \angle QSR + \angle QRS$$

$$\begin{aligned} &= \sin^{-1} \left(\frac{8.9 \sin 53^\circ}{\sqrt{235.46 - 222.5 \cos 53^\circ}} \right) + 53^\circ \\ &= 097.855309 \dots \\ &= 097.9^\circ \text{ (3s.f)} \end{aligned}$$

- (iv) To find the area of triangle QRS ,

$$\begin{aligned} \text{Area} &= \frac{1}{2} (QR)(RS) \sin \angle QRS \\ &= \frac{1}{2} (8.9)(12.5) \sin 53^\circ \\ &= 55.625 \sin 53^\circ \\ &= 44.42100 \dots \\ &= 44.4 \text{ m}^2 \end{aligned}$$

- (v) The angle of depression of R from P corresponds to $\angle QRP$ due to alternate angles

$$\therefore \tan \angle QRP = \frac{PQ}{RQ}$$

$$\tan 27^\circ = \frac{PQ}{8.9}$$

$$\begin{aligned} PQ &= 8.9 \tan 27^\circ \\ &= 4.534776 \dots \\ &= 4.53 \text{ m (3s.f)} \end{aligned}$$

- (b) The point of the greatest angle of elevation will correspond to the shortest distance of Q from RS . This shortest distance is the perpendicular distance from Q to RS

$$\therefore \text{Area of triangle } QRS = \frac{1}{2} (RS)(\text{perpendicular distance})$$

$$55.625 \sin 53^\circ = \frac{1}{2} (12.5)(\text{perpendicular distance})$$

$$\begin{aligned} \text{perpendicular distance} &= \frac{2(55.625 \sin 53^\circ)}{12.5} \\ &= 8.9 \sin 53^\circ \end{aligned}$$

Let the angle of elevation be θ , and by using the trigonometric ratio $\tan \theta$

$$\begin{aligned} \tan \theta &= \frac{\text{height of flagpole}}{\text{perpendicular distance}} \\ &= \frac{8.9 \tan 27^\circ}{8.9 \sin 53^\circ} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{8.9 \tan 27^\circ}{8.9 \sin 53^\circ} \right) \\ &= 32.537671 \dots \end{aligned}$$

$$= 32.5^\circ \text{ (1d.p)}$$

Tips / Thinking Process

This question is slightly more challenging than the rest of the common trigonometry questions as this question deals with a North line that is not pointing vertically upwards.

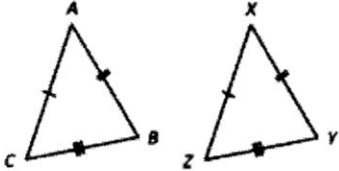
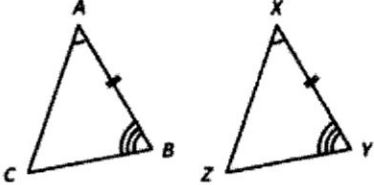
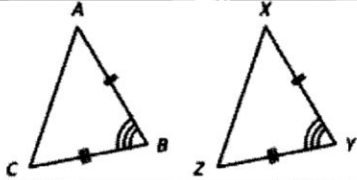
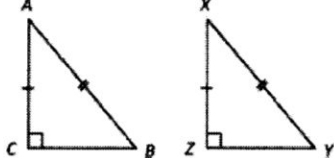
Be extra careful of the orientation of the North line as now the whole question needs to follow that new North line. Students need to be extra careful when dealing with bearings and always take reference to the question's North line.

All parts are already explained in full in the solutions above, so do read through the solutions for the explanations as to why the questions are done that way

8 Congruence and Similarity Tests

The entire chapter of congruence and similarity hinges on one main technique - being able to identify similar lengths and ratios between two figures. We will first start with 2D figures, before moving on to 3D figures.

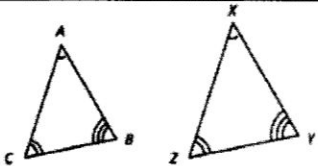
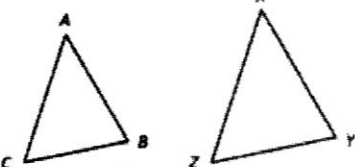
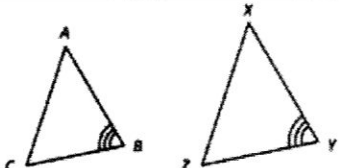
Congruency Tests for Triangles

Name	Test	Diagram Representation
'Side-side-side' SSS	$AB = XY$ $BC = YZ$ $AC = XZ$	
'Angle-side-angle' ASA	$AB = XY$ $\angle CAB = \angle ZXY$ $\angle CBA = \angle ZYX$	
'Side-angle-side' SAS	$AB = XY$ $BC = YZ$ $\angle ABC = \angle XYZ$	
'Right angle-hypotenuse-side' RHS	$AB = XY$ $BC = YZ$ $\angle ACB = \angle XZY = 90^\circ$	

Take Note!

For the SAS test, the angle chosen must be the included angle in-between the 2 chosen sides for the test.

Similarity Tests for Triangles

Name	Test	Diagram representation
'Angle-angle-angle' AAA	$\angle CAB = \angle ZXY$ $\angle ABC = \angle XYZ$ $\angle ACB = \angle XZY$	
Corresponding sides same ratio	$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$	
Ratio of 2 sides same + 1 included angle between the 2 chosen sides	$\angle ABC = \angle XYZ$ $\frac{AB}{XY} = \frac{BC}{YZ}$	

Take Note!

The naming conventions of Congruency & Similarity Questions are important! Many students make mistake when defining the sides that they want to use in their arguments

Example: To show that $\triangle ABC \cong \triangle XYZ$



To show that 2 sides are equal in length, the direction of how students define the length must stay consistent (Either all Clockwise or Anti-Clockwise)

$$AC = XZ \quad \text{or} \quad CA = ZX$$

To show that 2 angles are equal in size, the direction of how students define the angles must stay consistent (Either all Clockwise or Anti-Clockwise)

$$\angle ABC = \angle XYZ \quad \text{or} \quad \angle CBA = \angle ZYX$$

Justifications when answering questions

Formal justification must be provided at each step, no matter how trivial.

1. Equality of length/angle

- If the question states that the length/angles are equal, simply write the equal lengths/angles and state that it is “given”

$$AC = XZ \text{ (given)}$$

- If you require to perform some calculations to find the lengths, do not write “given”! Only do so when the question explicitly states so.

2. Shared length/angle

- If 2 triangles have a common side/angle (means that 2 triangles are stuck together and joined with a side/angle that both triangles have), simply write that the side/angle is common

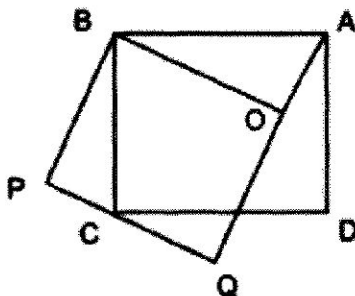
AC is a common side

3. Calculated angles

- If the question requires some calculations for the angles, reasons must be explicitly stated as to how you come about with said calculations.

Eg: $\angle ABC = \angle XYZ$ (alternate angles)

Worked Example 8.1



In the diagram, $ABCD$ is a square and $OBPQ$ is a rectangle. PCQ and AQ are straight lines.

- (a) Show that $\angle PBC$ is equal to $\angle OBA$
- (b) Prove that $\triangle PBC$ is congruent to $\triangle OBA$

[S4 MGS P1/2018 PRELIM Qn 18 (Modified)]

[Solution]

- (a) Let $\angle PBC = \alpha^\circ$

$$\angle PBO = 90^\circ \text{ (angles in a rectangle } OBPQ)$$

$$\angle CBO = 90^\circ - \alpha^\circ \text{ (complementary angles)}$$

$$\angle ABC = 90^\circ \text{ (angles in a square } ABCD)$$

$$\angle OBA = \angle ABC - \angle CBO$$

$$= 90^\circ - (90^\circ - \alpha^\circ)$$

$$= \alpha^\circ \text{ (complementary angles)}$$

$$\therefore \angle PBC = \angle OBA$$

- (b) $\angle PBC = \angle OBA$ (shown above) (A)

$$BC = BA \text{ (sides of a square are equal in length) (S)}$$

$$\angle BPC = 90^\circ \text{ (angle in a rectangle)}$$

$$\angle BOQ = 90^\circ \text{ (angle in a rectangle)}$$

$$\angle BOA = 90^\circ \text{ (} QOA \text{ is a straight line, supplementary angles)}$$

$$\therefore \angle BPC = \angle BOA \text{ (A)}$$

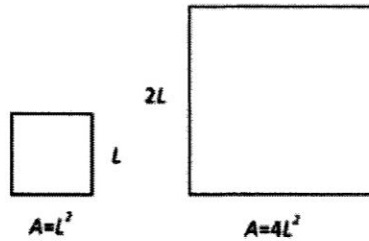
By the ASA congruency test, $\triangle PBC$ is congruent to $\triangle OBA$

Tips!

For part (b), a very common mistake is to say that PB is equal to OB . However, this is not correct as $OBPQ$ is a rectangle. It is stated explicitly in the question. Although the diagram does look like a square, do be careful as the diagram might not be as it seems.

Area of similar planar figures

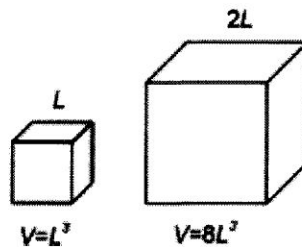
If 2 figures are similar, then the ratio of their areas is equal to the square of the ratio of the lengths of any pairs of corresponding sides.



$$\frac{A_1}{A_2} = \left(\frac{I_1}{I_2}\right)^2 = \frac{(I_1)^2}{(I_2)^2}$$

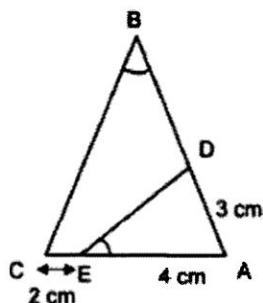
Volume of similar planar figures

If 2 solids are similar, then the ratio of their volumes is equal to the cube of the ratio of the lengths of any pairs of corresponding sides



$$\frac{V_1}{V_2} = \left(\frac{I_1}{I_2}\right)^3 = \frac{(I_1)^3}{(I_2)^3}$$

Worked Example 8.2



In the diagram, $\angle ABC = \angle AED$

- Explain why $\triangle ABC$ and $\triangle AED$ are similar
- Given that $AD = 3$ cm, $AE = 4$ cm and $EC = 2$ cm, calculate
 - BD
 - $\frac{\text{Area of } BCED}{\text{Area of } AED}$

[S4 CCHS(Y) P1/2017 PRELIM Qn 15 (Modified)]

[Solution]

- $\angle ABC = \angle AED$ (given) (A)

$$\angle BAC = \angle EAD \text{ (common angle) (A)}$$

By induction, $\angle BCA = \angle EDA$ (A)

Hence, using the AAA similarity test, $\triangle ABC$ is similar to $\triangle AED$

- Since $\triangle ABC$ is similar to $\triangle AED$

$$\frac{AB}{AE} = \frac{BC}{ED} = \frac{CA}{DA}$$

$$\frac{BD+AD}{4} = \frac{2+4}{3}$$

$$3(BD + AD) = 6(4)$$

$$3(BD + 3) = 24$$

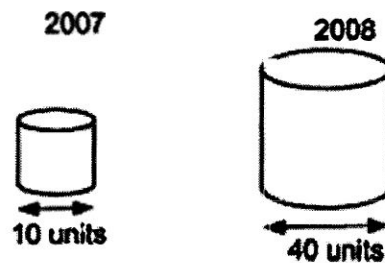
$$BD = 5 \text{ cm}$$

- Let $\triangle ABC$ of Figure 1, $\triangle AED$ be Figure 2

$$\frac{A_1 \text{ of } \triangle ABC}{A_2 \text{ of } \triangle AED} = \left(\frac{l_1}{l_2}\right)^2 = \left(\frac{6}{3}\right)^2 = \frac{36}{9}$$

$$\therefore \frac{\text{Area of } BCED}{\text{Area of } AED} = \frac{\text{Area of } ABC - \text{Area of } AED}{\text{Area of } AED} = \frac{36-9}{9} = 3$$

Worked Example 8.3



The diagram below is a graphical representation that shows the volume of petrol consumed by Country X in year 2007 and 2008. The report accompanying the diagram claims that the volume of petrol consumed has increased by 4 times from 2007 to 2008. By using the concept of similar figures and data analytics, explain briefly why the graphical representation above is inaccurate. Show calculations if necessary.

[S4 AHS P1/2018 PRELIM Qn 2]

[Solution]

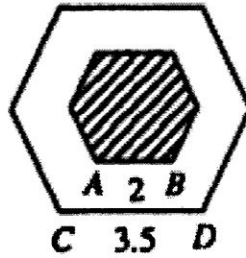
Let the cylinder representing 2007 be Diagram 1 and the other representing 2008 be Diagram 2.

By using the concept of volume of similar figures,

$$\begin{aligned}\frac{V_1}{V_2} &= \left(\frac{l_1}{l_2}\right)^3 \\ &= \left(\frac{10}{40}\right)^3 \\ &= \frac{1}{64}\end{aligned}$$

There is misrepresentation of information as the consumption has increased by 64 times instead of 4

Worked Example 8.4



In a recent concert held at the Singapore Indoor Stadium, the regular hexagonal central stage is divided into 2 sections. The shaded area represents a raised stage, similar to the central stage, while the remaining unshaded area represents a water stage.

- (a) Given that $AB = 2$ m and $CD = 3.5$ m, find the value of

$$\frac{\text{Area of raised stage}}{\text{Area of water stage}}$$

- (b) It is given that the area of the raised stage is 4 m^2 . Find the volume of water required to fill the water stage given that the depth is 5 cm.

[S4 VS P1/2017 MYE Qn 9]

[Solution]

- (a) Since the stages are similar,

$$\begin{aligned}\frac{\text{Area of raised stage}}{\text{Area of both stages}} &= \left(\frac{2}{3.5}\right)^2 \\ &= \frac{16}{49} \\ \therefore \frac{\text{Area of raised stage}}{\text{Area of water stage}} &= \frac{16}{49-16} \\ &= \frac{16}{33}\end{aligned}$$

- (b) Since the stages are similar,

$$\begin{aligned}\frac{\text{Area of raised stage}}{\text{Area of water stage}} &= \frac{16}{33} \\ \text{Area of water stage} &= 4 \div \frac{16}{33} \\ &= 8.25 \text{ m}^2 \\ \therefore \text{Volume of water} &= 8.25 \times 0.05 \\ &= \frac{33}{80} \text{ m}^3\end{aligned}$$

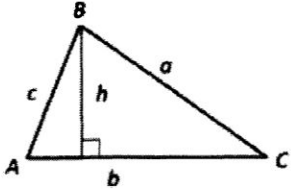
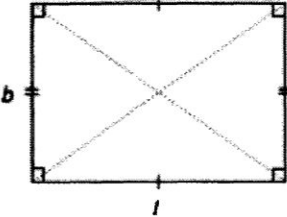
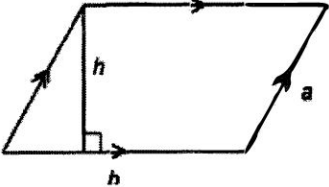
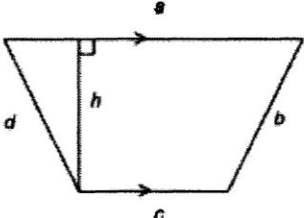
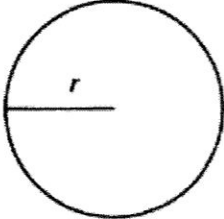
Tips!

