TEMASEK JUNIOR COLLEGE

Unit 7: Circular Motion

Learning Outcomes

Students should be able to:

7.1 Kinematics of uniform circular motion

7(a) express angular displacement in radians

7(b) show an understanding of and use the concept of angular velocity

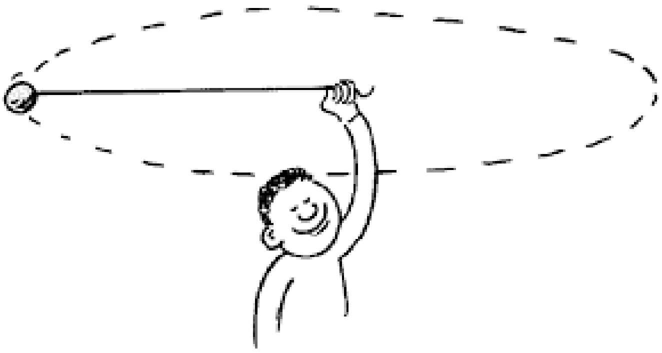
7(c) recall and use $v = r\omega$ to solve problems

7.2 Centripetal acceleration

7(d) show an understanding of centripetal acceleration in the case of uniform motion in a circle, and qualitatively describe motion in a curved path (arc) as due to a resultant force that is both perpendicular to the motion and centripetal in direction

7(e) recall and use centripetal acceleration $a = r\omega^2$ and $a = \frac{v^2}{r}$ to solve problems

7(f) recall and use $F = mr\omega^2$ and $F = \frac{mv^2}{r}$ to solve problems.



1 Introduction

We can find many examples of circular motion around us, e.g. bicycles and cars going round a corner and the Moon orbiting around the Earth.

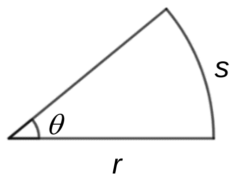
In the first part of this topic, we will describe the motion of objects moving in a circular path; in the subsequent parts, we will consider what causes motion to be circular.

2 Definition of Physical Quantities

2.1 Angular Displacement, θ *LO(a)*

In terms of the length of arc s and the radius of the circle r, angular displacement θ is expressed as

$$\boxed{\theta = \frac{s}{r}}$$



- Angular displacement is measured in **radian (rad).**
- For one complete revolution, $\theta = \dfrac{2\pi r}{r} = 2\pi$ rad (i.e. 2π rad = 360°)
- To convert θ in degrees to radians: $\dfrac{\theta(\text{rad})}{2\pi} = \dfrac{\theta(°)}{360°}$

Note: The radian is physically dimensionless as it is the ratio of two lengths.

One radian is defined as the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

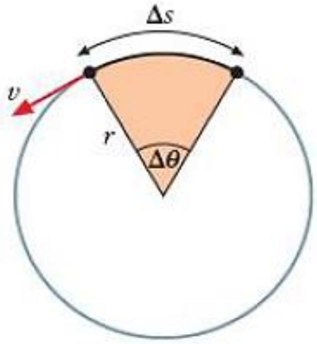
2.2 Angular Velocity, ω *LO(b),(c)*

Angular velocity ω is defined as the rate of change of angular displacement

i.e. $\omega = \dfrac{d\theta}{dt}$ unit of ω: rad s^{-1}

Consider an object moving with constant speed v in a circular path of radius r. In time Δt the object moves along an arc of length Δs and sweeps out an angle $\Delta\theta$.

$$\begin{aligned}\omega &= \frac{\Delta\theta}{\Delta t}\\ &= \frac{\Delta s/r}{\Delta t} && \text{(since } s = r\theta\text{)}\\ &= \frac{1}{r}\frac{\Delta s}{\Delta t}\\ &= \frac{v}{r} && \text{(since } v = \frac{\Delta s}{\Delta t}\text{)}\end{aligned}$$



Hence $\boxed{v = r\omega}$

For uniform circular motion, since v has the same magnitude throughout the motion, angular velocity is a constant (equal angle is swept out in equal time intervals). If the time taken to complete one revolution (i.e. to turn through an angle of 2π radians) is T, then

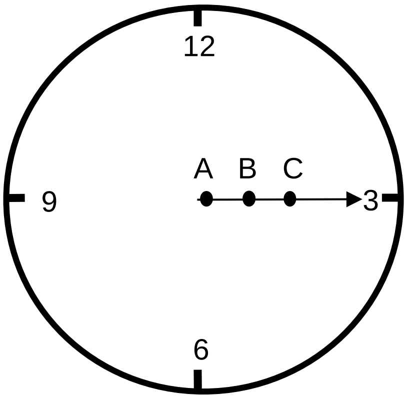
$$\boxed{\omega = \frac{2\pi}{T}}$$

T is known as the *period*.

