NANYANG JUNIOR COLLEGE

JC2 COMMON TEST 2

Higher 2

FURTHER MATHEMATICS **9649/02**

Paper 2

9th July 2021

3 Hours

Additional Materials: Answer Booklets

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

NANYANG JUNIOR COLLEGE
Internal Examinations

Section A: Pure Mathematics [50 marks]

1 The surface S has equation $z = \mathrm{f}(x,y)$ where $\mathrm{f}(x,y) = x^2y + xy^3 - 3xy$. The point A on S has coordinates $(1,2,4)$. The quadratic approximation for S at A is given by $z = Q(x,y)$.

Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. Show that $Q(x,y)$ can be written in the form

$$k + \mathbf{w}^T(\mathbf{u}-\mathbf{a}) + \frac{1}{2}(\mathbf{u}-\mathbf{a})^T \mathbf{H}(\mathbf{u}-\mathbf{a})$$

where k is a constant, **a** and **w** are 2×1 vectors, and **H** is a 2×2 symmetric matrix. [8]

What is the significance of **w** in relation to **a**? [1]

2 Let $\mathrm{f}(x) = 2x^{\frac{3}{2}} - 7x + 2x^{-\frac{1}{2}}$.

(i) Show that the equation $\mathrm{f}(x) = 0$ has two roots. [2]

(ii) Using a suitable interval $[k, k+1]$, where k is an integer, apply linear interpolation iteratively to get an approximation for the larger root. Justify that your answer is correct up to 2 decimal places. [4]

(iii) Using $x_1 = 0.1$ as a first approximation to the smaller root, apply the Newton-Raphson method iteratively to approximate the root. Justify that your answer is correct up to 2 decimal places. [3]

(iv) The Newton-Raphson method will fail when $x_n \in [a,b]$, where $b > a > 0$. Find the values of a and b, leaving your answer correct to 3 significant figures. [4]

3 A new flower bed is being designed for a large garden. With respect to a fixed point O and the initial line $\theta = 0$ extending due east from O, flowers are planted within the regions enclosed by the curve C. The curve C is defined by the polar equation $r = |1 + 2\cos\theta|$, where r represents the distance in metres from O and $0 \le \theta < 2\pi$.

(a) Sketch C, indicating all key features and symmetries. [3]

The two regions of the flower bed are fenced to separate them from their surroundings. Referring to your sketch in part **(a)**, label the smaller region as A and the larger region as B.

The cost per metre for fencing A is k times the cost per metre for fencing B.

(b) Given that the cost of fencing A is the same as that of B, determine the value of k. [4]

A sprinkler is to be positioned such that the entire flower bed falls within a metres from the sprinkler.

(c) State the optimal position, Q, in polar coordinates, to place the sprinkler for minimum a. By finding possible expressions for QP^2, where P is a point on C, deduce the minimum value of a. [5]

4 Show that the solutions of the equation $\left(1+z\right)^{2n}+\left(1-z\right)^{2n}=0$ are

$$\mathrm{i}\tan\left[\frac{\left(2k+1\right)\pi}{4n}\right]$$

where $k=0,1,2,\ldots,2n-1$. [4]

Show that

$$\tan\frac{\left(2k+1\right)\pi}{4n}=-\tan\frac{\left[2\left(2n-1-k\right)+1\right]\pi}{4n}.$$ [1]

Deduce from the above results that

$$\left(1+z\right)^{2n}+\left(1-z\right)^{2n}=2\prod_{k=0}^{n-1}\left\{z^2+\tan^2\left[\frac{\left(2k+1\right)\pi}{4n}\right]\right\}.$$ [5]

Hence by considering the coefficient of z^{2n-2}, prove that

$$\sum_{k=1}^{n}\sec^2\left[\frac{2k-1\ \pi}{4n}\right]=2n^2.$$ [6]

Section B: Probability and Statistics [50 marks]

5 **(a)** From time to time a firm manufacturing pre-packed furniture needs to check the mean distance between pairs of holes drilled by machine in pieces of chipboard to ensure that no change has occurred. It is known from experience that the standard deviation of the distance is 0.43 mm.