This is a very standard geometrical similarity of similar solids kind of question.

Although it is not explicitly stated that the hexagons are geometrically similar, do remember that regular polygons of the same number of sides are similar. This is due to them having the same interior angles

9 Mensuration and Circular Measure

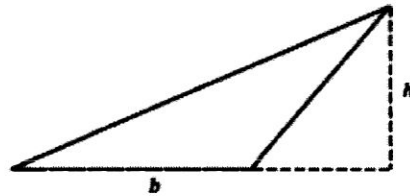
Plane Figures (recap)

Figure	Perimeter	Area	Diagram Representation
Triangle	$a + b + c$	$\frac{1}{2}bh$	
Square Rectangle	$2(l + b)$	bl	
Parallelogram Rhombus	$2(a + b)$	bh	
Trapezium	$a + b + c + d$	$\frac{1}{2}h(a + c)$	
Circle	πd	πr^2	

Things to take note about Plane Figures

1. Finding area of triangle

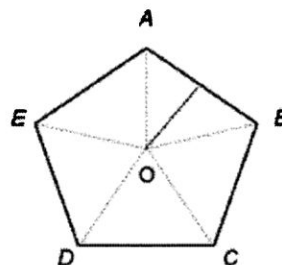
- Many students struggle to find the area of this triangle in exams.



- To find the area, we extend the base and draw a vertical line from the tip of the triangle h to this new extended base.
- Do note that the dotted lines above in the triangle are additional construction lines and NOT part of the triangle!

2. Area of regular polygon

- This can be found by dividing the polygon into identical triangles and then finding the area of one of these triangles.



- Since this is a regular polygon, all of the interior angles are the same. We can use the formula for 1 interior angle, and the formula $\frac{1}{2}ab \sin C$ to find the area of 1 triangle.

Assume the above shape is a regular pentagon of side length 4 cm.

Sum of interior angles = $(5 - 2) \times 180^\circ = 540^\circ$

$\angle OAB = \angle OBA = 540^\circ \div 10 = 54^\circ$

Let X be the foot of perpendicular from O to AB .

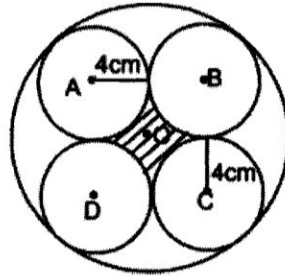
$$XA = 4 \text{ cm} \div 2 = 2 \text{ cm}$$

$$\cos 54^\circ = \frac{2}{OA} \Rightarrow OA = \frac{2}{\cos 54^\circ} = 3.4026 \text{ cm}$$

Hence,

$$\text{Area of pentagon} = \frac{1}{2}(3.4026)(2) \sin 54^\circ \times 10 = 27.5 \text{ cm}^2$$

Worked Example 9.1



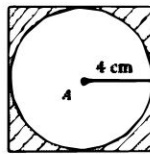
The figure shows 4 equal circles of centres A, B, C and D , each of radius 4 cm drawn in contact with one another bounded by a larger circle centre O

- Calculate the area of the shaded region
- Calculate the perimeter of the shaded region
- Show that the diameter of the larger circle is $(8 + 8\sqrt{2})$ cm

[S3 CCHS(M) P1/2011 EOY Qn 8]

[Solution]

(a)



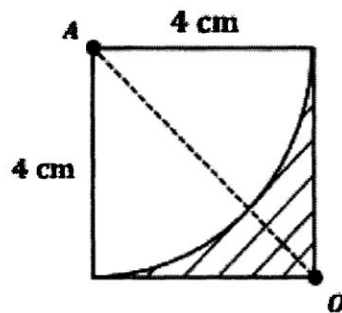
The shaded area can be rearranged around one small circle

$$\begin{aligned}\text{Shaded area} &= (4 + 4)^2 - \pi(4)^2 \\ &= 13.7 \text{ cm}^2 \text{ (3s.f.)}\end{aligned}$$

- (b) The perimeter of the shaded area corresponds to circumference

$$\begin{aligned}\text{Perimeter of shaded area} &= 2(\pi)4 \\ &= 25.1 \text{ cm (3s.f.)}\end{aligned}$$

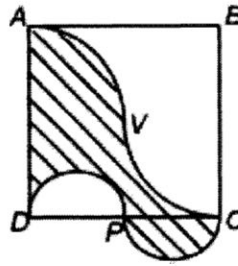
(c)



$$OA = \sqrt{4^2 + 4^2} = \sqrt{32}$$

$$\begin{aligned}\text{Diameter of larger circle} &= 4 + \sqrt{32} + \sqrt{32} + 4 \\ &= 8 + 8\sqrt{2} \text{ cm (shown)}\end{aligned}$$

Worked Example 9.2



In the diagram, $ABCD$ is a square of sides 4 cm with centre V . P is the midpoint of CD . Two identical semicircles are drawn on DP and PC as the diameter. AV and VC are two identical quarter circles. Taking the value of π to be $\frac{22}{7}$,

- Find the perimeter of the shaded region
- Find the area of the shaded region

[S4 PLMGS P1/2016 PRELIM Qn 23]

[Solution]

- Length of arc AV = Length of arc VC

$$= \frac{1}{4} \left(\frac{22}{7} \right) (4)$$

$$= \frac{22}{7} \text{ cm}^2$$

Length of arc DP = Length of arc PC

$$= \frac{1}{2} \left(\frac{22}{7} \right) (2)$$

$$= \frac{22}{7} \text{ cm}^2$$

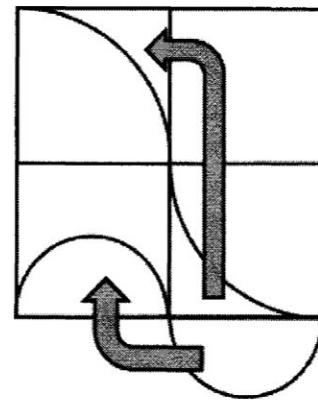
$$\therefore \text{Perimeter of shaded region} = 4 + 2 \left(\frac{22}{7} \right) + 2 \left(\frac{22}{7} \right)$$

$$= 16 \frac{4}{7} \text{ cm}^2$$

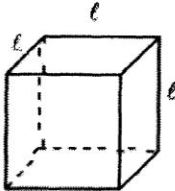
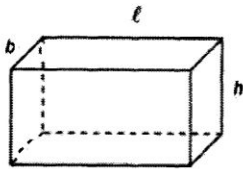
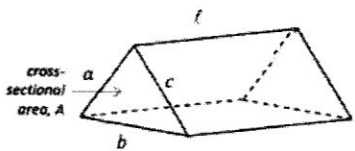
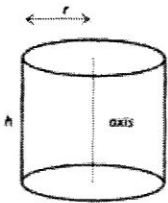
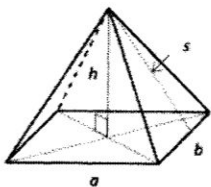
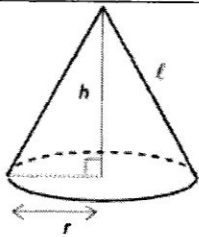
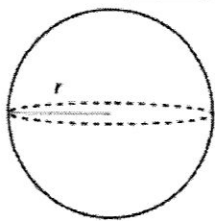
- This shaded region can be redrawn into a rectangle

$$\therefore \text{Area of shaded region} = 2 \text{ cm} \times 4 \text{ cm}$$

$$= 8 \text{ cm}^2$$



Solid Figures

Figure	Surface Area	Volume	Diagram Representation
Cube	$6l^2$	l^3	
Cuboid	$2(lb + lh + bh)$	lbh	
Prism	$2A + (a + b + c)l$	Al	
Cylinder	$2\pi rh + 2\pi r^2$	$\pi r^2 h$	
Pyramid	$\frac{1}{2}s(2a + 2b) + ab$	$\frac{1}{3}abh$	
Cone	$\pi rl + \pi r^2 *$	$\frac{1}{3}\pi r^2 h$	
Sphere	$4\pi r^2 *$	$\frac{4}{3}\pi r^3$	

* Note that the formulas in the asterisk above are provided in the formula sheet

Things to take note about Solids

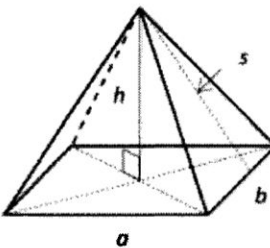
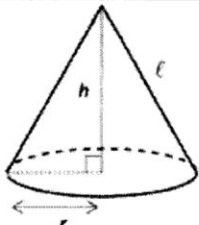
- **Open Solids**

- Certain questions have the solid as an open container (i.e. no top cover). Be careful not to blindly apply the formulas on the previous page - always read carefully!

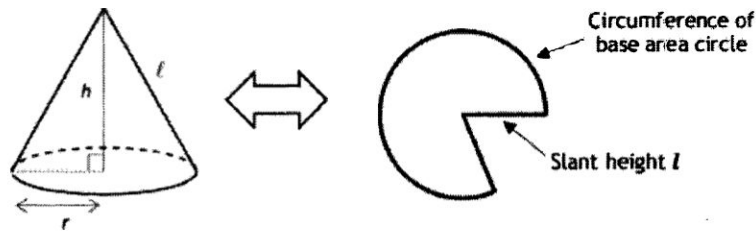
Figure	Surface Area	Breakdown of Formulas
Prism	$2A + (a + b + c)l$	Cross-sectional area: $2A$ Sides: $(a + b + c)l$
Cylinder	$2\pi rh + 2\pi r^2$	Cross-sectional area: $2\pi r^2$ Curved surface area: $2\pi rh$
Pyramid	$\frac{1}{2}s(2a + 2b) + ab$	Based area: ab Side profile triangles: $\frac{1}{2}s(2a + 2b)$
Cone	$\pi rl + \pi r^2$	Base area: πr^2 Curved surface area: πrl
Hemisphere	$3\pi r^2$	Based area: πr^2 Spherical surface area: $2\pi r^2$

- **Slant Height for Pyramid & Cone**

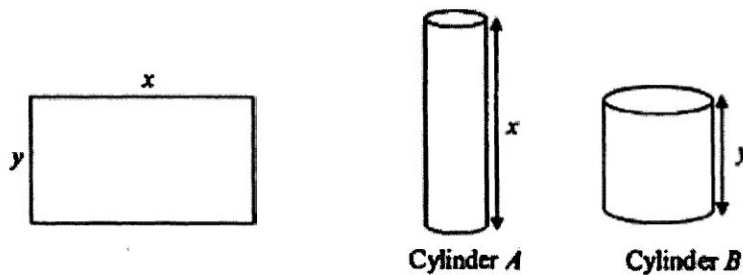
- Note that we can draw a right-triangle within the pyramid and cone. Hence, these slant heights can be calculated with Pythagoras' Theorem

Figure	Slant Height	Figure Representation
Pyramid	$\sqrt{h^2 + \left(\frac{1}{2}a\right)^2}$ $\sqrt{h^2 + \left(\frac{1}{2}b\right)^2}$	
Cone	$\sqrt{h^2 + r^2}$	

- Question extension for Cones
 - When cones are opened up or flattened out, they can be extended to arc length & area of sector questions



Worked Example 9.3



A rectangular card has length x cm and breadth y cm, where $x > y$. The card can be folded to form cylinder A of height x cm or cylinder B of height y cm. Show that the volume of cylinder B is greater than the volume of cylinder A .

[S4 BSS P1/2017 PRELIM Qn 16]

Solution

Cylinder A

$$2\pi r = y$$

$$r = \frac{y}{2\pi}$$

$$\begin{aligned} \text{Volume} &= \pi \left(\frac{y}{2\pi} \right)^2 x \\ &= y \left(\frac{xy}{4\pi} \right) \end{aligned}$$

Since $x > y$,

$$x \left(\frac{xy}{4\pi} \right) > y \left(\frac{xy}{4\pi} \right)$$

Cylinder B

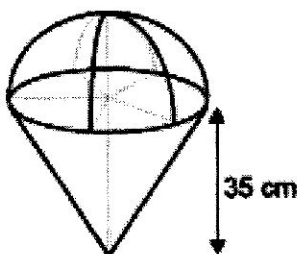
$$2\pi r = x$$

$$r = \frac{x}{2\pi}$$

$$\begin{aligned} \text{Volume} &= \pi \left(\frac{x}{2\pi} \right)^2 y \\ &= x \left(\frac{xy}{4\pi} \right) \end{aligned}$$

Therefore, Volume of cylinder $B >$ Volume of cylinder A (shown)

Worked Example 9.4



The figure shows an ice-cream cone mould that a company would like to use. Above is made up of a solid metallic cone of diameter 27 cm and height 35 cm and a solid hemisphere.

Taking $\pi = \frac{22}{7}$, calculate,

- (a) The mass of the solid, if the density of the metal is 0.7 g/cm^3 (give your answer in kg).
- (b) The total surface area of the solid, correct to the nearest cm^2 .
- (c) The new radius, if the original figure is melted down and reshaped into a sphere.

[S4 DHS P2/2011 PRELIM Qn 5]

Solution

- (a) Radius of cone = 13.5 cm

$$\begin{aligned}\text{Volume of cone portion} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\left(\frac{22}{7}\right)(13.5)^2(35) \\ &= 6682\frac{1}{2} \text{ cm}^3\end{aligned}$$

$$\begin{aligned}\text{Volume of hemispherical portion} &= \frac{2}{3}\pi r^3 \\ &= \frac{2}{3}\left(\frac{22}{7}\right)(13.5)^3 \\ &= 5155\frac{1}{14} \text{ cm}^3\end{aligned}$$

$$\begin{aligned}\text{Total volume of solid} &= 6682\frac{1}{2} + 5155\frac{1}{14} \\ &= 11837\frac{4}{7} \text{ cm}^3\end{aligned}$$

$$\begin{aligned}\therefore \text{Mass of solid} &= \frac{\left(11837\frac{4}{7}\right)(0.7)}{1000} \\ &= 8.2863 \dots \\ &= 8.29 \text{ kg (3s.f.)}\end{aligned}$$

(b) Let l be the slant height of the cone

$$l = \sqrt{35^2 + 13.5^2}$$

$$= \sqrt{1407\frac{1}{4}} \text{ cm}$$

$$\therefore \text{Curved surface area of cone} = \pi r l$$

$$= \left(\frac{22}{7}\right) (13.5) \left(\sqrt{1407\frac{1}{4}}\right)$$

$$= 42\frac{3}{7} \left(\sqrt{1407\frac{1}{4}}\right) \text{ cm}^2$$

$$\text{Curved surface area of hemisphere} = 2\pi r^2$$

$$= 2 \left(\frac{22}{7}\right) (13.5)^2$$

$$= 1145\frac{4}{7} \text{ cm}^2$$

$$\therefore \text{Total surface area of solid} = 42\frac{3}{7} \left(\sqrt{1407\frac{1}{4}}\right) + 1145\frac{4}{7}$$

$$= 2737.208 \dots$$

$$= 2737 \text{ cm}^2 \text{ (nearest cm}^2\text{)}$$

(c) We equate the volume of the solid to the volume of a sphere to find the radius

$$11837\frac{4}{7} = \frac{4}{3} \pi r^3$$

$$11837\frac{4}{7} = \frac{4}{3} \left(\frac{22}{7}\right) r^3$$

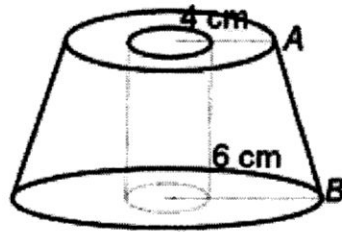
$$\therefore r = \sqrt[3]{\frac{\left(11837\frac{4}{7}\right)}{\left(\frac{88}{21}\right)}}$$

$$= 14.1 \text{ cm (3s.f.)}$$

Common Mistake

Premature rounding for this chapter (especially) will result in many inaccurate calculations as we move from part to part. Remember to always use 5sf, or if possible, exact form.