Note: Angular velocity is a vector quantity.

Example 1

(a) Which point A, B or C on the minute hand of a clock is moving with the greatest speed?
(b) Calculate the angular velocity of the minute hand.



(a) Each point on the minute hand moves with same angular velocity, ω.

Since $v = r\,\omega$,
point C, being farthest from the centre, has the greatest linear speed.

(b)
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{60\times 60} = 1.75\times 10^{-3}\ \text{rad s}^{-1}$$

Example 2

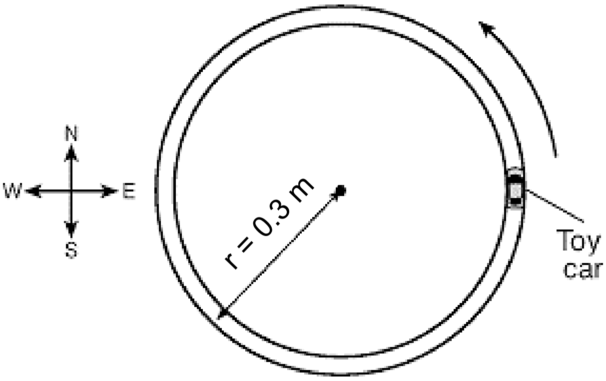
A model car moves round a circular track of radius 0.30 m at 2.0 revolutions per second.

Calculate (a) the period T,
(b) the angular speed ω,
(c) the linear speed v of the car.

(a) $T = \dfrac{1}{2.0} = 0.50$ s

(b) $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.50} = 12.6\ \text{rad s}^{-1}$

(c) $v = r\omega$
$= 0.30 \times 12.6 = 3.8\ \text{m s}^{-1}$

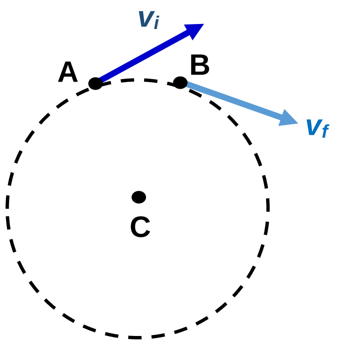


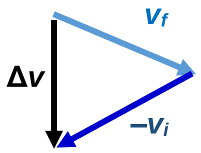
3 Uniform Motion in a Circle

3.1 Centripetal Acceleration *LO (d),(e),(f)*

In a <u>uniform</u> circular motion, although the linear speed remains unchanged, *the velocity is always changing* (since the direction changes with time). So, the motion is an *accelerated* one and by Newton's second law, a *resultant force* must be acting on it.

Alternatively, you can view it as a body moving along a circular path requires a resultant force to act on it. Otherwise it would move in a straight line in accordance with Newton's first law.





$\Delta \boldsymbol{v} = \boldsymbol{v}_f - \boldsymbol{v}_i = \boldsymbol{v}_f + (-\boldsymbol{v}_i)$

In the above diagram, consider a very short interval of time so that points A and B are close. As the particle moves from A to B, its velocity changes from $\boldsymbol{v}_i$ to $\boldsymbol{v}_f$. The change in velocity $\Delta \boldsymbol{v}$ is directed toward point C, the centre of the circle. Hence the acceleration is also directed towards the centre of the circle.

