RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 6B: Applications of Differentiation

Section A (Basic Questions)

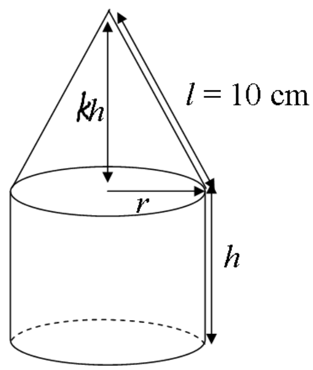
1 A curve C has parametric equations $x = t^2 + 4t$, $y = t^3 + t^2$.
The tangent to the curve at the point P where $t = 2$ is denoted by l.

(i) Find the cartesian equation of l. **[3]**

(ii) The tangent l meets C again at the point Q. Use a non-calculator method to find the coordinates of Q. **[4]**

[(ii) $y = 2x - 12$ (iii) $(-3, -18)$ **]**

2



A closed container is constructed using a sheet of metal with area 100π cm^2. The container comprises 2 shapes, a cone and a cylinder. The slant height, l, of the cone is 10 cm. Given that the cylinder has height h, and radius r, and the height of the cone is kh, where k is a positive constant.

(i) show that $h = \dfrac{100 - r^2 - 10r}{2r}$, **[1]**

(ii) use differentiation to show that the exact maximum volume of the container is given that $V = \left(\dfrac{1}{3}k + 1\right)\dfrac{2500\pi}{27}$ cm^3, proving that it is a maximum. **[6]**

[Volume of Cone $= \dfrac{1}{3}\pi r^2 h$, Curved Surface Area of Cone $= \pi r l$]

3 A curve C has equation $y^3 - 2xy^2 + 3x^2 - 3 = 0$.

(i) Find $\dfrac{dy}{dx}$ in terms of x and y. **[2]**

(ii) Find the equation of the normal to the curve at the point $P(2, 3)$. **[2]**

(iii) Given that C meets the y-axis at the point R and the normal in **(ii)** meets the y-axis at the point A, find the area of triangle APR in the form $a - \sqrt[3]{b}$, where a and b are integers to be determined. **[3]**

[(i) $\dfrac{dy}{dx} = \dfrac{2y^2 - 6x}{3y^2 - 4xy}$ **(ii)** $y = -\frac{1}{2}x + 4$ **(iii)** $a = 4$, $b = 3$**]**

Section B (Discussion Questions)

1 A curve C has equation $kxe^y + ke^x = y^2 + k^2$, where k is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of x, y and k. **[3]**

(ii) Explain why there is no point on C where the tangent is parallel to the x-axis. **[2]**

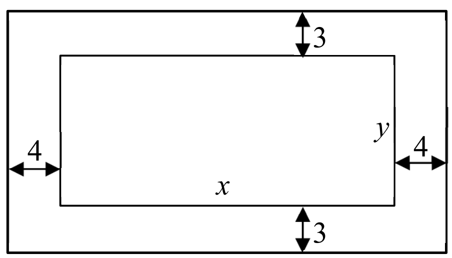
[**(i)** $\frac{dy}{dx} = \frac{ke^x + ke^y}{2y - kxe^y}$]

2 **(i)** Given that $x^2 - y^2 + 2xy + 4 = 0$, find $\frac{dy}{dx}$ in terms of x and y. **[4]**

(ii) For the curve with equation $x^2 - y^2 + 2xy + 4 = 0$, find the coordinates of each point at which the tangent is parallel to the x-axis. **[4]**

[**(i)** $\frac{dy}{dx} = \frac{x+y}{y-x}$ **(ii)** $\left(-\sqrt{2}, \sqrt{2}\right)$ and $\left(\sqrt{2}, -\sqrt{2}\right)$]

3 A designer wishes to create a piece of artwork with painted area of 1200 cm^2 on a rectangular piece of canvas. The painted area measures x cm by y cm and is surrounded by an unpainted border with top and bottom margins of 3 cm each, and side margins of 4 cm each on the canvas, as shown in the diagram below.



(i) By differentiation, find the dimensions of the canvas with the smallest area. **[6]**

(ii) What is the largest possible area of the canvas if $30 \le x \le 50$? **[2]**

At an exhibition, a spotlight illuminates a circular region of radius $\frac{2}{\sqrt{\pi}}$ cm on the artwork. The area of this circular region then increases at a constant rate of 20 cm^2 per minute.