Worked Example 9.5



A circus searched for a company to make them pedestals. The upper part of a solid wooden right circular cone was cut off, leaving the frustum illustrated in the diagram. The top and the base of the frustum are horizontal circles with radii 4 cm and 6 cm respectively. The slant height AB of the frustum is 7 cm. A cylindrical hole of radius 2 cm is drilled centrally through the frustum.

- (a) Calculate the height of the frustum
- (b) Given that the total mass of the wood removed when drilling the hole was 60 g, calculate the density of the wood in g/cm^3
- (c) Show that the slant height of the cone which was cut off is 14 cm
- (d) Hence, calculate the total surface area of the frustum

[S4 AHS P2/2011 PRELIM Qn 7]

[Solution]

- (a) Let the height of the frustum be h

$$7^2 = h^2 + 2^2$$

$$49 = h^2 + 4$$

$$h = \sqrt{45}$$

$$= 6.708203 \dots$$

$$= 6.71 \text{ cm}$$

- (b) Volume of wood removed $= \pi r^2 h$

$$= \pi(2)^2(\sqrt{45})$$

$$= 4\pi\sqrt{45} \text{ cm}^3$$

$$\text{Density of wood} = \frac{60}{4\pi\sqrt{45}}$$

$$= 0.711762 \dots$$

$$= 0.712 \text{ g/cm}^3$$

- (c) Let the slant height of the cone which was cut off be l . We know that the 2 cones are geometrically similar

$$\frac{l}{l+7} = \frac{4}{6}$$

$$6l = 4l + 28$$

$$2l = 28$$

$$l = 14 \text{ cm}$$

- (d) Surface area of the cylinder surface = $2\pi rh$

$$= 2\pi(2)(\sqrt{45})$$

$$= 4\pi\sqrt{45} \text{ cm}^2$$

$$\text{Area of the top portion of the frustum} = \pi(4)^2 - \pi(2)^2$$

$$= 16\pi - 4\pi$$

$$= 12\pi \text{ cm}^2$$

$$\text{Area of the base of the frustum} = \pi(6)^2 - \pi(2)^2$$

$$= 36\pi - 4\pi$$

$$= 32\pi \text{ cm}^2$$

$$\text{Area of the curved surface area} = \pi(6)(14 + 7) - \pi(4)(14)$$

$$= 126\pi - 56\pi$$

$$= 70\pi \text{ cm}^2$$

$$\text{Total surface area of the frustum} = 4\pi\sqrt{45} + 12\pi + 32\pi + 70\pi$$

$$= 442.439339 \dots$$

$$= 442 \text{ cm}^2 \text{ (3s.f)}$$

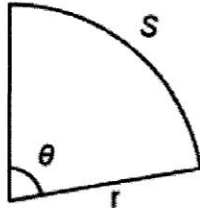
Tips!

This frustum is another common shape in mensuration. Remember that a frustum is formed by cutting off the top of a cone, leaving behind the stump.

Do note that the top part being cut off is also a cone, and this small cone is geometrically similar to the entire cone.

Circular Measure

The shape below is known as a sector, which is cutting a fraction out from a circle. Using fractions, we have the following formulas.



$$\text{Arc length} = \frac{\theta}{360^\circ} \times 2\pi r$$

$$\text{Area of sector} = \frac{\theta}{360^\circ} \times \pi r^2$$

Where $2\pi r$ and πr^2 gives the circumference of the circle and area of circle respectively.

Using **radians** to measure the angle instead of degrees, since $180^\circ = \pi$ radians, we have,

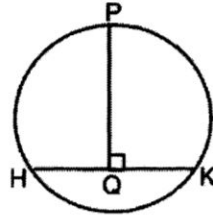
$$\text{Arc length} = \frac{\theta}{2\pi} \times 2\pi r = r\theta$$

$$\text{Area of sector} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta$$

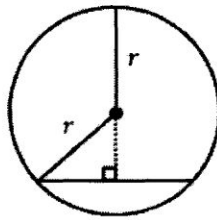
These formulas allow us to calculate the arc length and area of a sector much more easily. Remember however, that the angle has to be measured in radians instead of degrees!

2 Common (but difficult) figures in exams!

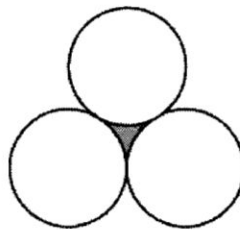
- Find the radius of the following figure (given lengths HK and PQ)



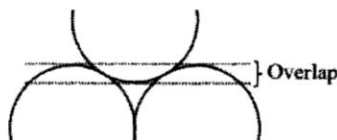
- Concept:** Recall that any line from the centre of the circle to the circumference is the radius. A right-triangle can be drawn in the figure and, by Pythagoras' Theorem, the radius can be found



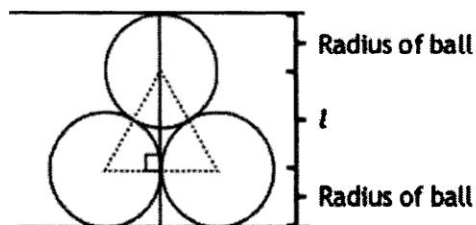
- Find the height of the following figure.



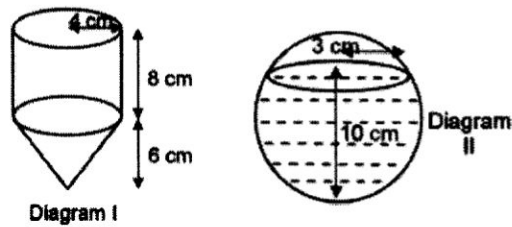
- Common mistake: Since there is a portion that overlaps (shown below), the height of the figure is not simply 2 times of the length of the diameter.



- To find the height, we must inscribe a triangle within the 3 circles where the sides of the triangles are made up with 2 radii of the circle.



Worked Example 9.6



An open container is made by joining a cylinder of radius 4 cm and a cone of height 6 cm as shown in Diagram I. The container is completely filled with water

- (a) If the height of the cylinder is 8 cm, find,
- the volume of water,
 - the total surface area of the container.
- (b) All the water from the container is poured into a spherical bowl such that the height of the water in the bowl is 10 cm and the radius of the water surface is 3 cm. Find,
- the radius of the spherical bowl,
 - the percentage of the bowl that is filled with water.

[S4 ANDSS P2/2009 PRELIM Qn 3]

[Solution]

- (a) (i) Volume of water = Volume of cone + Volume of cylinder

$$\begin{aligned}
 &= \frac{1}{3}\pi(4)^2(6) + \pi(4)^2(8) \\
 &= 503 \text{ cm}^3 \text{ (3s.f)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) Slant height} &= \sqrt{4^2 + 6^2} \\
 &= \sqrt{52} \text{ cm}
 \end{aligned}$$

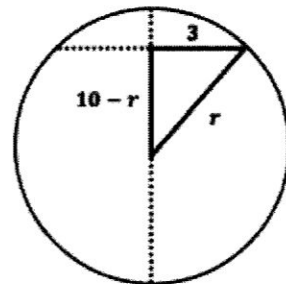
Total surface area = Surface area of cone + Surface area of cylinder

$$\begin{aligned}
 &= \pi(4)(\sqrt{52}) + 2\pi(4)(8) \\
 &= 292 \text{ cm}^2
 \end{aligned}$$

- (b) (i) We need to draw a right-angled triangle inscribed in the bowl

By Pythagoras' Theorem,

$$\begin{aligned}
 (10 - r)^2 + 3^2 &= r^2 \\
 100 - 20r + r^2 + 9 &= r^2 \\
 20r &= 109 \\
 r &= 5.45 \text{ cm}
 \end{aligned}$$

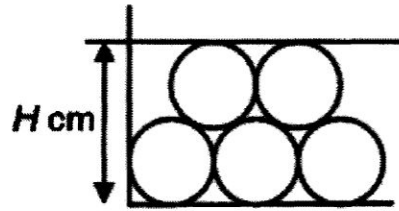


- (ii) We first need to find the volume of the sphere

$$\text{Volume of spherical bowl} = \frac{4}{3}\pi(5.45)^3$$

$$\text{Percentage of bowl that is filled} = \frac{160\pi}{\left[\frac{4}{3}\pi(5.45)^3\right]} \times 100\% = 74.1\% \text{ (3s.f)}$$

Worked Example 9.7

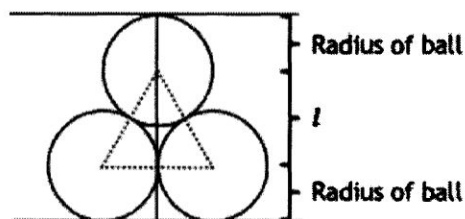


In a wood cutting competition, contestants are to cut out as many spherical balls of radius 6 cm as possible from a piece of wood. The piece of wood has length 96 cm, width 72 cm and height 45 cm.

- Contestant *A* sketches out how he intends to cut out the spherical balls. The diagram shows a portion of the piece of wood. Find the value of H .
- Find the maximum number of spherical balls that can be cut out by Contestant *A* using this method.
- Another Contestant *B* decides to cut the block into cubes first such that the balls are fully inscribed in each cube. Find the maximum number of spherical balls that can be cut using this method.
- Contestant *C* manages to cut out 150 balls. If he fully submerges the 150 balls into a cylindrical container of paint, find the increase in the height of the paint level if the radius of the container is 24 cm.
- Find the total area of these 150 identical spherical balls that will be coated by paint.

[Solution]

(a)



Let the vertical height in the centre be l

By Pythagoras' Theorem,

$$l = \sqrt{12^2 - 6^2}$$

$$= \sqrt{108}$$

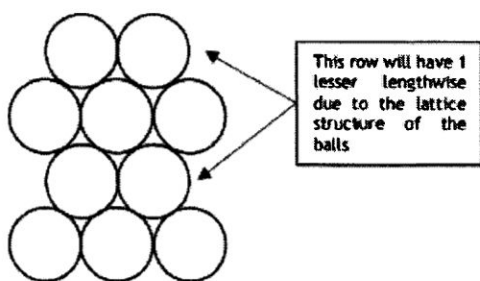
$$\therefore H = \sqrt{108} + 6 + 6$$

$$= 12 + \sqrt{108}$$

$$= 22.39230 \dots$$

$$= 22.4 \text{ cm (3s.f)}$$

(b)



$$\text{Length-wise maximum} = 96 \div 12$$

$$= 8$$

$$\text{Width-wise maximum} = 72 \div 12$$

$$= 6$$

$$\text{Height-wise maximum} = 45 \div (12 + \sqrt{108})$$

$$\approx 2$$

$$\therefore \text{Maximum number of balls} = (8 \times 6 \times 2) + (7 \times 6 \times 2)$$

$$= 180$$

(c) Since the balls are cut into cubes, the lengths of the cubes should be equal to the diameter of the balls

$$\text{Height-wise maximum} = 45 \div 12$$

$$= 3.75$$

$$\approx 3$$

$$\therefore \text{Maximum number of balls} = 8 \times 6 \times 3$$

$$= 144$$

(d) To find the increase in height, we need to first figure out how much will these 150 balls displace in the can of paint

$$\text{Volume of 150 balls} = 150 \left(\frac{4}{3} \pi r^3 \right)$$

$$= 150 \left(\frac{4}{3} \right) \pi (6)^3$$

$$= 43200\pi \text{ cm}^3$$

$$\therefore \text{Height increase} = 43200\pi \div \pi(24)^2$$

$$= 75 \text{ cm}$$

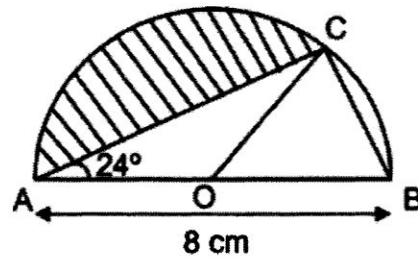
(e) Total surface area of all 150 balls = $150(4\pi r^2)$

$$= 150(4)\pi(6)^2$$

$$= 67858.401 \dots$$

$$= 67900 \text{ cm}^2 \text{ (3s.f)}$$

Worked Example 9.8



The diagram above shows a semicircle of diameter AB . It is given that $AB = 8$ cm and C is a point on the semicircle such that $\angle CAB = 24^\circ$. Find,

- (a) the area of $\triangle ABC$,
- (b) the perimeter of the shaded region subtended by the arc AC .

[S4 TKSS P2/2014 PRELIM Qn 4]

[Solution]

- (a) $\angle ACB = 90^\circ$ (angles in a semicircle)

Since this is a right-angled triangle, we can use trigonometric ratios to solve for the side AC

$$\cos \angle CAB = \frac{AC}{AB}$$

$$AC = 8 \cos 24^\circ$$

$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} (AC)(AB) \sin \angle CAB$$

$$= \frac{1}{2} (8 \cos 24^\circ)(8) \sin 24^\circ$$

$$= 11.89031 \dots$$

$$= 11.9 \text{ cm}^2 \text{ (3s.f)}$$

- (b) $OA = OC$ (radii of circle)

$\therefore \triangle AOC$ is an isosceles triangle

$$\angle AOC = 180^\circ - 24^\circ - 24^\circ$$

$$= 132^\circ \text{ (angles in an isosceles triangle)}$$

$$\text{Arc length } AC = r\theta$$

$$= \left(\frac{8}{2}\right) \frac{132^\circ}{180^\circ} \times \pi$$

$$= \frac{44}{15} \pi \text{ cm}$$

$$\therefore \text{Perimeter of shaded region} = \frac{44}{15} \pi + 8 \cos 24^\circ$$

$$= 16.5 \text{ cm (3sf)}$$

10 Circle Properties

Symmetrical Properties of circles

Theorem	Property	Diagram Representation
Chord Theorem	If two chords are equidistant from the centre of the circle ($OE = OF$) then the length of the chords are the same ($AB = CD$)	
Perpendicular Bisector Theorem	If the length of two chords are the same ($AB = CD$) then they are equidistant from the centre of the circle ($OE = OF$)	
	A line segment drawn from the centre to the midpoint of the chord is perpendicular to the chord If $AM = MB$, then $AB \perp OM$	
Tangent Theorem	The perpendicular bisector of a chord passes through the centre	
	Every point on the perpendicular bisector of a line segment is equidistant from the end points of the segment $AC = BC$	

Angle Properties of Circle

Property	Diagram Representation
Angles in the same segment $\angle BAO = \angle DCO$	

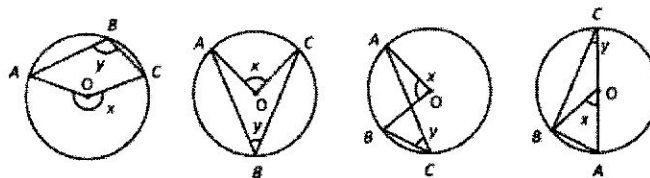
Tip!