This acceleration is known as **centripetal acceleration** and is given by

$$\boxed{a = \frac{v^2}{r} = r\omega^2}$$

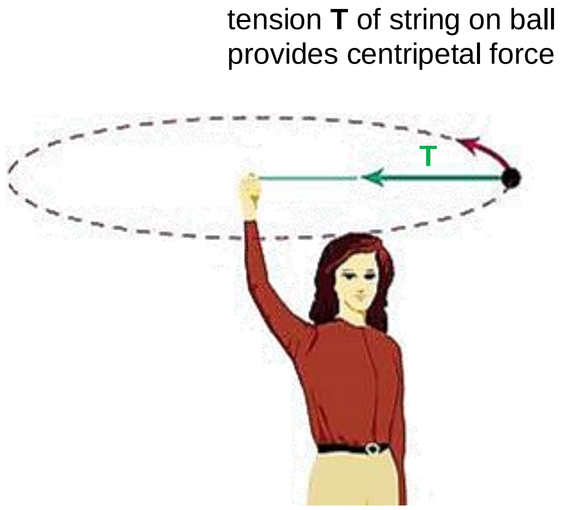
3.2 Centripetal Force *LO (f)*

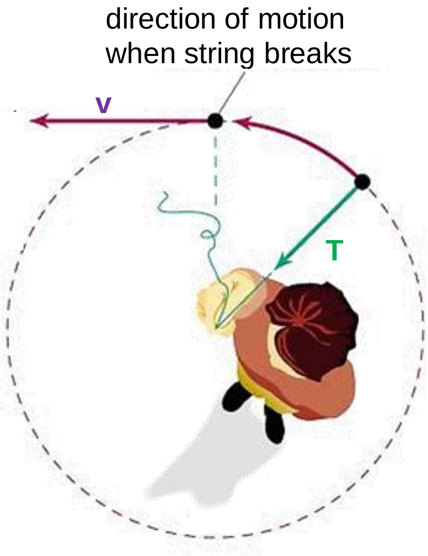
Since $F = ma$, there must be a *net force directed towards the centre of the circle* to provide the acceleration. The force causing the circular motion is known as the **centripetal force** and is given by

$$\boxed{F = \frac{mv^2}{r} = mr\omega^2}$$

Note:

1. For an object to move in a circular path, a physical force must be present to *provide for* the centripetal force. **The centripetal force is just a resultant force**. As such, the centripetal force should *not* be drawn as an additional force in force diagrams.





Hence for the above ball swinging in a <u>horizontal</u> circle, only the **tension T** in the string providing the centripetal force is drawn, not the resultant centripetal force. If the string were to break, there would be no centripetal force and the ball would fly off in a direction tangent to the circular path.

2. Since the centripetal force acts at right angles to the motion, the centripetal force cannot do any work. (Recall: work done by a force is defined as the product of the force and displacement in the direction of the force). Hence the centripetal force does not change the K.E. or speed of the object.

<u>Example 3</u> Conical pendulum

A mass of 0.50 kg is made to describe a circular path in a fixed horizontal plane. The string of length 25 cm makes an angle of 30° to the vertical.

Calculate (a) the tension in the string,
(b) the linear speed of the mass, and
(c) the period of oscillation of the mass.

(a) $T\cos\theta = mg$

$T\cos 30^{\circ} = 0.50 \times 9.81$

$T = 5.7$ N

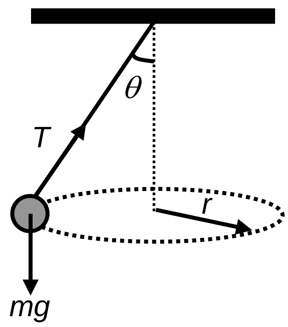
(b) $T\sin\theta = \dfrac{mv^2}{r} = \dfrac{mv^2}{L\sin\theta}$

$5.7\sin 30^{\circ} = \dfrac{0.50v^2}{(0.25)\sin 30^{\circ}}$

$v = 0.84$ m s^{-1}

(c) $T = \dfrac{dist}{speed} = \dfrac{2\pi r}{v}$

$= \dfrac{2\pi \times 0.25\sin 30^{o}}{0.84} = 0.93$ s



Conical pendulum – a mass on a string oscillating in a circular path.

Example 4 **Car turning around a bend**

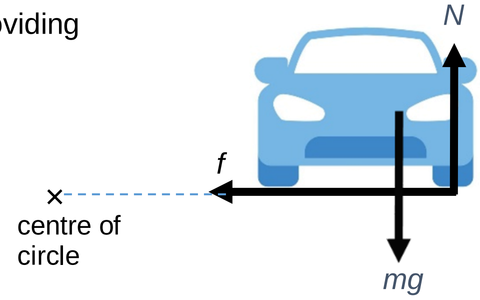
A 1200 kg car is turning a bend at 8.0 m s^{-1}, and it travels along an arc of a circle in the process. The radius of the circle is 9.0 m.

(a) Calculate the horizontal force exerted by the road on the tyres that is required to hold the car in the circular path.

(b) If the maximum frictional force between the road and the wheels is only 4.8×10^3 N, state and explain what will happen.

(c) Calculate the maximum speed for safe turning.

f is the sideways frictional force by road on tyres providing the centripetal force F_C.