(iii) Find the rate of change of the radius **after** 3 minutes. **[4]**

[**(i)** 48cm by 36cm **(ii)** 1748 cm^2 **(iii)** 0.705 cm/min]

4 A curve C has parametric equations

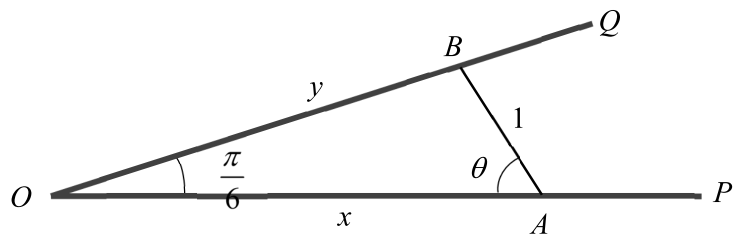
$$x = e^t, \quad y = t - \ln t, \qquad \text{where } t > 0.$$

(i) Show that there is no tangent to C parallel to the y-axis and find the equation of the tangent to C that is parallel to the x-axis. **[5]**

(ii) Describe the behavior of the tangent to C as $t \to 0$. **[1]**

[**(i)** $y = 1$]

5 In the figure (not drawn to scale), POQ is a rail and $\angle POQ = \frac{\pi}{6}$. AB is a rod of fixed length 1 m which is free to slide on the rail with end A on OP and end B on OQ. It is given that $OA = x$ m, $OB = y$ m and $\angle OAB = \theta$ radians.



(i) Express x and y in terms of θ. **[2]**

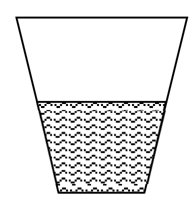
(ii) Given that S is the area of triangle OAB, find S in terms of θ. **[1]**

Given that B is moving towards O at a rate of 0.2 ms^{-1} at the instant when $\theta = \frac{\pi}{4}$,

(iii) find the rate of change of S at this instant. **[4]**

[(i) $x = 2\sin\left(\frac{5\pi}{6} - \theta\right)$, $y = 2\sin\theta$ **(ii)** $S = \sin\left(\frac{5\pi}{6} - \theta\right)\sin\theta$ **(iii)** $-0.122\ \text{ms}^{-1}$**]**

6 Water is poured at a constant rate of 20 cm^3 per second into a cup which is shaped like a truncated cone as shown in the figure. The upper and lower radii of the cup are 4 cm and 2 cm respectively. The height of the cup is 6 cm.



(i) Show that the volume of water inside the cup, V cm^3 is related to the height of the water level, h cm by the equation

$$V = \frac{\pi}{27}(h+6)^3 - 8\pi .$$

(ii) How fast will the water level be rising when h is 3 cm? Express your answer in exact form.

[(ii) $\frac{20}{9\pi}$ cms^{-1} **]**

7 A curve C is defined by the parametric equations $x=\tan\theta,\ y=\sec\theta$ for $0<\theta<\frac{\pi}{2}$.

(i) Find the cartesian equation of C.

The tangent and normal at $P\left(\tan\theta,\ \sec\theta\right)$ meets the x-axis at Q and R respectively.

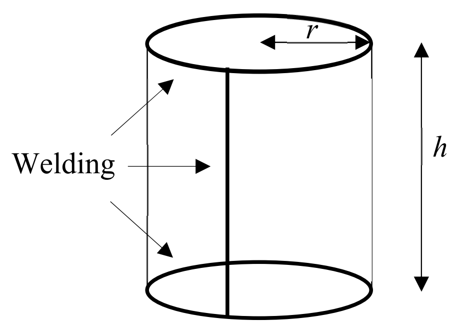
(ii) Show that the area, A of the circle passing through P, Q and R can be expressed as

$$A=\pi\left(\tan\theta+\frac{1}{2\tan\theta}\right)^2.$$

(iii) Show that $\left(t+\frac{1}{2t}\right)^2-2\geq 0$ for all $t\in\mathbb{R}, t\neq 0$. Deduce the minimum value of A.

[(i) $x^2+1=y^2$ **(iii)** 2π **]**

8 A closed cylindrical can with radius *r* cm and height *h* cm has fixed volume 20π cm^3. The material for the top and bottom faces costs \$0.50 per cm^2 and the material for the curved surface costs \$0.30 per cm^2. It also costs \$0.80 per cm to weld the top and bottom faces onto the cylinder and \$0.60 per cm to weld the seam up the curved surface of the cylinder (see diagram).



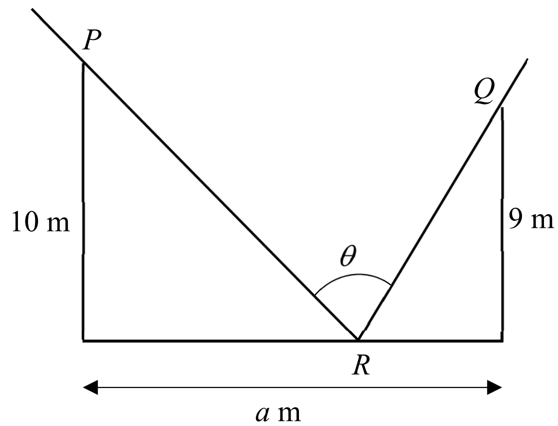
(i) The total cost of the can is \$*C*. Show that $C=\pi r^2+3.2\pi r+\frac{12\pi}{r}+\frac{12}{r^2}$. **[3]**

(ii) Use differentiation to find the values of *r* and *h* which give a minimum value of *C*, proving that *C* is a minimum. State this value of *C*. **[7]**

(iii) It is given instead that $0.5\leq r\leq 2$. Find the corresponding range of values of *C*. **[2]**

[(ii) $r=1.61$, $h=7.68$, \$52.37 **(iii)** $52.37\leq C\leq 129.21$**]**

9 A straight street of fixed width a metres on the horizontal ground is bounded on its parallel sides by two vertical walls, one of height 10 metres, and the other of height 9 metres. The intensity of light at point R at ground level on the street is proportional to the angle θ, in radians, where $\theta = \angle PRQ$ as shown in the diagram.