Always look out for the “butterfly” shape to use this property. The only condition is that all 4 vertices of this “butterfly” must touch the circumference of the circle

Angle at centre is 2 times the angle at the circumference $\angle AOC = \angle ABC$ $x = 2y$	
--	--

Tip!

There are many variations of this theorem. Always look out for this arrow shape to use this property. This arrow head must also consist of the centre of the circle



Common Mistakes

Many students get tricked by the third figure as it looks like a “butterfly” - angle in same segment. Recall that angle in the same segment only applies if all 4 vertices touch the circumference.

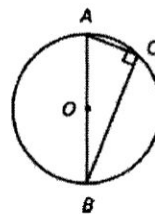
Many students get also tricked by this figure



Many students think that $\alpha = 2\beta$ when in actual fact there is no special relationship between α and β UNLESS if the 2 lines above are tangents that extend to a point, then

$$\alpha + \beta = 180^\circ$$

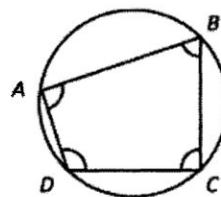
Angle in a semicircle
 $\angle ACB = 90^\circ$



Tip!

This property can only be used if AOB is the diameter of the circle. We should not assume unless it is explicitly given in the question!

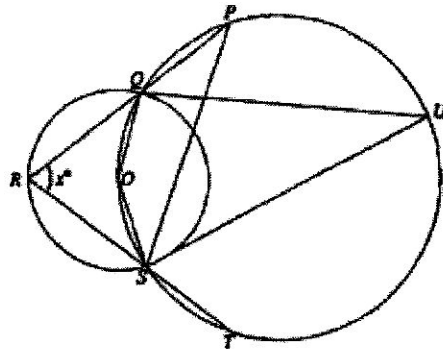
Opposite angles of a cyclic quadrilateral
are 180° (supplementary)
 $\angle ADC + \angle ABC = 180^\circ$



Tip!

This property can only be used when the quadrilateral is a cyclic quadrilateral - all 4 vertices must lie on the circumference of the circle. We should not assume unless it is explicitly given in the question!

Worked Example 10.1



In the diagram, circle QRS and circle PUT meet at points Q and S . O lies on circle PUT and is the centre of the circle QRS . RQP and RST are straight lines. Given that $\angle QRS = x^\circ$,

showing all reasons clearly,

(a) Find $\angle QUS$ in terms of x

(b) Determine if $\triangle RPS$ is an isosceles triangle

[S4 FSS P1/2017 MYE Qn 10]

[Solution]

(a) $\angle QOS = 2x^\circ$ (angle at centre is 2 times angle at circumference)

$$\angle QUS = 180^\circ - \angle QOS \text{ (angles in opposite segments)}$$

$$= 180^\circ - 2x^\circ$$

(b) $\angle RPS = \angle QUS$ (angles in same segment)

$$= 180^\circ - 2x^\circ$$

$$\angle PSR = 180^\circ - x^\circ - (180^\circ - 2x^\circ) \text{ (angle sum of triangle)}$$

$$= x^\circ$$

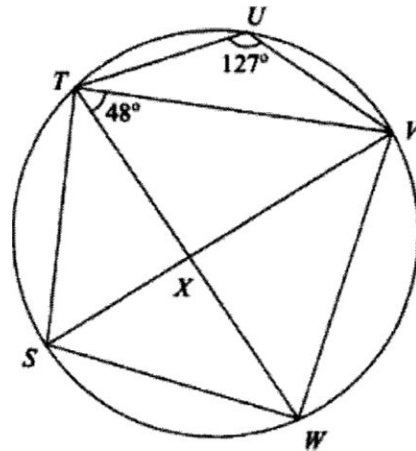
Since $\angle PSR = \angle PRS = x^\circ$, this implies that $\triangle RPS$ is an isosceles triangle

Thinking Process

Do recall that to prove that a triangle is an isosceles triangle, we need to show that either 2 lengths are equal, or 2 of the angles are equal.

We can first let an angle be unknown (say, x), and afterwards prove that another angle ends up with the same unknown

Worked Example 10.2



A circle is given in the diagram above, where S, T, U, V and W are points on a circle. SV is the diameter and it intersects TW at X . $\angle VTW = 48^\circ$ and $\angle VUT = 127^\circ$

(a) Find, stating your reasons clearly,

- (i) $\angle VSW$,
- (ii) $\angle TVS$,
- (iii) $\angle WXV$.

(b) Is X the centre of the circle? Explain your answer, stating your reasons clearly.

[S4 SCGS P1/2018 PRELIM Qn 19]

[Solution]

(a) (i) $\angle VSW = \angle VTW$ (angles in the same segment)
 $= 48^\circ$

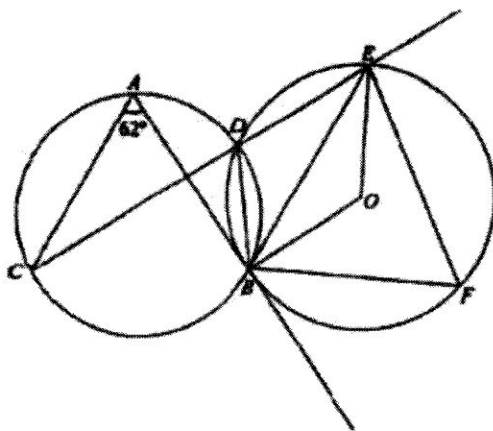
(ii) $\angle STV = 90^\circ$ (right angle in semicircle)
 $\angle TSV = 180^\circ - 127^\circ$ (angles in opposite segments)
 $= 53^\circ$
 $\angle TVS = 180 - 90^\circ - 53^\circ$ (angle sum of triangle)
 $= 37^\circ$

(iii) $\angle WXV = 37^\circ + 48^\circ$ (exterior angle of triangle)
 $= 85^\circ$

(b) $\angle TSW = 53^\circ + 48^\circ$
 $= 101^\circ \neq 90^\circ$

TW is not a diameter of the circle due to the property (right angle in semicircle).
Hence, since a diameter must cut the centre of the circle, X is not the centre of the circle.

Worked Example 10.3



In the diagram shown above, AB is a tangent to the circle with centre O . It is given that $\angle CAB = 62^\circ$ and CDE is a straight line.

(a) By stating your reasons clearly, find,

- (i) $\angle CDB$,
- (ii) $\angle BDE$,
- (iii) $\angle EFB$,
- (iv) $\angle BOE$,
- (v) $\angle EBO$.

(b) Find $\angle ABE$. Hence write a statement about the lines CA and BE . Justify your statement.

[S4 CCHS(M) P2/2016 PRELIM Qn 8]

[Solution]

(a) (i) $\angle CDB = \angle CAB$ (angles in the same segment)

$$= 62^\circ$$

(ii) $\angle BDE = 180^\circ - \angle CDB$ (adjacent angles on straight line)

$$= 180^\circ - 62^\circ$$

$$= 118^\circ$$

(iii) $\angle EFB = 180^\circ - 118^\circ$ (angles in opposite segments)

$$= 62^\circ$$

(iv) $\angle BOE = 62^\circ \times 2$ (angle at centre is 2 times angle at circumference)

$$= 124^\circ$$

(v) To find $\angle EBO$, we note that $\triangle EBO$ is an isosceles triangle as OE and OB are radii of the circle

$$\angle EBO = \frac{(180^\circ - 124^\circ)}{2} \quad \begin{array}{l} \text{(angle sum of triangle)} \\ \text{(base angles of isosceles triangle)} \end{array}$$

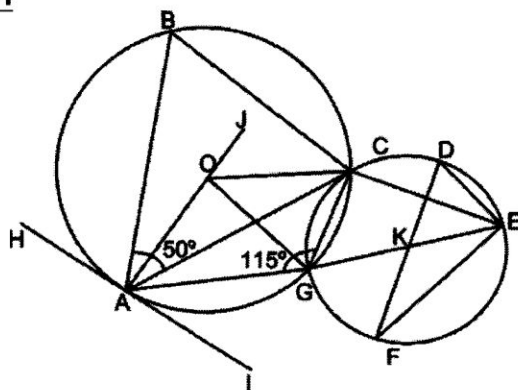
$$= 28^\circ$$

$$\begin{aligned} \text{(b) } \angle ABE &= 90^\circ - 28^\circ \text{ (tangent perpendicular to radius)} \\ &= 62^\circ \end{aligned}$$

$$\angle ABE = \angle CAB = 62^\circ$$

This implies that CA and BE are parallel due to the property of alternate angles between parallel lines

Worked Example 10.4



The points A, B, C, D, E, F and G lie on the circumference of two circles. The centre of the smaller circle is marked K and shown in the diagram above. AOJ is a straight line and HAI is a tangent to the bigger circle. $\angle BAC = 50^\circ$ and $\angle AGC = 115^\circ$.

- (a) Calculate, stating your reasons clearly,
- $\angle IAC$
 - $\angle JOC$
- (b) Show that AOJ bisects the $\angle BAC$
- (c) Given that $\angle DFE = \angle DEC = 27.5^\circ$, prove that CG and KD are parallel lines

[Solution]

- (a) (i) $\angle ABC = 180^\circ - 115^\circ$ (angles in opposite segments)
 $= 65^\circ$
 $\therefore \angle IAC = \angle ABC$ (alternate segment theorem)
 $= 65^\circ$
- (ii) $\angle AOC = 2 \times \angle ABC$ (angle at centre is 2 times angle at circumference)
 $= 2 \times 65^\circ$
 $= 130^\circ$
 $\therefore \angle JOC = 180^\circ - 130^\circ$ (adjacent angles on straight line)
 $= 50^\circ$

(b) Since $\angle JOC = 50^\circ$,

$$\angle JAC = \frac{50^\circ}{2} \quad (\text{angle at centre is 2 times angle at circumference})$$

$$= 25^\circ$$

$$\angle BAJ = 50^\circ - 25^\circ$$

$$= 25^\circ$$

$$= \angle JAC$$

Since $\angle BAJ = \angle JAC$, the line AJ cuts the angle $\angle BAC$ exactly in half, implying that AJ is an angle bisector of $\angle BAC$, bisecting $\angle BAC$

(c) $\angle DEF = 90^\circ$ (right angle in semicircle)

Let the intersection point of lines CE and DF be L

$$\angle LEF = 90^\circ - \angle DEC$$

$$= 62.5^\circ$$

$$\angle KLF = 180^\circ - \angle LEF - \angle DFE \quad (\text{angle sum of triangle})$$

$$= 180^\circ - 62.5^\circ - 27.5^\circ$$

$$= 90^\circ$$

$$\angle GCE = 90^\circ \quad (\text{right angle in semicircle})$$

Since $\angle KLF = \angle GCE = 90^\circ$, this implies that these 2 angles are corresponding angles, implying that CG is parallel to KD

Tips!

For any such questions involving Circle Properties, recall all the properties and treat them like a checklist. After going through one property for one question, it is unlikely that it will appear again. Use all the special memorization techniques (listed below) to help identify the property needed to find the angle.

Always be on the lookout for the common shapes like:

- Arrow: Angle at centre is 2 times angle at circumference
- Butterfly: Angles in the same segment
- Diameter: Angles in a semicircle
- 4-sided figure inscribed in the circle: Angles in opposite segments

11 Set Notation

Sets can be specified in 3 unique ways

1. Listing

- Elements are separated by commas and enclosed within braces “{ }”

$$A = \{1, 2, 3, 4\}$$

2. Description (not very common)

- Elements are being described in words.

$$A = \{\text{students playing basketball}\}$$

3. Set Builder Notation

- All elements will satisfy a certain property stated within the set.

$$A = \{x: \text{prime numbers between } 0 < x < 20\}$$

Tip!

When attempting questions involving the ‘Set Builder Notation’, always write out all the elements in the set first, to avoid confusion, and prevent missing out any terms.

Set Notation

Symbol	Name	Definition
ξ	Universal Set	Set which contains all the elements relevant to the question
A'	Complement	All the elements that are not in set A
$A \subseteq B$	Subset	A set A is a subset of B if and only if each element in A is also an element in B
$A \subset B$	Proper Subset	A set A is a proper subset of B if and only if each element in A is also an element in B , but B has at least 1 element that is not in A .

Conversely, the notations “ $\not\subseteq$ ” and “ $\not\subset$ ” implies that the sets are not subsets / proper subsets respectively.

Set Notation (continued)

Symbol	Name	Definition
$A = B$	Equality	2 sets are equal if and only if they contain the same elements
$\emptyset$ OR $\{ \}$	Empty/Disjoint	A set containing no elements
$A \cap B$	Intersection	The set of elements common in both sets
$A \cup B$	Union	The set of elements found in A or B or both sets

Common Mistakes

1. Null Set

- Given an empty set A ,

$$n(A) = \emptyset \quad \mathbf{X}$$

- LHS: $n(A)$ refers to the number of elements in set A
- RHS: $\emptyset$ refers to the empty set (null set)

The correct expressions should be:

$$n(A) = 0$$

$$A = \emptyset$$

2. Unique Sets

- Do take note that the following sets are not equal

$$\{\{a\}, \{b\}\} \neq \{a, b\}$$

- LHS: The set containing the sets $\{a\}$ and $\{b\}$
- RHS: A set containing elements a and b

Worked Example 11.1

Given that

$$\varepsilon = \{x: x \text{ is an integer, } 20 \leq x \leq 99\}$$

$$A = \{x: x = 2m + 1, \text{ where } m \text{ is a positive integer}\}$$

$$B = \{x: x \text{ is divisible by } 7\}$$

$$C = \{x: x \text{ is a multiple of } 6\}$$

$$D = \{x: x \text{ has a remainder of } 2 \text{ when divided by } 4\}$$

- (a) State the value of $n(\varepsilon)$ and $n(A \cap D)$.
- (b) List the elements in B .
- (c) Describe, as simply as possible, the meaning of $B \cap C \neq \emptyset$ in words.