(a) $F_c = \dfrac{mv^2}{r}$

$= 1200 \times 8.0^2/9.0$

$= 8.5 \times 10^3$ N

(b) The maximum frictional force of 4.8×10^3 N is less than the centripetal force 8.5×10^3 N required to turn around the bend. Hence the car will *skid outwards*.

(c) For safe turning,

$f_{max} = F_{Cmax}$

$4.8 \times 10^3 = \dfrac{mv^2}{r} = \dfrac{(1200)v^2}{9.0}$

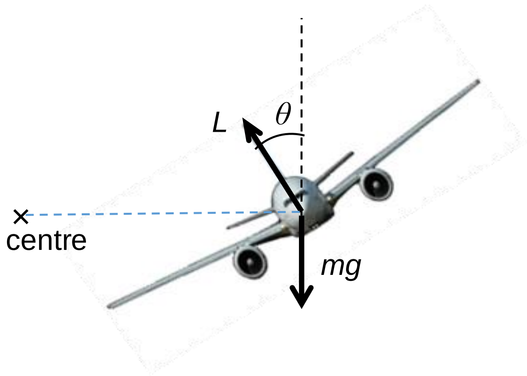
$v_{max} = 6.0$ m s^{-1}

controlling a skid

3.2.1 Providing centripetal force using banking

A banked turn is one in which the vehicle banks or inclines towards the inside of the turn. It is a common manoeuvre that increases the availability of forces that provide for the centripetal force required for a turn.

Case Study 1: Airplane turning in a horizontal plane



Front view of airplane making a turn to its right.

Lift L arises as a result of the difference in air speeds above and below the wings of the airplane. L acts in a direction perpendicular to the wings.

θ = banking angle

Horizontally,

$L \sin\theta = \dfrac{mv^2}{r}$ (1)

At constant height,

$L \cos\theta = mg$ (2)

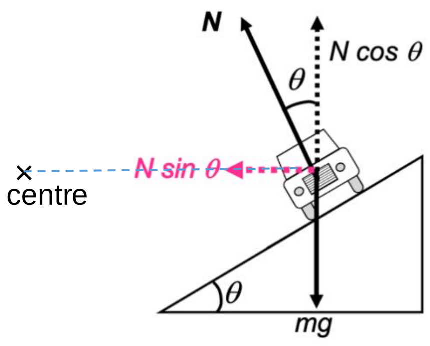
(1)÷(2): $\tan\theta = \dfrac{v^2}{rg}$

How aircraft turn

This concept can also be applied to land transportation in the form of **banked** roads to improve the handling of turns. As illustrated in Example 4, a vehicle negotiating a bend typically uses the sideways frictional force between the tyres and the road to provide the required centripetal force. However, there is a limit to this frictional force for a particular surface. This means that there is a maximum speed v for which the car can turn *without skidding*.

To increase this maximum (safe) speed, certain bends on roads are inclined or banked. This way, the horizontal component of the normal force N exerted by the road on the wheels of the car provides for the additional centripetal force required. We can obtain an expression for the banking angle θ such that banking alone provides all the required centripetal force.

Case Study 2: Car turning on banked road



Front view of car turning on banked road.

If* banking alone provides all the centripetal force,

$N \sin\theta = \dfrac{mv^2}{r}$ (1)

$N \cos\theta = mg$ (2)

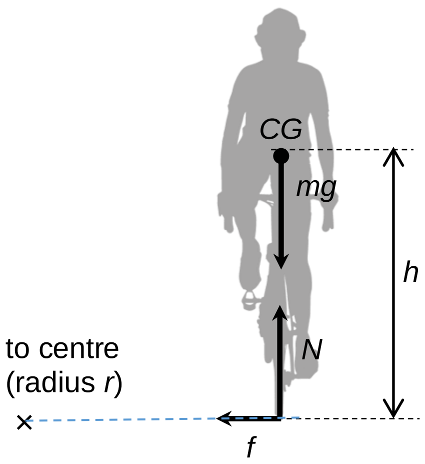
(1)÷(2): $\tan\theta = \dfrac{v^2}{rg}$

Banking of Roads

*banking may actually *overprovide* or *underprovide* for the required centripetal force

Like a car turning round a bend, a cyclist also relies on the sideways frictional force between the wheels of his bicycle and the road to provide the centripetal force. As before, if friction, $f <$ centripetal force, skidding occurs. However, even if f is sufficiently large, a cyclist needs to perform an additional leaning action to achieve stability. Why is that so?