It is given that x is the distance of R from the base of the wall of height 10 metres.

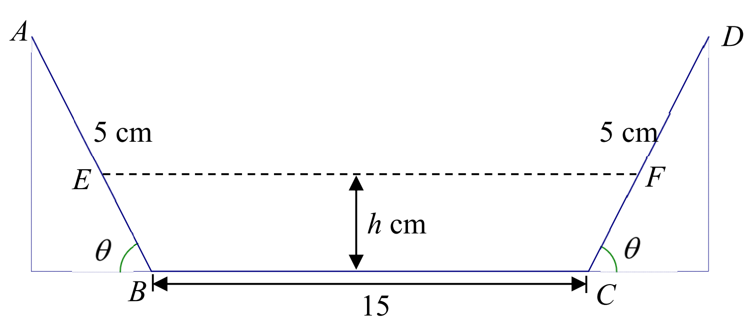
(i) Find an exact expression for θ in terms of x, a and π. **[1]**

(ii) Show that $\dfrac{d\theta}{dx} = \dfrac{10}{100+x^2} - \dfrac{9}{(a-x)^2+81}$. Find, in terms of a, the value of x when the intensity of light at R is maximum. **[6]**

(iii) The point R moves across the street towards the wall of height 10 metres at a speed of 0.5 ms^{-1}. Given that $a = 20$, find the rate of change of θ at the instant when R is at the midpoint of the street. Give your answer correct to 4 significant figures. **[2]**

[(i) $\theta = \pi - \tan^{-1}\dfrac{10}{x} - \tan^{-1}\dfrac{9}{a-x}$ **(ii)** $x = 10a - 3\sqrt{10(a^2+1)}$ **(iii)** -0.0001381 rad s^{-1}**]**

10 The diagram below shows the cross-section $ABCD$ of a gutter made using a sheet of metal 25cm wide. The sheet of metal is bent such that AB and CD are both 5cm long and inclined to BC at an angle θ radians.



(i) Show that the area of the cross-section of the gutter, S cm^2, is given by

$$S = 25\sin\theta\left(3+\cos\theta\right).$$

Find the maximum possible cross-sectional area that can be constructed, showing clearly how you obtain the value.

(ii) Water fills the gutter in such a way, that at time t seconds, the depth of water is h cm and $EBCF$ represents the cross-section of the portion of the gutter filled with water as shown in the diagram.

Given that $\theta = \dfrac{\pi}{4}$, find an expression for the angle FBC in terms of h.

When the depth of water in the gutter is 1.5 cm, the water level is rising at a rate of 0.4 cm s^{-1}. Find the rate at which angle FBC is increasing at this instant.

[(i) 78.7 cm^2 **(ii)** $\angle FBC = \tan^{-1}\left(\dfrac{h}{15+h}\right)$; 0.0219 rads^{-1} **]**

11 Fig 1 shows a glass bauble consisting of a square prism inscribed in a spherical glass shell with fixed radius a cm. The corners of the square prism are in contact with the spherical shell. The centre, O, of the prism coincides with that of the sphere. The corners of the square base are labelled A, B, C and D, and θ is the angle that OA makes with the base of the prism.

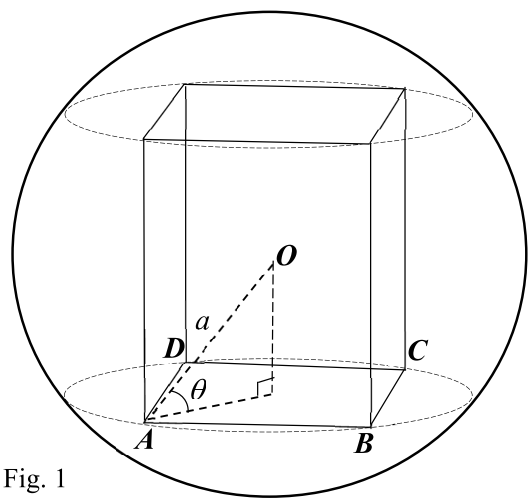


Fig. 1

(i) Show that the volume V of the square prism is given by $V = 4a^3\cos^2\theta\sin\theta$. **[3]**

(ii) Use differentiation to find, in terms of a, the dimensions of the square prism such that the value of V is maximised. You do not need to show that this value is a maximum. **[5]**

(iii) State the geometric shape of the square prism in part **(ii)**. **[1]**

[(ii) $\dfrac{2}{\sqrt{3}}a$ **]**

End of Tutorial