[S4 VS P2/2015 Prelim Q3]

[Solution]

$$(a) \quad n(\varepsilon) = 99 - 20 + 1$$

$$= 80$$

$$n(A \cap D) = 0$$

$$(b) \quad B = \{21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98\}$$

$$(c) \quad \text{There are elements which are multiples of } 42.$$

Worked Example 11.2

Given that

$A = \{\text{points lying on the line } 2x + y = 8\}$

$B = \{\text{points lying on the line } 3x - 4 = 12\}$

$C = \{\text{points lying on the line } mx - 4y = c\}$

- (a) Is $(-1, 6) \in A$? Explain your answer.
- (b) Find the element p such that $p \in (A \cap B)$.
- (c) Write down a possible value of m and of c such that $B \cap C = \emptyset$.

[S4 CGSS P1/2018 Prelim Q3]

[Solution]

- (a) By substituting $(-1, 6)$, $\text{LHS} = 2(-1) + 6 = 4$, $\text{RHS} = 8$

$\text{LHS} \neq \text{RHS}$. Hence, $(-1, 6) \notin A$

- (b) For p to be an element in $A \cap B$, p has to be the intersection point of the 2 lines.

$$2x + y = 8 \dots \dots (1)$$

$$3x - 4y = 12 \dots \dots (2)$$

From Equation (1),

$$y = 8 - 2x \dots \dots (3)$$

Substituting Equation (3) into Equation (2),

$$3x - 4(8 - 2x) = 12$$

$$11x = 44$$

$$x = 4$$

Substituting $x = 4$ into Equation (3),

$$y = 8 - 2(4) = 0$$

$$p = (4, 0)$$

- (c) Since $B \cap C = \emptyset$ this implies that the 2 lines cannot intersect. For 2 lines that cannot intersect, this means that the lines are parallel with the same gradient.

$$m = 3$$

Since the lines cannot intersect, the y -intercept can be any value other than the y -intercept of the first line.

C can be any real value other than 12.

Thinking Process

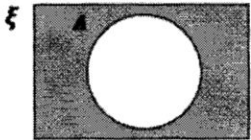
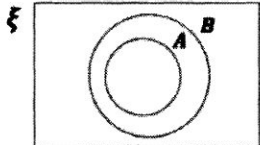
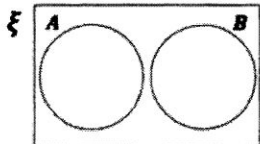
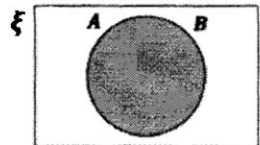
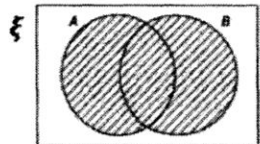
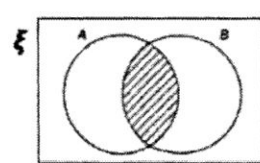
This is a unique question that ties in a lot to Coordinate Geometry.

For part (a), to check if the point is lying on the line, substitute the point into the equation of the line and check if $LHS = RHS$.

For part (b), since we are looking for an element in $A \cap B$, this implies that the lines can intersect. To find the intersection point, we need to use the methods in simultaneous equations.

Explanation for part (c) is in the solution itself.

Venn Diagrams

Set	Name	Diagram Representation
A'	Complement	
$A \subset B$	Proper Subset	
$A \cap B = \emptyset$	Disjoint	
$A = B$	Equality	
$A \cup B$	Union	
$A \cap B$	Intersection	

Worked Example 11.3

It is given that

$$\varepsilon = \{x: x \text{ is an integer between 0 and 9 inclusive}\}$$

$$A \subset \varepsilon \text{ and } B \subset \varepsilon$$

$$\{0, 2\} \subset (A' \cap B)$$

$$7 \in A \cap B$$

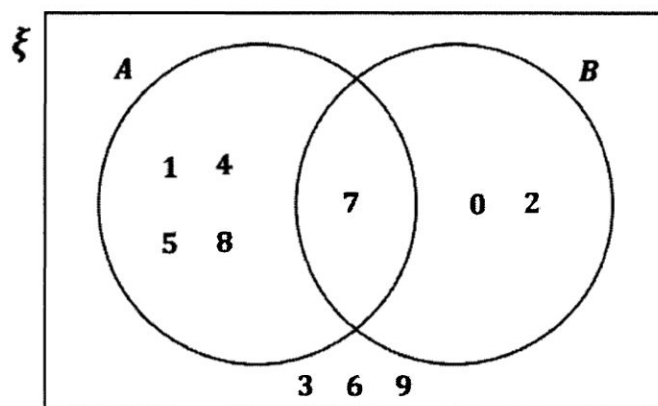
$$\{1, 4, 5, 8\} \subset ((A \cup B) \cap B')$$

$$3, 6, 9 \in (A \cup B)$$

Draw the Venn diagram to represent all the information given

[S4 AHS P1/2018 Prelim Q7(a)]

[Solution]



Thinking Process

Although this looks like a rather complicated question, it is quite simple as it only requires students to understand the notations and do step by step. For example in

$$(A \cup B) \cap B'$$

$A \cup B$ implies the union of A and B, comprising of the region in both sets (circles). Since it is intersecting with B' (everywhere except B), this means that we are only interested in the region that is in A but not in B.

12 Probability

Probability is the branch of Mathematics that deals with the calculating the likelihood of a given event's occurrence. Since "likelihood" is like a "percentage chance" of it happening, the probability value is between 0 and 1, inclusive.

For any event X ,

$$P(X) = \frac{\text{Number of times event } X \text{ occurs}}{\text{Total number of possible events}}$$

Eg:

$$P(\text{rolling a 6 on a fair die}) = \frac{\text{Number of faces containing a 6}}{\text{Total number of possible faces}} = \frac{1}{6}$$

Properties of probability

Event	Probability
Sample space consisting of N possible outcomes that are equally likely	$\frac{1}{N}$
Outcome of an event is not possible	0
Outcome of an event is definitely happening	1

Probability Diagram

Diagram which shows all possible outcomes of an experiment, usually used where each outcome is equally likely to happen.

Sum of faces of 2 die

-	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

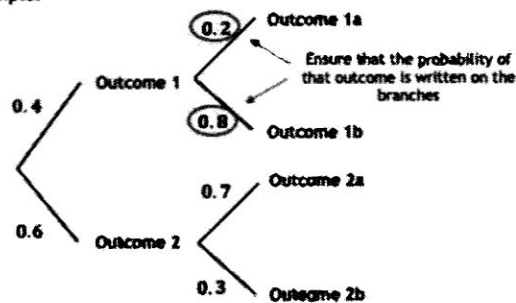
Number of faces showing an even number

-	1	2	3	4	5	6
1	0	1	0	1	0	1
2	1	2	1	2	1	2
3	0	1	0	1	0	1
4	1	2	1	2	1	2
5	0	1	0	1	0	1
6	1	2	1	2	1	2

Tree Diagram

Each of the n possible outcomes are represented as branches of the tree diagram.

Example:



To find the probability of any outcome occurring, we multiply the probabilities along the branches.

eg: $P(\text{outcome 1a}) = 0.4 \times 0.2 = 0.08$.

Notice that

$$P(\text{outcome 1a}) + P(\text{outcome 1b}) + P(\text{outcome 2a}) + P(\text{outcome 2b}) = 1!$$

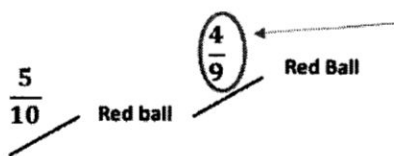
What is “without replacement”?

Replacement involves putting back the object that was taken out.

Imagine 5 red balls and 5 black balls in a bag, and two are taken out without replacement.

Why $\frac{4}{9}$?

Since there is no replacement, and the first ball taken out is red, there are 9 balls remaining, 4 of which are red.



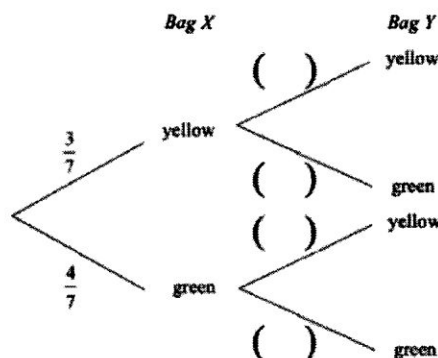
Rules of Probability

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. $P(A') = 1 - P(A)$
3. When considering two separate events (case 1 and case 2), we add the probabilities.
4. When considering two events within the same case (happening simultaneously), we multiply the probabilities.

Worked Example 12.1

Two bags contain coloured marbles. Bag X contains 3 yellow marbles and 4 green marbles. Bag Y contains 5 yellow marbles and 6 green marbles. A marble is drawn at random from Bag X and put into Bag Y . A marble is then drawn at random from Bag Y .

- (a) Complete the tree diagram to show this information.

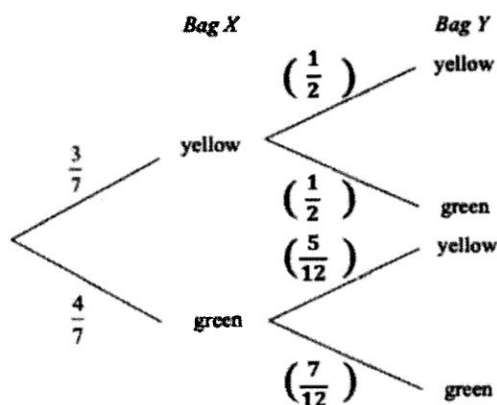


- (b) Find the probability that two marbles drawn are of different colours.
- (c) If the marble drawn from Bag Y has the same colour as the marble drawn from Bag X , it is set aside and another marble is drawn at random from Bag Y . Find the probability that all the three marbles drawn have the same colour.

[S4 SCGS P1/2018 Prelim Q21]

[Solution]

- (a)



(b) $P(\text{different colours}) = \frac{3}{7} \left(\frac{1}{2} \right) + \frac{4}{7} \left(\frac{5}{12} \right) = \frac{19}{42}$

(c) $P(\text{same colours}) = \frac{3}{7} \left(\frac{1}{2} \right) \left(\frac{5}{11} \right) + \frac{4}{7} \left(\frac{7}{12} \right) \left(\frac{6}{11} \right) = \frac{43}{154}$

Tips/Thinking Process

This is a classic question that always appears during examinations. Some questions will provide the tree diagram and ask students to fill up some segments of it. Some might ask you to draw the tree diagram yourselves.

For Part (c), we want to multiply the probability of each smaller branch happening, to get the probability of the entire branch. For eg:

$$\frac{3}{7} \left(\frac{1}{2} \right) \left(\frac{5}{11} \right)$$

- $\frac{3}{7}$ is choosing a yellow from Bag X
- $\frac{1}{2}$ is choosing a yellow from Bag Y
- $\frac{5}{11}$ is choosing another yellow from Bag Y. The total remaining number of marbles in the bag (since no replacement) is 11. And there are 5 yellow marbles in the bag.

Addition Law

Applies for 2 or more mutually exclusive events. When 2 events are mutually exclusive, it means that they cannot happen at the same time.

We usually add probabilities together when we consider different cases.

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

Multiplication Law

Applies for 2 or more independent events. When 2 events are independent, it means that the occurrence of one of them has no influence on the occurrence of the other.

We usually multiply probabilities together when we consider within the same case. For example in 10.1 above, we multiplied the probabilities of each event happening within the branch.

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$

Worked Example 12.2

The pages of a book are numbered from 1 to 50 inclusive. A page is chosen at random. Write down, giving your answer as a fraction in its simplest form, the probability that,

- (a) the page chosen will contain only a single digit,
- (b) the page chosen will contain at least one digit “4”,
- (c) the sum of the digits of the page chosen is more than 8.

[S4 AHS P2/2011 Prelim Q7]

[Solution]

- (a) Pages that only have 1 digit are between pages 1 and 9.

$$\therefore P(\text{Page numbers between 1 and 9}) = \frac{9}{50}$$

- (b) Pages that contain at least the digit “4”:

4, 14, 24, 34, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49 (Total: 14)

$$\begin{aligned}\therefore P(\text{Page that contain at least the digit “4”}) &= \frac{14}{50} \\ &= \frac{7}{25}\end{aligned}$$

- (c) Pages that the sum of digits is > 8 :

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Highlighted cells above show the numbers where the sum of digits is more than 8.

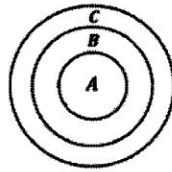
We can see that there are 15 such possible events.

$$\begin{aligned}\therefore P(\text{Page that the sum of digits} > 8) &= \frac{15}{50} \\ &= \frac{3}{10}\end{aligned}$$

Tips

Although it is not explicitly stated, drawing the diagrams in your solutions will help to visualise the situation. It does not need to be the neatest diagram, as long as it is understandable, it will be accepted as a working.

Worked Example 12.3



An archery target comprises of 3 concentric circles, as shown in the diagram. The inner circle, A , has radius of x cm, the next circle, B , has radius $x + 20$ cm and the outermost circle, C , has radius $x + 30$ cm.

- (a) Write down the areas of A and B in terms of x and π .
- (b) Find, in terms of x , the probability that an arrow land within B and outside A .
- (c) The probability that an arrow land with B and outside A is $\frac{2}{9}$. Use this information to form an equation in terms of x , and show that this simplifies to $x^2 - 120x - 900$.
- (d) Hence, find the area of the 3 circles.

[S4 TKGS P2/2019 Prelim Q8]

[Solution]

(a) Area of circle $A = \pi x^2 \text{ cm}^2$

Area of circle $B = \pi(x + 20)^2 \text{ cm}^2$

(b) Probability that arrows lands inside B (including A): $\frac{\pi(x+20)^2}{\pi(x+30)^2}$

Probability that arrow lands inside A : $\frac{\pi x^2}{\pi(x+30)^2}$

Probability that arrow lands inside B and outside A : $\frac{\pi(x+20)^2}{\pi(x+30)^2} - \frac{\pi x^2}{\pi(x+30)^2}$

$$\therefore \frac{\pi(x+20)^2}{\pi(x+30)^2} - \frac{\pi x^2}{\pi(x+30)^2}$$

$$= \frac{\pi[x^2 + 40x + 400] - \pi x^2}{\pi(x+30)^2}$$

$$= \frac{x^2 + 40x + 400 - x^2}{(x+30)^2}$$

$$= \frac{40x + 400}{(x+30)^2}$$

(c) Probability that arrow lands inside B inside A :

$$\frac{40x+400}{(x+30)^2} = \frac{2}{9}$$

$$9(40x + 400) = 2(x + 30)^2$$

$$360x + 3600 = 2(x^2 + 60x + 900)$$

$$2x^2 + 120x + 1800 - 360x + 3600 = 0$$

$$2x^2 - 240x - 1800 = 0$$

$$x^2 - 120x - 900 = 0 \text{ (shown)}$$

(d) $x^2 - 120x - 900 = 0$

$$x = \frac{-(-120) \pm \sqrt{(-120)^2 - 4(1)(-900)}}{2(1)}$$

$$x = \frac{120 \pm \sqrt{18\,000}}{2}$$

$$x = -7.0820(\text{N.A}) \quad \text{or} \quad x = 127.0820$$

$$\text{Area of circle } A = \pi x^2$$

$$= \pi(127.0820)^2$$

$$= 50736.23353 \dots$$

$$= 50\,700 \text{ cm}^2 \text{ (3s.f.)}$$

$$\text{Area of circle } B = \pi(x + 20)^2$$

$$= \pi(127.0820 + 20)^2$$

$$= 67962.47063 \dots$$

$$= 68\,000 \text{ cm}^2 \text{ (3s.f.)}$$

$$\text{Area of circle } C = \pi(x + 30)^2$$

$$= \pi(127.0820 + 30)^2$$

$$= 77518.06698 \dots$$

$$= 77\,500 \text{ cm}^2 \text{ (3s.f.)}$$

Worked Example 12.4

A bag contains 7 black marbles and 3 white marbles. John takes 2 marbles at random from the bag, without replacement. Find the probability that

- (a) Both marbles are black.
- (b) The first marble drawn is black and the second is white.
- (c) One marble is black and the other is white.
- (d) The second marble is white.