Case Study 3: Cyclist going round a bend

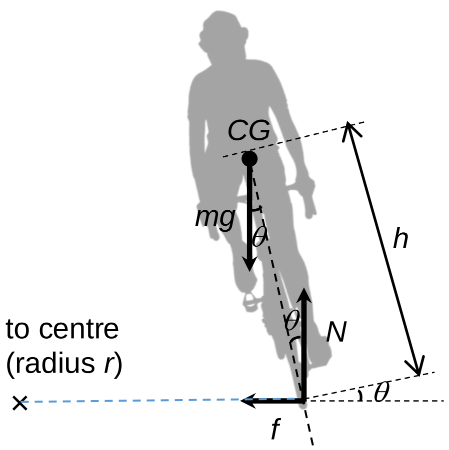


Front view of cyclist making turn without leaning.

Assuming f sufficiently provides for centripetal force,

if the cyclist remains upright, taking moments about CG, there is a net clockwise moment due to friction = $f \times h$.

Cyclist will topple outwards.



Front view of cyclist making turn while leaning inwards.

If however he leans inwards with angle θ such that

Horizontally: $f = \dfrac{mv^2}{r}$

Vertically: $N = mg$

Taking moments about CG,

$f\,(h \cos\theta) = N\,(h \sin\theta)$ where θ is the leaning angle

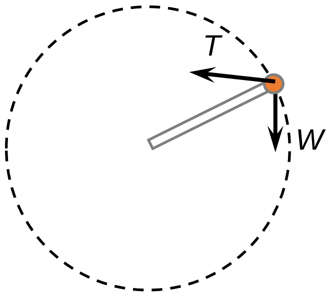
then $\tan\theta = \dfrac{f}{N} = \dfrac{\frac{mv^2}{r}}{mg} = \dfrac{v^2}{rg}$

Thus when a cyclist turns around a bend, he must lean inwards (ie. incline towards the centre of curvature) to avoid toppling.

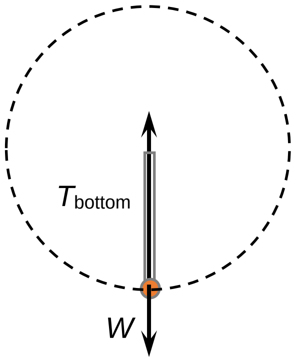
Bikes - how far to lean

3.2.1 Uniform vertical circular motion

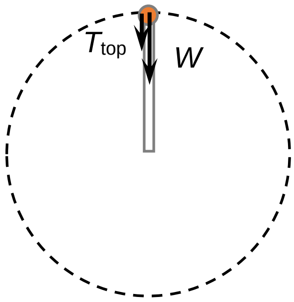
If a mass at the end of a *rigid* rod is rotated in a vertical plane, it can be made to perform uniform circular motion. This is due to the *variable* nature of the tension acting on the ball by the rigid rod, which continually changes its magnitude and direction to meet the centripetal force requirements.



Forces acting on a ball at a position in uniform vertical circular motion



Forces acting on the ball at the bottom of the circular motion.



Forces acting on the ball at the top of the circular motion.

Since the ball is moving in a circle at constant speed, the resultant force always acts towards the centre of the circle and is constant in magnitude.

The tension due to the rigid rod however will not be acting along the rod except when the ball is at the top or the bottom of the circle.

At the top, $T_{top} + W = \dfrac{mv^2}{r}$

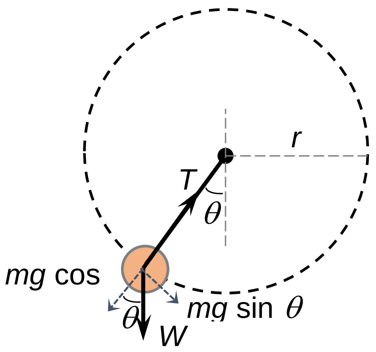
At the bottom, $T_{bottom} - W = \dfrac{mv^2}{r}$

The capsule of ferris wheel such as the Singapore Flyer is a classic example of a vertical yet uniform circular motion

4 Non-Uniform Motion in a Circle

In <u>non-uniform</u> circular motion, the object moves in a circular path with <u>varying speed</u>. Since the speed is changing, there is <u>tangential acceleration</u> in addition to centripetal acceleration.

