NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2
Higher 3

MATHEMATICS 9820/01

Preliminary Examination **25 September 2025**

Paper 1 **3 hours**

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

You are reminded of the need for clear presentation in your answers. Up to 2 marks may be deducted for improper presentation.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages and **2** blank pages.

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1 **(a)** Explain geometrically why for any positive real numbers a, b, c and d, if $\frac{a}{b} < \frac{c}{d}$, then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$ [3]

(b) Prove that if $0 < x < \frac{\pi}{2}$, then

$$\sin x < 2\tan\frac{x}{2} < \tan x.$$ [4]

(c) Prove that if a and b are distinct positive real numbers, then

$$4 < \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)^2 < 2\left(\frac{a}{b} + \frac{b}{a}\right).$$ [4]

2 **(a)** Given that $\mathrm{f}: \mathbb{Z} \to \mathbb{Z}$ satisfies

$$\mathrm{f}(2m) + 2\mathrm{f}(n) = \mathrm{f}^2(m+n)$$

for all integer values of m and n, find all possible functions f. [6]

(b) Given that $x > y$, find exactly the values of the real variables x and y such that

$$x + y = 4,$$
$$x^5 + y^5 = 464.$$ [6]

3 **(a)** A soccer player is training for a competition over a period of 20 weeks, starting on a Sunday. Each day, he trains for at least 1 hour. In order not to overtrain and injure himself, the player decides not to train for more than 13 hours in any calendar week – a week beginning with Sunday and ending with Saturday. The number of hours trained each day is a positive integer. Show that there exists a succession of consecutive days during which the player will have trained for exactly 19 hours. [5]

(b) **(i)** Explain how the integer solutions of the inequality $x_1 + x_2 + x_3 + x_4 \le 50$ can be related to the solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 50$. [1]

(ii) Use the principle of inclusion and exclusion to find the number of integer solutions of $x_1 + x_2 + x_3 + x_4 \le 50$ such that $3 \le x_1 \le 12$, $0 \le x_2 \le 7$, $-8 \le x_3 \le 12$ and $6 \le x_4 \le 20$. [7]

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4 **(a)** A curve C has the following property.

At any point P on the curve C, the normal line cuts the x- and y-axes at A and B respectively, and B is always the midpoint of AP.

Find an equation for C. [6]

(b) The Weierstrass substitution suggests that

$$\sin x = \frac{2t}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}, \text{ where } t = \tan\frac{x}{2}.$$

You may use the above results without proof in solving the following question.

A student forgot the Product Rule for differentiation and wrongly assumed $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)$, where u and v are functions of x. However, he was lucky enough to get the correct answer.

Given that $u(x) = \mathrm{e}^{3\sin x}$, show that

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{3\cos x}{3\cos x - 1}.$$ [2]

Hence find $v(x)$. [5]

5 The nth Fermat number, F_n, is defined by

$$F_n = 2^{2^n} + 1, \text{ for non-negative integers } n.$$

(i) Write down the values of F_0, F_1, F_2 and F_3. [1]

(ii) By considering $k=1, k=2$ and $k=3$, conjecture a formula, in terms of F_k, for the product $F_0F_1F_2...F_{k-1}$. [2]

(iii) Prove, by mathematical induction, that the conjecture in part **(ii)** holds for all positive integers k. [5]

(iv) Deduce that any two distinct Fermat numbers are coprime. [3]

(v) It is given that $641 = 2^4 + 5^4 = \left(5\times 2^7\right)+1$.

(a) Find the remainder when $5^4 \times 2^{28}$ is divided by 641. [2]

(b) Hence, show, without the use of a calculator, that $F_5 \equiv 0 \pmod{641}$. [2]

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

Investigating Catalan Numbers

The Catalan numbers are a sequence of positive integers that are very useful in counting problems. The n^{th} Catalan number can be expressed using binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

C_n counts the number of ways p_{n+1} to associate – or to group – the product $x_1x_2...x_{n+1}$ into products of two expressions using parenthesis. Hence, we can write $C_n = p_{n+1}$, for $n \geq 1$.

A Recursive Formula for p_{n+1}

We define $p_1 = 1$.

For $n = 1$, we have a two-fold product, *ab*, which has one way to do the above grouping using parenthesis, namely (ab), so that $p_2 = 1$.

For $n = 2$, we can associate the three-fold product abc as $((ab)c),(a(bc))$. Hence, $p_3 = 2$.

For $n = 3$, we can associate the four-fold product $abcd$ in five different groupings, namely

$$(a(b(cd))),\ (a((bc)d)),\ ((a(bc))d),\ ((ab)(cd)),\ (((ab)c)d),$$

so $p_4 = 5$.

Thus, the value of p_{n+1} can be obtained by first grouping the product $x_1x_2...x_{n+1}$ into two subproducts of lengths r and $n-r+1$ for $r = 1,2,...,n$. As there are p_r ways to group an *r*-fold product and p_{n-r+1} ways to group an $(n-r+1)$-fold product, we see that

$$p_{n+1} = p_1p_n + p_2p_{n-1} + ... + p_np_1, \text{ for } n \geq 2.$$

Relations between groupings

The relations between the different ways to associate an $(n+1)$-fold product can be represented with a graph called the *associahedron*, of order $n+1$. This is a diagram whose vertices are the p_{n+1} groupings of an $(n+1)$-fold product, and two vertices are joined by an edge if each can be transformed into the other by a single application of the associative law $(x(yz)) = ((xy)z)$.