[S4 CSS P1/2010 Prelim Q17]

[Solution]

$$\begin{aligned}\text{(a) } P(\text{both black}) &= \frac{7}{10} \times \frac{6}{9} \\ &= \frac{7}{15}\end{aligned}$$

$$\begin{aligned}\text{(b) } P(1^{\text{st}} \text{ black, } 2^{\text{nd}} \text{ white}) &= \frac{7}{10} \times \frac{3}{9} \\ &= \frac{7}{30}\end{aligned}$$

$$\begin{aligned}\text{(c) } P(\text{one black, other white}) &= P(1^{\text{st}} \text{ black, } 2^{\text{nd}} \text{ white}) + P(1^{\text{st}} \text{ white, } 2^{\text{nd}} \text{ black}) \\ &= \frac{7}{30} + \frac{3}{10} \times \frac{7}{9} \\ &= \frac{7}{15}\end{aligned}$$

$$\begin{aligned}\text{(d) } P(2^{\text{nd}} \text{ white}) &= P(1^{\text{st}} \text{ white, } 2^{\text{nd}} \text{ white}) + P(1^{\text{st}} \text{ black, } 2^{\text{nd}} \text{ white}) \\ &= \frac{3}{10} \times \frac{2}{9} + \frac{7}{30} \\ &= \frac{3}{10}\end{aligned}$$

13 Statistical Representations

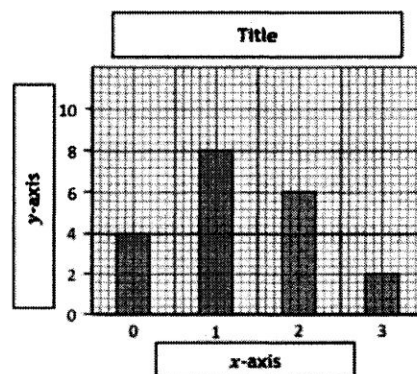
Statistical Diagrams (recap)

1. Tables

A table is a diagram in which numerical information is classified and arranged into rows and columns.

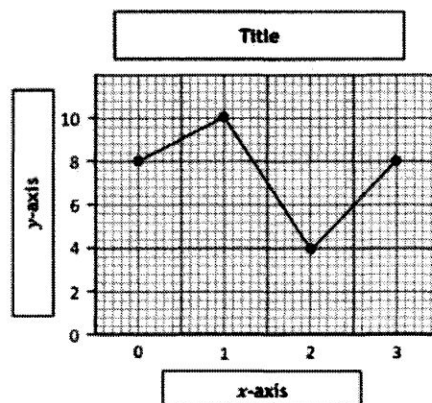
2. Bar Graphs

A bar graph is a diagram in which numerical information is represented by vertical or horizontal bars.



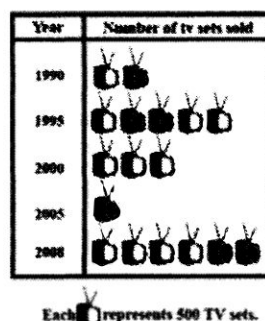
3. Line Graphs

A line graph is a diagram in which numerical information is plotted and subsequently joined by straight line.



4. Pictograms

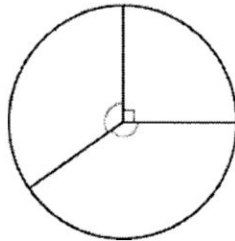
A pictogram is a diagram in which symbols are used to display results.



5. Pie Charts

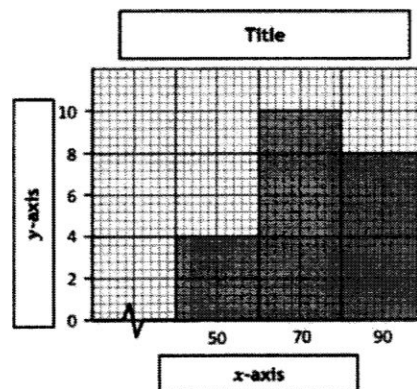
A pie chart is a diagram in which numerical information is represented by sectors of a circle. The angles of the sectors are proportional to the frequencies.

$$\text{Angle} = \frac{\text{Frequency}}{\text{Total Frequency}} \times 360^\circ$$



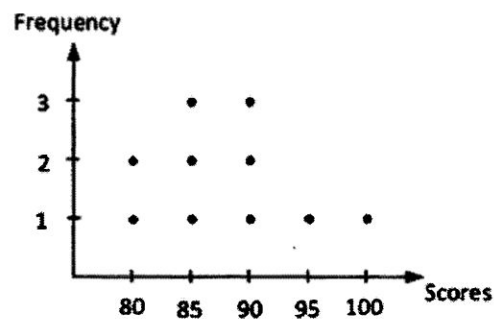
6. Histograms (with equal class intervals)

A histogram is a vertical bar chart in which numerical information is represented by vertical bars.



7. Dot Diagrams

A dot diagram is a diagram which illustrates the pattern of a distribution, showing whether the data are spread out evenly or if they tend to cluster about any point.



8. Stem-and-leaf diagrams

A stem-and-leaf diagram is a plot in which numerical data is organized into rows according to place value designations.

1	1 2 3
2	1 2
3	1 2 3 4
4	1 2
5	1

Legend: 1 | 1 = 11

Worked Example 13.1

The following are marks obtained by 24 students in a Mathematical test.

32	20	44	21	43	34	36	22
55	30	59	40	26	37	35	51
33	51	49	51	58	45	36	46

- Draw a stem-and-leaf diagram to illustrate the results.
- Find the median mark.
- The pass mark is 50. The teacher decided to moderate the marks by multiplying each of the students' mark by 2 and then subtracting 20 from result.
 - Calculate the increase in the number of students who passed after the moderation of marks.
 - John's mark did not change after the moderation. What was his mark?

[S4 SPSS P1/2011 Prelim Q18]

[Solution]

(a)

Stem	Leaf
2	0 1 2 6
3	0 2 3 4 5 6 6 7
4	0 3 4 5 6 9
5	1 1 1 5 8 9

Legend

2 | 0 = 20

- (b) The median is calculated by using the 12th and 13th mark.

$$\therefore \text{Median mark} = \frac{37+40}{2} = 38.5$$

- (c) (i) Let x be the unknown mark.

$$2x - 20 = 50$$

$$x = 35$$

Hence, the initial mark must be 35 and above to be considered a pass after the moderation.

Number of new people that pass = 10

(ii) Let y be John's mark.

Since there is no change in the mark,

$$2y - 20 = y$$

$$y = 20$$

Tips!

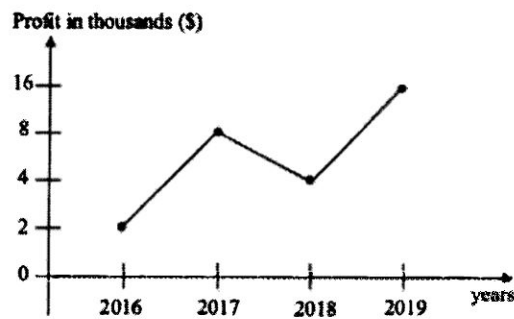
This is a common example of a question where it gives students a bunch of data, and students need to convert this data into a stem-and-leaf diagram.

Do note that the legend drawn in part (a) is very important as many students will forget to write this legend during examinations. Students will lose marks for not writing them.

Evaluation of Statistical Diagrams

Statistical Representation	Purpose	Advantage	Disadvantage
Tables	To classify different results	Provide the exact result Allow for easy manipulation into other statistical representation	Difficult to draw out relationship between results by looking at only numbers
Bar graphs	To compare between 2 variables	Easy to construct and understand	Unable to compare more than 2 variables
Pictograms	To give a simple overview of the results	Able to get a quick overview of the results	Provide simple and less accurate information Construction is difficult
Line Graphs	To show the relationship between 2 variables	Shows the relationship between 2 variables Easy to construct	Not useful if 2 variables have no relations
Pie Chart	To create a visual representation of the proportionality of the different results	Especially useful in representing proportions, percentages, and fractions	Construction is difficult
Histograms	To compare between 2 sets of data	Easy to construct Able to see the frequency of the results through the area of each bar	Unable to compare more than 2 variables

Worked Example 13.2



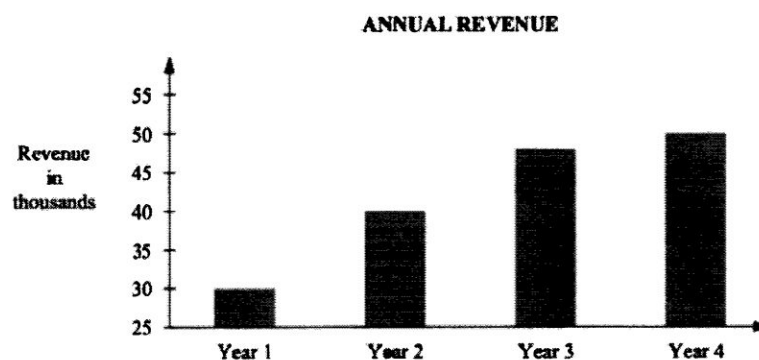
The line graph below shows the profit made by Company X over the 4 years. State one aspect of the graph that may be misleading and explain how this may lead to a misinterpretation of the graph.

[S4 NCHS P1/2018 Prelim Q15]

[Solution]

The vertical axis is inconsistent, hence making the increase from 2016 to 2017 look the same as the increase from 2018 to 2019, but the actual is lesser.

Worked Example 13.3



The bar chart shows the revenue of a company in its first 4 years of operation. State one aspect of the graph that may be misleading and explain how this may lead to a misinterpretation of the graph.

[S4 YTSS P1/2018 Prelim Q7]

[Solution]

The vertical axis does not start from 0, making it seem that the revenue of Year 2 is thrice more than Year 1.

Tips/Thinking Process

Common example of a question that always comes out, misinterpretation of data. Usually, the mistake in the graph will be very obvious.

For line graphs, either the lines go out of the boundary of the axis, or the axis do not run in numerical order, always use these examples to explain.

For bar charts, always check if the axes start at 0 or if the bars are sized proportionally. These are always the key 2 issues for such questions.

Measures of Central Tendency

Mean: The mean is the average of a set of numbers.

Mode: The mode is the value which occurs most frequently.

Median: The median is the middle value of a set of data when it is arranged in ascending order. If there happens to be 2 middle numbers, the median is taken to be the average of the two numbers.

Worked Example 13.4

5 different integers between 19 and 30 were written. The mean is 25 and the median of 26. They have a range of 7. Write down the 5 integers.

[S4 TKSS P1/2018 Prelim Q6]

[Solution]

Since we know that the median is 26 and the range is 7, we have 4 integers already:

21, __, 26, 27, 28

Since we know the mean,

$$25 = \frac{21+n+26+27+28}{5}$$

$$n = 23$$

$$\therefore 21, 23, 26, 27, 28$$

Tips/Thinking Process

Simple and logical question. Recall the definitions of mean and median.

Worked Example 13.5

The data below shows the amount of time taken, in minutes, to complete a piece of Mathematics homework by 10 students.

20 12 15 15 26 35 40 40 24 18

Calculate

- (a) The mean time taken by the students.
- (b) The median time taken.
- (c) Given that one more student took 41 minutes to complete the homework, calculate the mean time taken for all the 11 students.

[S4 NLSS P1/2010 Prelim Q19]

[Solution]

$$\begin{aligned}\text{(a) Mean time} &= \frac{(20+12+15+15+26+35+40+40+24+18)}{10} \\ &= 24.5\end{aligned}$$

(b) Rearranging the numbers in an ordered sequence

12 15 15 18 20 24 26 35 40 40

$$\begin{aligned}\text{Median} &= \frac{20+24}{2} \\ &= 22\end{aligned}$$

$$\begin{aligned}\text{(c) Mean time for all 11 students} &= \frac{(24.5 \times 10 + 41)}{11} \\ &= 26\end{aligned}$$

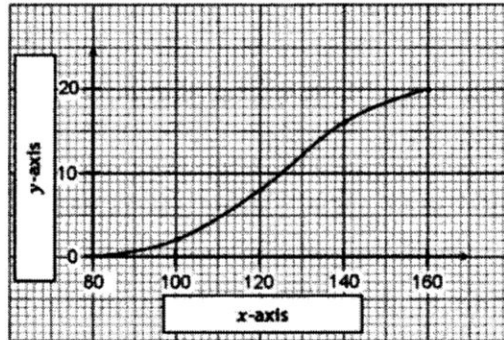
Tips!

For part (b), students often forget to rearrange the values in numerical order before finding the median. And for this case, since the median is an even number, we must take the average of the middle 2 numbers.

Measures of Dispersion

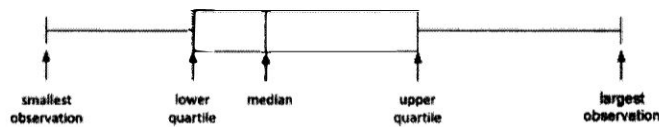
1. Cumulative Frequency diagram

A cumulative frequency diagram is a plot used to determine the number of observations that lie below a certain value.



2. Box-and-whisker plots

A box-and-whisker plot is a plot used to show the range, interquartile range and median of a sample. It provides a quick picture of the range and distribution of numerical data.



3. Range

The range is the difference between the highest and lowest data value.

4. Inter-quartile range

The inter-quartile range is the difference between the 75th percentile and the 25th percentile.

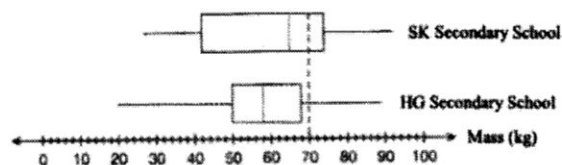
5. Mean & Standard Deviation of grouped data

$$\bar{x} = \frac{\sum f x}{\sum f}$$

$$\sigma = \sqrt{\frac{\sum f x^2}{\sum f} - \left(\frac{\sum f x}{\sum f}\right)^2}$$

- x is the mid-value of the class interval.
- f is the frequency of the interval.