Most vertical circular motions such as that of a mass tied to a string (instead of a rigid rod) are non-uniform by nature. The only two forces acting on the mass are the tension of the string, which only points towards the centre of the centre, and the weight of the mass which points vertically downwards throughout the motion.



Mass on a string in vertical circular motion

The tangential acceleration, a_t is given by

$$mg \sin\theta = ma_t$$
$$\therefore\ a_t = g \sin\theta$$

The centripetal acceleration, a_c is given by

$$a_c = \frac{v^2}{r}.$$

$$T - mg\cos\theta = \frac{mv^2}{r}$$
$$\therefore\ T = \frac{mv^2}{r} + mg\cos\theta$$

The centripetal acceleration changes in magnitude as its speed changes due to the tangential acceleration.

<u>Example 5</u> Water in a bucket

A pail containing water (mass m) is whirled in a vertical circle of radius r. Calculate the minimum speed at which the pail must be whirled for the water to remain in the pail.

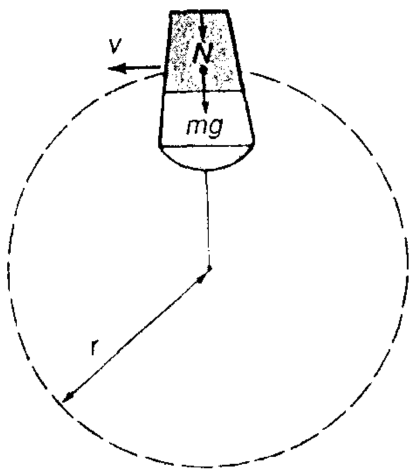
At the top of circle,

$$N + mg = \frac{mv^2}{r}$$
$$\therefore\ N = m\left(\frac{v^2}{r} - g\right)$$

Water will remain in pail if $N \geq 0$.

$$\Rightarrow m\left(\frac{v^2}{r} - g\right) \geq 0$$
$$\Rightarrow\ v \geq \sqrt{rg}$$

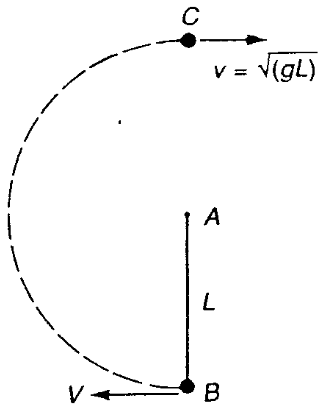
$$\therefore\ v_{min} = \sqrt{rg}$$



Spinning water bucket

Example 6 **Suspended particle at the end of string**

A particle is suspended from a point A by an inextensible string of length L. It is projected from point B with a velocity V, perpendicular to AB, which is just sufficient for it to reach the point C.



(a) Show that, if the string is just to be taut when the particle reaches C, its speed v there is $\sqrt{(gL)}$.

(b) Determine, in terms of g and L, the speed V with which the particle should be projected from B.

(c) Hence determine in terms of g and L, the tension in the string at B.

(a) At top, $T + mg = \dfrac{mv^2}{L}$

For string to be just taut at C, $T = 0$ N.

$mg = \dfrac{mv^2}{r}$

$v = \sqrt{gL}$

(b) By conservation of energy, $KE_B + GPE_B = KE_C + GPE_C$

$$\tfrac{1}{2}mV^2 + 0 = \tfrac{1}{2}mv^2 + mg(2L)$$
$$= \tfrac{1}{2}m(gL) + mg(2L)$$
$$= \tfrac{5}{2}mgL$$
$$\therefore\ V = \sqrt{5gL}$$

(c) At bottom,

$T - mg = \dfrac{mV^2}{L}$

$T = mg + m\left(\dfrac{5gL}{L}\right) = 6\ mg$

<u>Tutorial Questions</u> (Solutions to Questions 2 and 5 are provided.)

Angular Displacement and Angular Velocity

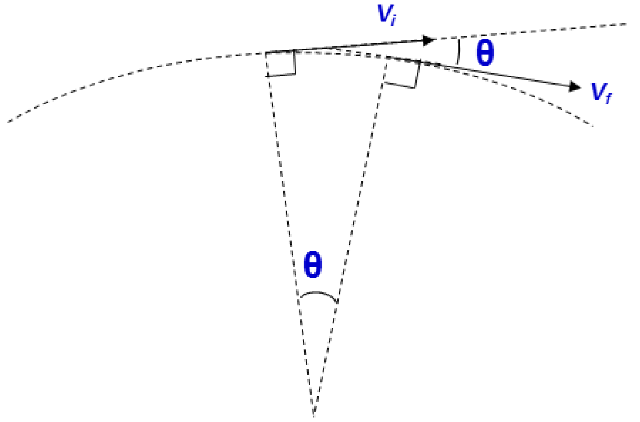
1 Calculate the angular velocities of (i) the second hand, (ii) the minute hand, and (iii) the hour hand of the clock.