The associahedron of order 5 has 14 vertices corresponding to the 14 groupings of a five-fold product. The diagram below shows this associahedron with 2 missing groupings, indicated as g_1 and g_2.

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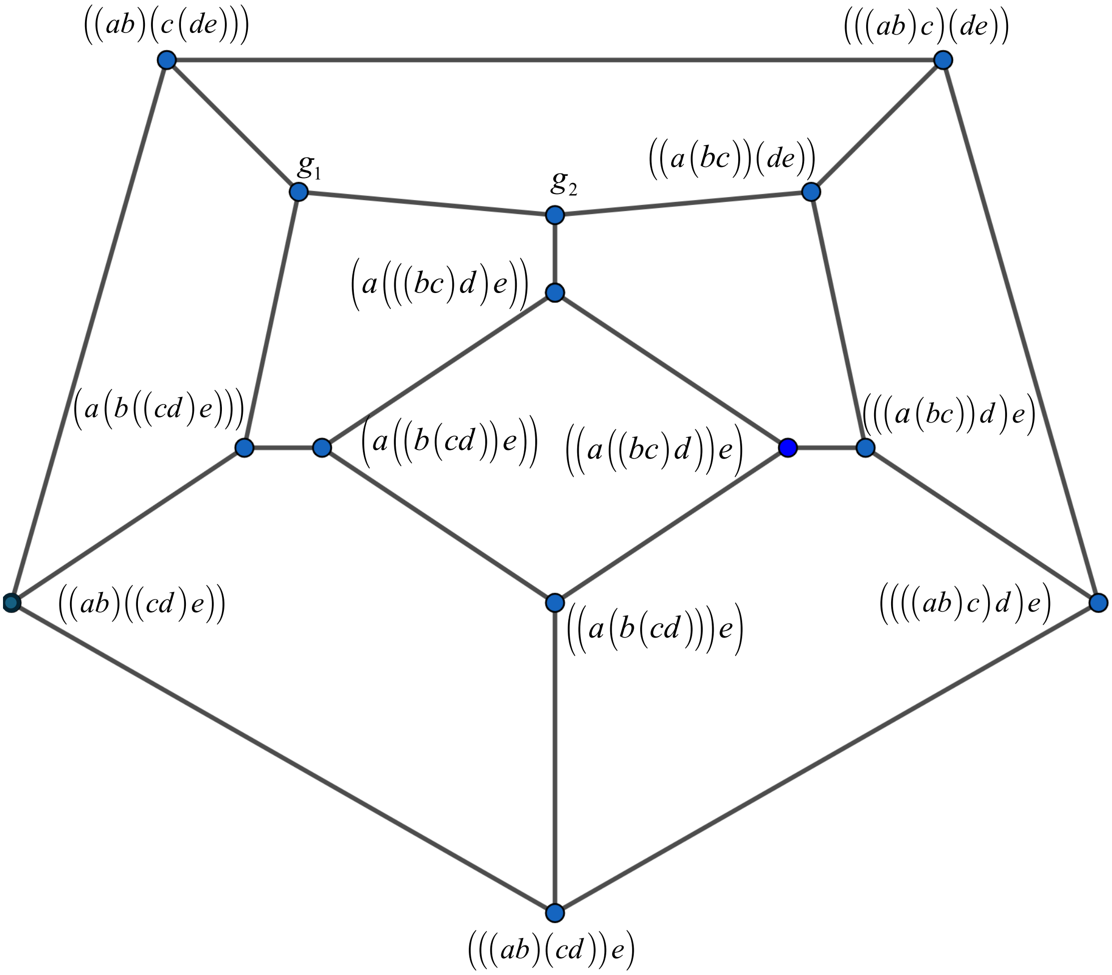

A Formula for p_{n+1} in terms of n

Here is another way to look at groupings of products. Each grouping of an $(n+1)$-fold product corresponds to a string of n O's (for open parenthesis) and n C's (for close parenthesis) in which the number of O's never falls below the number of C's. For example, the string OOCOOCCC corresponds to the grouping $((ab)(c(de)))$.

Replacing each O by a 1 and each C by a (-1) yields a sequence $y_1, y_2, ..., y_{2n}$ of n 1's and n (-1)'s in which

$$y_1 + y_2 + ... + y_k \geq 0 \quad \text{for } 1 \leq k \leq 2n.$$

We call such sequences *good*; thus, the good sequence

$$1, 1, -1, 1, 1, -1, -1, -1$$

corresponds to the grouping $((ab)(c(de)))$. So, the number of good sequences of length $2n$ equal the number of groupings p_{n+1} of an $(n+1)$-fold product. That number is the Catalan number C_n.

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6 **(i)** Verify that the number of ways to associate a five-fold product into products of two expressions using parenthesis is 14. [1]

(ii) Show that $P_{2i+1} \equiv 0 \pmod 2$ for $i \in \mathbb{Z}^{+}$. [1]

(iii) **(a)** Find the expressions for g_1 and g_2 in the associahedron of order 5. [2]

(b) Explain how you can use the associahedron of order 5 to find the associahedron of order 4. [1]

(iv) Determine a bijection between the set of non-good sequences of n 1's and n (-1)'s and the set of sequences of $(n+1)$ 1's and $(n-1)$ (-1)'s. [4]

(v) Hence, prove that $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$. [3]

(vi) Prove that $C_n \mid n+2$ if and only if $n \le 3$, where $n \ge 1$. [4]

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