Worked Example 13.6



The following box-and-whisker diagrams show the distribution of the mass of 300 students from each school, SK Secondary School and HG Secondary School respectively.

- What is the median mass for each school?
- Compare the mass of the students from SK Secondary School and HG Secondary School in 2 ways.
- Mary commented that there are more students in HG Secondary School than SK Secondary School who weigh more than or equal to 70 kg. Do you agree with Mary? Support with a reason.

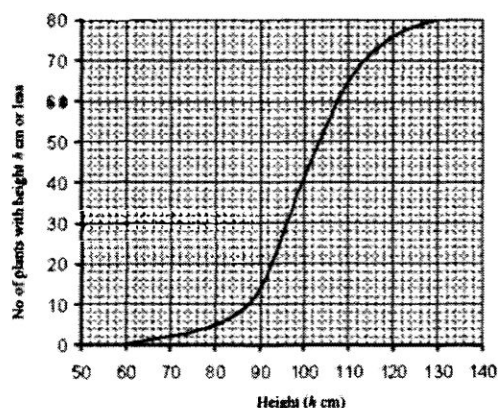
[S4 NCHS P2/2018 Prelim Q9(a)]

[Solution]

- SK Secondary School: Median = 65 kg
HG Secondary School: Median = 58 kg
- Generally, the students for SK Secondary School are heavier compared to the students in HG Secondary School because of the higher median.

For SK, the interquartile range is 32 kg. For HG, the interquartile range is 18 kg. The mass of the students in HG Secondary School is more consistent than that of SK Secondary School because of the smaller interquartile range.
- Disagree. There are more than 25% of the students from SK Secondary School weigh more than 70 kg while there are less than 25% of the students from HG Secondary School that weigh more than 70 kg.

Worked Example 13.7



The cumulative frequency curve shows the height distribution of 80 plants.

- (a) Use the graph to find
- (i) the value of m , if 32.5% of the plants have heights more than m cm.
 - (ii) the probability that two plants chosen will each have a height more than 118 cm.
- (b) (i) The height of the 80 plants was also recorded in the following frequency table.

Height (h cm)	Number of plants
$60 < h \leq 70$	2
$70 < h \leq 80$	a
$80 < h \leq 90$	9
$90 < h \leq 100$	27
$100 < h \leq 110$	23
$110 < h \leq 120$	b
$120 < h \leq 130$	4

Find the value of a and of b .

- (ii) Hence, find the mean and standard deviation of the height of the 80 plants.

[S4 NCHS P2/2018 Prelim Q9(b)]

[Solution]

(a) (i) Number of plants less than or equal to m cm = 67.5% of 80 = 54 plants

(ii) To find the probability,

$$\begin{aligned} P(\text{plants have height more than 118 cm}) &= \frac{6}{80} \times \frac{5}{59} \\ &= \frac{3}{632} \end{aligned}$$

(b) (i) $a = 3$

$$b = 12$$

(ii) To find the mean,

$$\begin{aligned} \bar{X} &= \frac{\sum fx}{\sum f} \\ &= \frac{2(65)+3(75)+9(85)+27(95)+23(105)+12(115)+4(125)}{80} \\ &= 99.75 \text{ cm (exact value)} \end{aligned}$$

To find the standard deviation,

$$\begin{aligned} \sigma &= \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} \\ &= \sqrt{\frac{2(65)^2+3(75)^2+\dots+12(115)^2+4(125)^2}{80} - (99.75)^2} \\ &= 12.646639 \dots \\ &= 12.6 \text{ cm (3s.f.)} \end{aligned}$$

14 Matrices

A matrix is a rectangular array of numbers, symbols or expressions, arranged in rows and columns. The individual items in a matrix are called its elements.

A matrix is typically characterized by their size, written in the form of $(m \times n)$, where m represents the number of rows and n represents the number of columns in the matrix.

A matrix is usually written as follows:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

where the letters represent the elements. The matrix above has an order of 3×3 , consisting of 3 rows and 3 columns.

Special Matrices

Type	Description	Example
Column Matrix	Consists of only one column	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
Row Matrix	Consists of only one row	$\begin{pmatrix} 1 & 2 \end{pmatrix}$
Square Matrix	Consists of same number of rows and columns	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
Identity Matrix	Square matrix where the diagonals are 1's and the remaining entries are 0's.	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Operations on Matrices

1) Addition

- Addition of matrices is an operation where two matrices of the same dimension are added together to produce a new matrix.
- For addition to work, there must be the same number of rows and columns on both matrices.
- The addition is performed element-wise, meaning each element of the first matrix is added to the corresponding element of the second matrix.
- Eg:

$$- A + B = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 1+2 & 3+4 \\ 5+6 & 7+8 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 11 & 15 \end{pmatrix}$$

2) Subtraction

- Subtraction of matrices is an operation where two matrices of the same dimension are subtracted to produce a new matrix.
- Similar to addition, there must be the same number of rows and columns on both matrices for subtraction of matrices to work.
- The subtraction is performed element-wise, meaning each element of the second matrix is subtracted from the corresponding element of the first matrix.
- Example: $C - D = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5-1 & 6-2 \\ 7-3 & 8-4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$

3) Scalar Multiplication

- Scalar multiplication is an operation in which each element of a matrix is multiplied by a fixed number, also known as a scalar.
- This operation can be performed on matrices of any size and it results in a new matrix of the same dimensions where each element is the product of the scalar and the corresponding element of the original matrix.
- Example: $2C = 2 \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 2 \times 5 & 2 \times 6 \\ 2 \times 7 & 2 \times 8 \end{pmatrix} = \begin{pmatrix} 10 & 12 \\ 14 & 16 \end{pmatrix}$

4) Multiplication of Matrix

- Multiplication of matrix is an operation where two matrices are combined to produce a third matrix.
- This operation can only be valid if the number of columns in the first matrix is the same as the number of rows in the second matrix.
- Your final product will be made up with the same number of rows as the first matrix and the same number of columns as the second matrix.
- Each element in the resulting matrix is the sum of the product of the elements in the row from the first matrix and the column from the second matrix.
- Example: $A \times B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

Tip

If dimension of A is $(m \times n)$ and dimension of B is $(n \times p)$, then the resultant matrix of AB has dimension of $(m \times p)$.

Note: There is no resultant matrix BA , unless $p = m$.

Other properties to take note of:

1. Always take note of the order of the matrix. $A - B \neq B - A$
2. Associative Law: $(AB)C = A(BC)$
3. Distributive Law: $A(B + C) = AB + AC$ (left hand multiplication)
 $(A + B)C = AC + BC$ (right hand multiplication)

Worked Example 14.1

- (a) Find the matrix A in the following: $\begin{pmatrix} -6 & 5 \\ 0 & 2 \\ 3 & -1 \end{pmatrix} - A = \begin{pmatrix} 0 & 2 \\ -4 & -3 \\ 3 & 6 \end{pmatrix}$
- (b) Given that $\begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 2x-3 \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$, find the value of x and y .

[Solution]

$$\begin{aligned}
 \text{(a) Matrix } A &= \begin{pmatrix} -6 & 5 \\ 0 & 2 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ -4 & -3 \\ 3 & 6 \end{pmatrix} \\
 &= \begin{pmatrix} -6-0 & 5-2 \\ 0-(-4) & 2-(-3) \\ 3-3 & -1-6 \end{pmatrix} \\
 &= \begin{pmatrix} -6 & 3 \\ 4 & 5 \\ 0 & -7 \end{pmatrix}
 \end{aligned}$$

$$\text{(b) Given that } \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 2x-3 \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix},$$

$$-3 = 3(2x-3) + 4(y) \Rightarrow -3 = 6x + 4y - 9 \Rightarrow 6x + 4y = 6 \dots (1)$$

$$7 = -2(2x-3) + (-1)(y) \Rightarrow 7 = -4x - y + 6 \Rightarrow y = -4x - 1 \dots (2)$$

Subbing equation (2) into (1),

$$6x + 4(-4x - 1) = 6 \Rightarrow -10x - 4 = 6 \Rightarrow -10x = 10 \Rightarrow x = -1$$

Subbing $x = -1$ into equation (2),

$$y = -4(-1) - 1 = 3$$

Worked Example 14.2

Given that $P = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} -3 & 4 \\ -5 & 2 \end{pmatrix}$, find

(a) $Q^2 - 2P$,

(b) The matrix R such that $3P - 2R = QI$, where I is a 2×2 identity matrix.

[Solution]

(a) $Q^2 - 2P$

$$\begin{aligned} &= \begin{pmatrix} -3 & 4 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ -5 & 2 \end{pmatrix} - 2 \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -3 \times (-3) + 4 \times (-5) & -3 \times 4 + 4 \times 2 \\ -5 \times (-3) + 2 \times 4 & -5 \times 4 + 2 \times 2 \end{pmatrix} - \begin{pmatrix} -2 & -4 \\ 2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} -11 & -4 \\ 23 & -16 \end{pmatrix} - \begin{pmatrix} -2 & -4 \\ 2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} -11 - (-2) & -4 - (-4) \\ 23 - 2 & -16 - 6 \end{pmatrix} \\ &= \begin{pmatrix} -9 & 0 \\ 21 & -22 \end{pmatrix} \end{aligned}$$

(b) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$3P - 2R = QI$$

$$3 \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} - 2R = \begin{pmatrix} -3 & 4 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}$$

$$3 \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} -3 & 4 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} = 2R$$

$$\begin{pmatrix} -3 & -6 \\ 3 & 9 \end{pmatrix} - \begin{pmatrix} -3 \times (-1) + 4 \times 1 & -3 \times (-2) + 4 \times 3 \\ -5 \times (-1) + 2 \times 1 & -5 \times (-2) + 2 \times 3 \end{pmatrix} = 2R$$

$$\begin{pmatrix} -3 & -6 \\ 3 & 9 \end{pmatrix} - \begin{pmatrix} 7 & 18 \\ 7 & 16 \end{pmatrix} = 2R$$

$$2R = \begin{pmatrix} -3 - 7 & -6 - 18 \\ 3 - 7 & 9 - 16 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & -24 \\ -4 & -7 \end{pmatrix}$$

$$R = \frac{1}{2} \begin{pmatrix} -10 & -24 \\ -4 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -12 \\ -2 & -\frac{7}{2} \end{pmatrix}$$

Worked Example 14.3

Peter runs a stall in a food court. From Monday to Friday, for the morning shift, he employs 3 people for the counter and 4 people for the kitchen. For the afternoon shift, 5 people for the counter and 6 people for the kitchen are employed.

This information can be represented by the matrix A , $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$.

On Saturday and Sunday, for the morning shift, he employs 4 people for the counter and 5 people for the kitchen. For the afternoon shifts, he employs 6 people for the counter and 7 people for the kitchen.

- (a) Represent the information for the staff employed on Saturday and Sunday by the matrix B , using the same arrangement as matrix A .
- (b) Evaluate matrix $C = 5A - 2B$. What do the elements in matrix C represent?
- (c) The wages per session for each morning shift staff is \$50 and that for the afternoon shift is \$70. Find matrix W , where $W = C \begin{pmatrix} 50 \\ 70 \end{pmatrix}$. What do the elements in matrix W represent?
- (d) Evaluate matrix T , where $T = (1 \ 1)W$. What do the elements in matrix T represent?

[Solution]

(a) $B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$

(b)
$$\begin{aligned} C &= 5 \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} - 2 \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 5 \times 3 & 5 \times 4 \\ 5 \times 5 & 5 \times 6 \end{pmatrix} - \begin{pmatrix} 2 \times 4 & 2 \times 5 \\ 2 \times 6 & 2 \times 7 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 20 \\ 25 & 30 \end{pmatrix} - \begin{pmatrix} 8 & 10 \\ 12 & 14 \end{pmatrix} \\ &= \begin{pmatrix} 15 - 8 & 20 - 10 \\ 25 - 12 & 30 - 14 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 13 & 16 \end{pmatrix} \end{aligned}$$

The elements in matrix C represent how many more workers are employed on weekdays as compared to on weekends / the difference in number of workers employed on weekdays as compared to on weekends for each role and shift respectively.

(c)
$$\begin{aligned} W &= \begin{pmatrix} 7 & 10 \\ 13 & 16 \end{pmatrix} \begin{pmatrix} 50 \\ 70 \end{pmatrix} \\ &= \begin{pmatrix} 7 \times 50 + 10 \times 70 \\ 13 \times 50 + 16 \times 70 \end{pmatrix} \\ &= \begin{pmatrix} 1050 \\ 1770 \end{pmatrix} \end{aligned}$$

Since the first matrix is a 2×2 and second matrix is a 2×1 , you are able to apply multiplication and the final product will be 2×1 .

The elements in matrix W represents the difference in wages paid to workers employed on weekdays as compared to weekends for morning and afternoon shift respectively.

$$(d) T = (1 \ 1) \begin{pmatrix} 1050 \\ 1770 \end{pmatrix} = (1050 + 1770) = (2820)$$

The element in T represent the total difference in wages paid to _____ employed on weekdays as compared to on weekends.

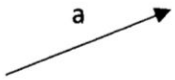
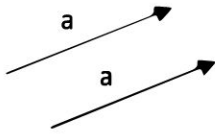
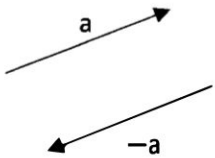
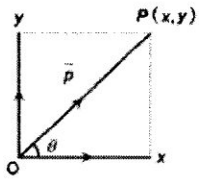
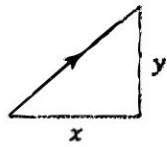
Note!

In real-world matrix applications, when responding to questions about what elements represent, it's essential to use the term "respectively" for clarity.

When multiplying a matrix that consists of $(1 \ 1)$, with another matrix, the resulting product is typically a single term that often represents a cumulative measure like the total amount, sum, or total hours, depending on the context of the data in the second matrix.

15 Vectors

Type of Vectors

Type	Definition	Notation	Diagram Representation
Vectors	Consists of a magnitude and direction	$\begin{pmatrix} x \\ y \end{pmatrix}, \overrightarrow{AB}, a$	
Equal Vectors	Same magnitude and direction	$\overrightarrow{AB} = \overrightarrow{PQ}$ $ \overrightarrow{AB} = \overrightarrow{PQ} $	
Negative Vectors	Equal magnitude but opposite in direction	$\overrightarrow{AB} = -\overrightarrow{BA}$	
Position Vectors	Position of a point relative to the origin	$\overrightarrow{OP}$	
Column Vectors	Vector written with a horizontal and vertical component	$\begin{pmatrix} x \\ y \end{pmatrix}$	

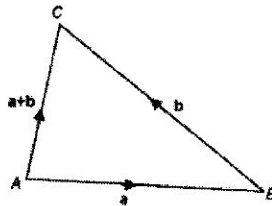
Vector Properties

1. Magnitude of a Vector

$$|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$$

Pythagoras' Theorem

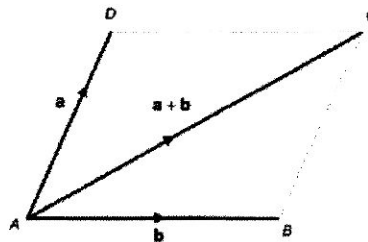
2. Vector Addition Triangle Rule



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\overrightarrow{AC} = a + b$$

Parallelogram Rule

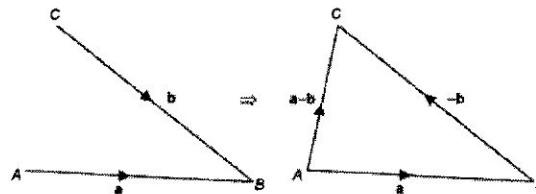


$$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$$

$$\overrightarrow{AC} = a + b$$

3. Vector Subtraction

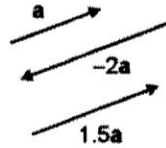
To subtract 2 vectors, add the first vector to the negative of the second vector.