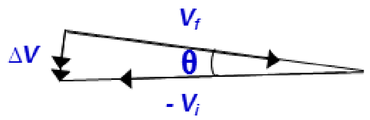
(i) The period for the second hand of clock is 60 s.

(ii) The period for the minute hand of clock is $60 \times 60 = 3600$ s.

(iii) The period for the hour hand of clock is $12 \times 3600 = 43200$ s.

2 An object travelling in a circle of radius *r* at a constant speed *v* is accelerating. By drawing a vector diagram, explain how this is possible.





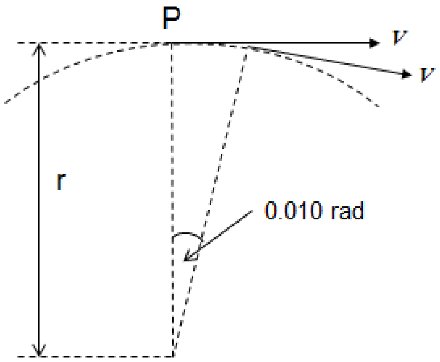
Ans:

$\Delta v = v_f + (-v_i)$, Δv is always directed towards centre of circle.

Hence, although linear speed is constant, direction is changing, so motion is accelerated.

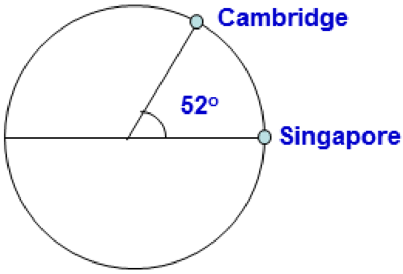
Uniform Circular Motion

3 An object P moves at a constant speed *v* through an arc of a circle of radius *r*. The arc subtends an angle 0.010 rad at the centre of the circle, as shown below.



(i) Determine, in terms of *v*, the magnitude of the change in velocity.
(ii) State the direction of the acceleration of P.
(iii) Deduce, in terms of r and v, the time taken for P to travel 0.010 rad.
(iv) Hence show that the magnitude of the acceleration of P is $\dfrac{v^2}{r}$

4 Singapore is on the Equator. Cambridge is at latitude of 52° N as shown in the diagram below.



A student at Singapore has a centripetal acceleration a_S because of the Earth's rotation about its axis. The centripetal acceleration of another student at Cambridge is a_C. The radius of the Earth's equator is 6.38×10^6 m. What are the magnitudes of the centripetal accelerations a_S and a_C?
[3.4×10^{-2} m s^{-2}, 2.1×10^{-2} m s^{-2}]

5 Calculate the minimum radius of a circle in which a car travelling at 20 m s^{-1} can turn if the maximum frictional force between the tires and the road is 3600 N and the mass of the car is 1200 kg. At what angle should this bend be banked to avoid the need for a centripetal frictional force at this point? [130 m, 17^0]

Ans:

$f_{max} = mv^2 / r_{min}$

$3600 = 1200\,(20^2) / r_{min}$

$r_{min} = 130$ m

$N \cos\theta = mg$

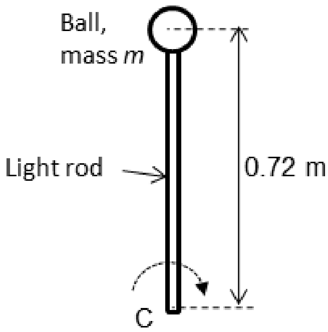
$N \sin\theta = mv^2 / r$

$\Rightarrow \tan\theta = v^2 / rg$

$\Rightarrow \theta = \tan^{-1}(20^2 / 130 \times 9.81)$

$= 17°$

6 A small ball of mass m is fixed to one end of a light rigid rod. The ball is made to move at constant speed around the circumference of a vertical circle with centre at C, as shown in figure below.



When the rod is vertical with the ball above C, the tension in the rod is given by

$$T = 2mg$$

where g is the acceleration of free fall.

(a) (i) Explain why the centripetal force on the ball is greater than 2 *mg*.