(i) The firm intends to take a random sample size of size n to construct a 99% confidence interval for the mean of the population. Calculate the minimum value of n such that the width of this interval must be no more than 0.60 mm. [2]

(ii) Given that the firm chooses to take a random sample of size n, where n takes on the value found in **(a)(i)**, explain if there is a need to make any assumption(s) to construct the confidence interval. [1]

(b) Out of 248 cars parked in a carpark, 72 were fitted with an anti-theft device on the steering wheel. Assuming that the cars form a random sample of parked cars, calculate an approximate 95% confidence interval for the population proportion of parked cars fitted with an anti-theft device on the steering wheel, giving the end-points of the interval correct to 3 decimal places. Give a reason why the assumption of randomness might not be valid. [4]

6 The continuous random variable X has probability density function given by

$$\mathrm{f}(x)=\begin{cases}0 & x<0,\\ \dfrac{a}{2^x} & x\geq 0.\end{cases}$$

where a is a positive constant.

(i) Show that X has an exponential distribution, and find the value of a. [3]

The variable Y is related to X by $Y=2^X$.

(ii) Find the distribution function of Y and hence find its probability density function. [4]

7 **(a)** **(i)** Show that $\dfrac{a^i}{i} = \int_0^a y^{i-1}\,\mathrm{d}y$. [1]

(ii) Hence show that if X is a geometric random variable with parameter p, then

$$\mathrm{E}\left(\frac{1}{X}\right) = \frac{-p\ln(p)}{1-p}.$$ [4]

(b) If X_1, X_2, …, X_n are independent and identically distributed geometric random variables with parameter p, show that the minimum of these n variables follows a geometric distribution, with parameter p_n, where p_n is a constant to be determined in terms of n and p. [4]

(c) A game is played among n players. Each player shall toss a coin that shows head with probability p repeatedly until a head appears where p is small. The game will end when one of the players obtains a head. Find the expected number of tosses needed for the game to end, leaving your answer in terms of n and p. [2]

8 Queuing Theory is an important branch of stochastic processes where different queueing models are studied. Some quantities of interest are the average waiting time for a customer and number of customers in the model in the long run. In a typical queuing model, we will need to have the distribution of the arrival time of the customers, the number of counters serving the customers and the serving time for each counter. The number of customers entering the queue is modelled by a Poisson process with rate λ per minute. The serving time, in minutes, of the counter follows an independent exponential distribution with parameter μ. When the customer is being served at the counter, he will leave the queue.

(i) Show that the waiting time for the next customer to enter the queue follows an exponential distribution with parameter λ. [2]

A ticketing office has only one counter. It will start operation at 0900 hrs. Customers start entering the queue at 0800 hours. It is known that the sum of n independent and identically distributed exponential random variables follow the gamma distribution with density function $\dfrac{1}{(n-1)!}\lambda^n x^{n-1}\mathrm{e}^{-\lambda x}$, where $x > 0$.

(ii) Show that the probability that there are at most n customers in the queue when the counter starts its operation is given by $\displaystyle\int_{60}^{\infty} \frac{1}{n!}\lambda^{n+1}x^n\mathrm{e}^{-\lambda x}\,\mathrm{d}x$. [2]

At 0900 hrs, there are n customers in the queue.

(iii) Find the expected waiting time for the last customer in the queue in terms of n and μ. [2]

(iv) Find the probability that after t minutes, there will be n customers in the queue and the first customer in the queue at 0900 hours is still being served. Leave your answer in terms of λ, μ and t. [4]

A queuing model is said to be in **steady state** when the system's performance such as average queue length and waiting times becomes stable and independent of time.

(v) Let P_n be the probability that there are n customers in the queue, where $n = 0, 1, 2, \ldots$. It is known that P_n satisfies the recurrence relation $P_n = \dfrac{\lambda}{\mu}P_{n-1}$, for $n = 1, 2, \ldots$. Find an expression for P_n in terms of λ, μ and n, and explain why this model will be in steady state. [4]

9 A farmer is interested in the distribution of the height of a certain species of plant in his plantation. He collected a random sample of 100 plants and the data is summarised as follows:

Height (cm)	50 – 55	55 – 60	60 – 65	65 – 70	70 – 75	75 – 80	80 – 85	85 – 90
Frequency	7	14	15	25	17	10	8	4

(a) Use a χ^2 test at the 5% level of significance to determine whether a normal distribution is an adequate model for the data. [5]

The farmer believes that the average height of this species is 70 cm. To test his belief, he obtained another random sample of 20 plants and found that the sample mean is 72.5 cm. In a 5% level of significance one-tailed test, he rejected the null hypothesis.

(b) Find the range of values of the sample standard deviation to arrive at the farmer's conclusion. [4]

(c) Another sample of 20 plants was taken and it had the same sample mean and standard deviation as the previous. Explain, with calculation, how this will affect the conclusion of the test if the two samples are combined. [2]

––––END OF PAPER––––