The reason as to why we take the negative of the second vector as we need to flip the direction of the vector to get $\overrightarrow{AC}$.

4. Scalar Multiplication

Vector a multiplied by a scalar k results in a vector ka



Properties of vector ka :

- Has the same direction as a
- Has magnitude $|k|$ times of a
- Parallel to a
- Same direction as a if $k > 0$
- Opposite direction as a if $k < 0$

5. Collinear Vectors

If A, B and C are collinear, then

$$\overrightarrow{AB} = k\overrightarrow{BC}$$

Formulae for Column Vectors

Formulae	Working
Magnitude	$ \overrightarrow{AB} = \sqrt{x^2 + y^2}$
Addition	$a + b = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} x + p \\ y + q \end{pmatrix}$
Subtraction	$a - b = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} x - p \\ y - q \end{pmatrix}$
Scalar Multiplication	$ka = \begin{pmatrix} kx \\ ky \end{pmatrix}$
Scalar Multiplication with 2 vectors	$ma + nb = m \begin{pmatrix} c \\ d \end{pmatrix} + n \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} mc + ne \\ md + nf \end{pmatrix}$

Tips!

1. Converting coordinates to position vectors

- Sometimes coordinates are given in questions. Students must know that coordinates are essentially position vectors as these coordinates are always written with respect to the origin.
- Example: P is a coordinate $(3,1)$

$$\overrightarrow{OP} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

2. Parallel Vectors

- Do note that parallel vectors are vector that are multiplied by a scalar k .
- Example: If $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$

$$\overrightarrow{AB} = k\overrightarrow{CD}$$

3. Tips for adding vectors

- When adding vectors together, always ensure that the vectors follow the following rule:

$$\overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{AB}$$

- Ensure that the underlined letters are from the same point. Vectors are only allowed to add if the middle 2 points are the same.

4. Flipping vectors by taking the negative

- Do note that students will need to flip the vectors in order to do the additions successfully. Take the negative to flip the vectors.

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \overrightarrow{BA} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

Worked Example 15.1

P is the point $(7, 1)$ and Q is the point $(2, 6)$.

(a) Find $|\overrightarrow{PQ}|$.

(b) If $\overrightarrow{SP} = 3\overrightarrow{PQ}$, find the coordinates of S .

(c) Given that $\overrightarrow{OR} = \begin{pmatrix} h+2 \\ 5 \end{pmatrix}$, find the value of h if $\overrightarrow{QR}$ is parallel to $\overrightarrow{PQ}$.

[S4 DSS P2/2017 MYE Q7(a)]

[Solution]

$$\begin{aligned}\text{(a) } \overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= \begin{pmatrix} -7 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 5 \end{pmatrix} \\ |\overrightarrow{PQ}| &= \sqrt{(-5)^2 + (5)^2} \\ &= \sqrt{50} \\ &= 7.07106 \dots \\ &= 7.07 \text{ units (3s.f.)}\end{aligned}$$

$$\begin{aligned}\text{(b) } \overrightarrow{SP} &= 3\overrightarrow{PQ} \\ \overrightarrow{SO} + \overrightarrow{OP} &= 3\overrightarrow{PQ} \\ \overrightarrow{SO} &= 3\overrightarrow{PQ} - \overrightarrow{OP} \\ &= 3\begin{pmatrix} -5 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -22 \\ 14 \end{pmatrix} \\ S &= (-22, 14)\end{aligned}$$

$$\begin{aligned}\text{(c) } \overrightarrow{OR} &= k\overrightarrow{PQ} \\ \begin{pmatrix} h+2 \\ 5 \end{pmatrix} &= k\begin{pmatrix} -5 \\ 5 \end{pmatrix} \\ k &= 1 \\ h &= -7\end{aligned}$$

Tips/Thinking Process

This is a classic question that always comes out in Paper 1. This question tests on the fundamental concepts of magnitude, scale factors, and parallel vectors.

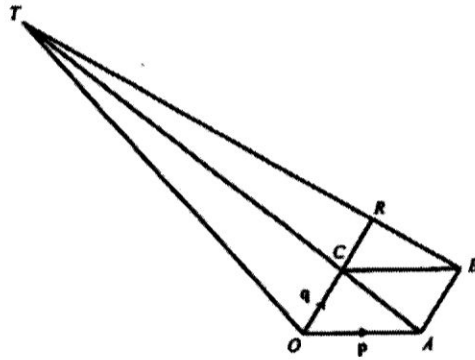
Trick to solving scale factor questions:

Always write out the vectors in terms of the vectors that are already provided to you.

Do note that if coordinates are given, convert this coordinate into a position vector.

$$P \text{ is the point } (7, 1) \Rightarrow \overrightarrow{OP} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

Worked Example 15.2



$OABC$ is a parallelogram. $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OC} = \mathbf{q}$ and $\overrightarrow{CT} = 4\overrightarrow{AC}$. ACT , BRT and OCR are straight lines.

(a) Express each of the following, as simply as possible, in terms of $\mathbf{p}$ and/or $\mathbf{q}$.

(i) $\overrightarrow{OB}$

(ii) $\overrightarrow{OT}$

(iii) $\overrightarrow{BT}$

(b) Given that $\overrightarrow{BR} = \frac{4}{5}\mathbf{q} - \mathbf{p}$, find k if $\overrightarrow{OC} = k\overrightarrow{CR}$

(c) Find the value of

$$\frac{\text{Area of } \triangle BCR}{\text{Area of } \triangle CRT}$$

[S4 DSS P2/2017 MYE Q7(b)]

[Solution]

(a) (i) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$$= \overrightarrow{OA} + \overrightarrow{OC}$$

$$= \mathbf{p} + \mathbf{q}$$

(ii) $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$$= -\overrightarrow{OA} + \overrightarrow{OC}$$

$$= -\mathbf{p} + \mathbf{q}$$

$$\overrightarrow{OT} = \overrightarrow{OA} + \overrightarrow{AT}$$

$$= \mathbf{p} + 5\overrightarrow{AC}$$

$$= \mathbf{p} + 5(\mathbf{q} - \mathbf{p})$$

$$= 5\mathbf{q} - 4\mathbf{p}$$

$$\begin{aligned}
 \text{(iii) } \overrightarrow{BT} &= \overrightarrow{TO} + \overrightarrow{OB} \\
 &= \overrightarrow{OT} - \overrightarrow{BO} \\
 &= 5\mathbf{q} - 4\mathbf{q} - (\mathbf{p} + \mathbf{q}) \\
 &= 4\mathbf{p} - 5\mathbf{q}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \overrightarrow{OR} &= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BR} \\
 &= \mathbf{p} + \mathbf{q} + \frac{4}{5}\mathbf{q} - \mathbf{p}
 \end{aligned}$$

$$= \frac{9}{5}\mathbf{q}$$

$$\overrightarrow{CR} = \overrightarrow{OR} - \overrightarrow{OC}$$

$$= \frac{9}{5}\mathbf{q} - \mathbf{q}$$

$$= \frac{4}{5}\mathbf{q}$$

$$\therefore \overrightarrow{OC} = \frac{5}{4}\overrightarrow{CR}$$

$$k = \frac{5}{4}$$

$$\text{(c) } \overrightarrow{BR} = \frac{1}{5}\overrightarrow{BT}$$

$$\overrightarrow{BR} = \frac{1}{4}\overrightarrow{RT}$$

$$\therefore \frac{\text{Area of } \triangle BCR}{\text{Area of } \triangle CRT} = \frac{\frac{1}{2} \times \overrightarrow{BR} \times h}{\frac{1}{2} \times \overrightarrow{RT} \times h}$$

$$= \frac{\frac{1}{4}\overrightarrow{RT}}{\overrightarrow{RT}}$$

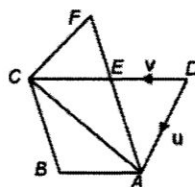
$$= \frac{1}{4}$$

Thinking Process

Part (c) usually pose some problems for some students. Whenever attempting such questions, always choose a common height for the diagrams, this is so that the height can be easily cancelled off during the calculations.

Most of the time for such questions, it will be the ratio of the lengths that is in question. Always use the ratios provided to help solve the question.

Worked Example 15.3



In the diagram, $ABCD$ is a trapezium and BC is parallel to AD .

It is given that $\overrightarrow{DA} = u$, $\overrightarrow{DE} = v$. It is also given that $4DE = 5EC$ and $AE:EF = 3:n$.

(a) Express, as simply as possible, in terms of u and/or v .

(i) $\overrightarrow{EC}$

(ii) $\overrightarrow{DB}$

(iii) $\overrightarrow{AE}$

(b) Express $\overrightarrow{EF}$ in terms of u, v and n .

(c) Given that FC and DB are parallel, find the value of n .

(d) Find the ratio of

$$\frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle ACE}$$

(e) Given that the area of triangle ADE is 90 units², find the area of triangle EFC .

(f) Given that the column vector $\overrightarrow{AB}$ is $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ and the position vector of B relative to the origin O is $\begin{pmatrix} 8 \\ 1 \end{pmatrix}$. Find,

(i) the coordinates of the point A ,

(ii) the value of $|\overrightarrow{AB}|$.

[S4 CCHS(Y) P2/2009 Prelim Q8(Modified)]

[Solution]

(a) (i) $5\overrightarrow{EC} = 4\overrightarrow{DE}$

$$\overrightarrow{EC} = \frac{4}{5}\overrightarrow{DE}$$

$$= \frac{4}{5}v$$

(ii) $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB}$

$$= \overrightarrow{DA} + \overrightarrow{EC}$$

$$= u + \frac{4}{5}v$$

(iii) $\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$

$$= -u + v$$

(b) $AE:EF = 3:n$

$$\overrightarrow{EF} = \frac{n}{3+n}(v-u)$$

(c) $\overrightarrow{FC} = \overrightarrow{FE} + \overrightarrow{EC}$

$$\begin{aligned} &= -\frac{n}{3+n}(v-u) + \frac{4}{5}v \\ &= \left(\frac{4}{5} - \frac{n}{3+n}\right)v + \left(\frac{n}{3+n}\right)u \end{aligned}$$

Since we know that $\overrightarrow{FC}$ is parallel to $\overrightarrow{DB}$,

$$k(\overrightarrow{FC}) = \overrightarrow{DB}$$

$$k\left[\left(\frac{4}{5} - \frac{n}{3+n}\right)v + \left(\frac{n}{3+n}\right)u\right] = \frac{4}{5}v + u$$

$$k\left(\frac{4}{5} - \frac{n}{3+n}\right)v + k\left(\frac{n}{3+n}\right)u = \frac{4}{5}v + u$$

Comparing coefficients,

$$k\left(\frac{n}{3+n}\right) = 1$$

$$\frac{nk}{3+n} = 1$$

$$k = \frac{3+n}{n} \dots \dots \dots (1)$$

$$k\left(\frac{4}{5} - \frac{n}{3+n}\right) = \frac{4}{5} \dots \dots \dots (2)$$

Substitute Equation (2) into Equation (1),

$$\left(\frac{4}{5} - \frac{n}{3+n}\right)\left(\frac{3+n}{n}\right) = \frac{4}{5}$$

$$\left(\frac{4(3+n)-5n}{5(3+n)}\right)\frac{3+n}{n} = \frac{4}{5}$$

$$\frac{12+4n-5n}{5n} = \frac{4}{5}$$

$$5(12+4n-5n) = 4(5n)$$

$$60-5n = 20n$$

$$25n = 60$$

$$n = 2\frac{2}{5}$$

This part maybe a little more challenging than most questions. When forming the equation for the parallel vectors, students should note that they have only 1 equation with 2 unknowns.

Based on the equation form, students should realise that they can compare coefficients to form 2 equations to get the value of n .

$$\begin{aligned}
 \text{(d)} \quad \frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle ACE} &= \frac{\left(\frac{1}{2}\right)(\text{Base})(\text{Height})}{\left(\frac{1}{2}\right)(\text{Base})(\text{Height})} \\
 &= \frac{\left(\frac{1}{2}\right)(\overline{DE})(\overline{EA})}{\left(\frac{1}{2}\right)(\overline{CE})(\overline{EA})} \\
 &= \frac{\overline{DE}}{\overline{CE}} \\
 &= \frac{5}{4}
 \end{aligned}$$

$$\text{(e)} \quad \frac{\text{Area of } \triangle ADE}{\text{Area of } \triangle ACE} = \frac{5}{4}$$

$$\text{Since } \triangle ADE = 90 \text{ units}^2,$$

$$\frac{90}{\text{Area of } \triangle ACE} = \frac{5}{4}$$

$$\text{Area of } \triangle ACE = \frac{90 \times 4}{5}$$

$$= 72 \text{ units}^2$$

$$\frac{\text{Area of } \triangle EFC}{\text{Area of } \triangle ACE} = \frac{\left(\frac{1}{2}\right)(\text{Base})(\text{Height})}{\left(\frac{1}{2}\right)(\text{Base})(\text{Height})}$$

$$= \frac{\left(\frac{1}{2}\right)(\overline{EF})(\overline{EC})}{\left(\frac{1}{2}\right)(\overline{AE})(\overline{EC})}$$

$$= \frac{\overline{EF}}{\overline{AE}} = \frac{2\frac{2}{5}}{3} = \frac{4}{5}$$

$$\text{Since } \triangle ACE = 72 \text{ units}^2,$$

$$\frac{\text{Area of } \triangle EFC}{72} = \frac{4}{5}$$

$$\text{Area of } \triangle EFC = \frac{72 \times 4}{5}$$

$$= 57\frac{3}{5} \text{ units}^2$$

$$\text{(f) (i)} \quad \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$$

$$= \overrightarrow{OB} - \overrightarrow{AB}$$

$$= \begin{pmatrix} 8 \\ 1 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{Coordinates of } A = (1, -1)$$

$$\text{(ii)} \quad |\overrightarrow{AB}| = \sqrt{7^2 + 2^2}$$

$$= \sqrt{53} \text{ units}$$

This part is the same as the previous example. Always write out the area of the triangles in terms of the ratios provided in the questions, and a common height which can be removed via cancellations during post calculations.

This part will be the most challenging part of whole the question.

Students must be able to form their own ratios and relationships between the triangles given.

To form this relationship, always use triangles that have a common height. When there is a common height, the ratios of the vectors will be used to calculate the area of the triangles.