(ii) State, in terms of mg, the magnitude of the centripetal force.

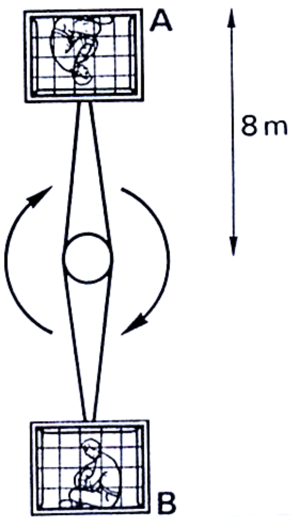
(iii) Determine the magnitude of the tension, in terms of *mg*, in the rod when the rod is vertical, with the ball below point C.

(b) The distance from the centre of the ball to point C is 0.72 m. Use your answer in (a)(ii) to determine, for the ball,
  (i) the angular speed,
  (ii) the linear speed.

(c) The ball has a constant angular speed.
  (i) Explain why work has to be done for the ball to move from the position where it is vertically above point C to the position where it is vertically below C.
  (ii) Calculate the work done in (i) for a ball of mass 240 g.

[3 mg, 4 mg, 6.4 rad s^{-1}, 4.6 m s^{-1}, 3.4 J]

7 In a ride at an entertainment park, two people, each of mass 80 kg, sit in cages which travel at constant speed in a vertical circle of radius 8.0 m as shown below. Each revolution takes 4.2 s. When a cage is at the top of the circle (position A) the person in it is upside down.



(a) For the person in cage A, calculate the magnitudes of
  (i) the angular velocity,
  (ii) the linear speed,
  (iii) the centripetal acceleration,

(b) (i) Draw a vector diagram to show the directions of the following forces acting on the person in cage A: The weight *W* of the person, and the force *F* exerted by the cage on the person.
  (ii) Draw the corresponding diagram for the person at the bottom of the circle (position B).
  (iii) What must be the value of the resultant of these two forces at both A and B?
  (iv) Explain why the person remains on the floor of the cage at the top of the circle.
  (v) State the position at which the force the floor exerts on the person has its maximum value. Calculate the magnitude of this force.

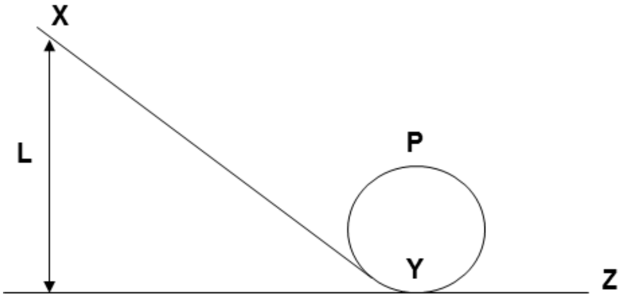
(c) Draw a vector diagram showing *W*, *F* and their resultant when the line joining the cages is horizontal. Numerical values are not required for this part of the question, but the force vectors should be drawn so that they have approximately correct relative sizes.

[1.5 rad s^{-1}, 12 m s^{-1}, 18 m s^{-2}, 1440 N, 2220 N]

Non-Uniform Circular Motion

8 A car travels over a humpback bridge of radius of curvature of 45 m. Calculate the maximum speed of the car if its wheels are to just stay in contact with the bridge. [21 m s^{-1}]

9 The diagram shows a toy smooth runway. After release from a point X, a small model car of mass 0.75 kg runs down the slope, loops the loop and travels on towards Z. The radius of the loop is 0.25 m.



(a) What is the minimum speed with which the car must pass point P at the top of the loop if it to remain in contact with the runway?

(b) What is the minimum value of *L* which allows the speed calculated in (a) to be achieved?

(c) With this minimum value of *L*, what is the force the model car exerts on the runway at point Y (the bottom of the loop)?

[1.6 m s^{-1}, 0.63 m, 44 N]

10 A stone of mass 500 g is attached to a string of length 50 cm which will break if the tension exceeds 20 N. The stone is whirled in a vertical circle, with the centre of the axis of rotation being at a height 100 cm above the ground.

(a) The angular velocity is slowly increased until the string breaks. Predict the position in the circular motion at which the string is most likely to break.

(b) Calculate the angular velocity at which this break first occurs.

(c) Hence, calculate the horizontal displacement of the stone when it hits the ground.

[7.8 rad s^{-1}, 1.2 m]