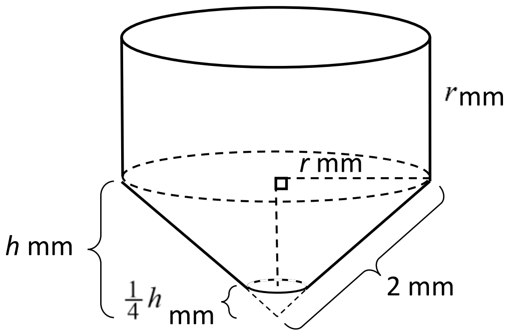
RAFFLES INSTITUTION
H2 Mathematics 9758
2025 Year 6 Term 3 Revision 1 (Zeta)
Topic(s): Calculus I
(Differentiation, Applications, Maclaurin and Graphing (Solutions)

1 **HCI Prelim 9758/2023/01/Q11**

[The volume of a right circular cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A student makes a printer nozzle for his self-built 3D printer. The printer nozzle consists of a hollow inverted right circular cone of negligible thickness with radius r mm where $0 < r < 2$, height h mm and slanted edge 2 mm joined to a hollow cylinder of negligible thickness with radius r mm and height r mm. A height of $\frac{1}{4}h$ mm is cut off from the vertex of the cone for the nozzle opening (see diagram).

The volume of the printer nozzle is V mm^3.

(a) Show that $V = \pi r^3 + \frac{21}{64}\pi r^2\sqrt{4-r^2}$. [3]

The student wants V to be a maximum.

(b) It is given that $r = r_1$ gives the maximum value of V. Show that r_1 satisfies the equation $4537r^4 - 18736r^2 + 3136 = 0$. [4]

(c) Show that one of the positive roots of the equation in part **(b)** does not give a stationary value of V. Suggest a reason why this value is a solution to the equation in part **(b)** even though it does not give a stationary value of V. [3]

(d) Given that r_1 is the largest positive root of the equation in part **(b)**, state the value of r_1 and find the corresponding value of h. (You need not show that your answer gives a maximum.) [1]

(e) With reference to the value of r_1 found in part **(d)**, comment on the practicality of having a maximum volume for the printer nozzle. [1]

Solution:

(a) By Pythagoras' Theorem,

$$r^2 + h^2 = 2^2$$
$$\therefore \quad h = \sqrt{4-r^2} \quad \text{(reject } h = -\sqrt{4-r^2} \text{ since } h > 0\text{)}$$

Let r_0 be radius of cut off cone.

Using similar $\triangle$s, $\frac{r_0}{r} = \frac{1}{4} \Rightarrow r_0 = \frac{r}{4}$

$$\begin{aligned}\therefore \ V &= \pi r^2 (r) + \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi r_0{}^2 (\frac{1}{4}h) \\ &= \pi r^3 + \frac{1}{3}\pi r^2 \sqrt{4-r^2} - \frac{1}{192}\pi r^2 \sqrt{4-r^2} \\ &= \pi r^3 + \frac{21}{64}\pi r^2 \sqrt{4-r^2} \qquad \text{(shown)}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}r} &= 3\pi r^2 + \frac{21}{64}\pi \left[2r\sqrt{4-r^2} + r^2 (\frac{1}{2\sqrt{4-r^2}})(-2r) \right] \\ &= 3\pi r^2 + \frac{21}{64}\pi r \left[2\sqrt{4-r^2} - \frac{r^2}{\sqrt{4-r^2}} \right] \\ &= 3\pi r^2 + \frac{21}{64}\pi r \frac{8-3r^2}{\sqrt{4-r^2}}\end{aligned}$$

At maximum value of V, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$, since $0 < r < 2$

$$\therefore \ 3\pi r^2 + \frac{21}{64}\pi r \frac{8-3r^2}{\sqrt{4-r^2}} = 0$$

$$3\pi r \left[r + \frac{7}{64}\left(\frac{8-3r^2}{\sqrt{4-r^2}} \right) \right] = 0$$

$$r + \frac{7}{64}\left(\frac{8-3r^2}{\sqrt{4-r^2}} \right) = 0 \quad \text{(since } 0 < r < 2\text{)}$$

$$r = \frac{7}{64}\left(\frac{3r^2 - 8}{\sqrt{4-r^2}} \right)$$

$$64r\sqrt{4-r^2} = 21r^2 - 56$$

$$4096r^2(4-r^2) = (21r^2 - 56)^2$$

$$16384r^2 - 4096r^4 = 441r^4 - 2352r^2 + 3136$$

$$\therefore \ 4537r^4 - 18736r^2 + 3136 = 0 \quad \text{(shown)}$$

(c) Using GC, since $r > 0$,

$r = 0.41806129085388$ or $r = 1.9886743863834$

When $r = 0.41806129085388$, $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 3.294435715 \neq 0$

Hence $r = 0.41806129085388$ does not give a stationary value of V.

The squaring of the equation to remove the square root created additional roots to the equation. These additional roots may not give rise to stationary values of V.

(d) $r_1 = 1.99$ (3 s.f.)

$h = \sqrt{4 - r_1^{\,2}} = 0.2125421957 = 0.213$ (3 s.f.)

(e) The value of $r_1 = 1.99$ is almost the same length as the slant height 2 of the inverted cone. This means that the inverted cone is essentially flat and non-existent, leaving the printer nozzle in the shape of a cylinder only. Hence it is not realistic to have a maximum volume for the printer nozzle.

Ideal measurements/values/conditions based on theoretical calculations may not be practical or feasible in real life.

2 NYJC Prelim 9758/2022/01/Q7

A curve is defined parametrically by

$$x = \frac{at}{t+1},\ y = \frac{at^2}{t+1},$$

where a is a non-zero constant and $t \in \mathbb{R}, t \neq -1$.

(i) Show that the equation of the normal to the curve at the point T with parameter t is given by

$$t(t+1)(t+2)y + (t+1)x = at(t^3 + 2t^2 + 1).$$ [4]

(ii) The normal to the curve at the point $P\left(-a, \dfrac{1}{2}a\right)$ meets the curve again at point Q. Find the exact coordinates of Q in terms of a. [4]

(iii) Let M be the mid-point of PQ. Find a cartesian equation of the curve traced by M if a varies. [2]

Solution:	
(i)	$x=\dfrac{at}{t+1}$ $\qquad y=\dfrac{at^2}{t+1}$ $\dfrac{dx}{dt}=\dfrac{a(t+1)-at}{(t+1)^2}=\dfrac{a}{(t+1)^2}$ $\qquad \dfrac{dy}{dt}=\dfrac{2at(t+1)-at^2}{(t+1)^2}=\dfrac{at^2+2at}{(t+1)^2}$ $\therefore \dfrac{dy}{dx}=t^2+2t$ Equation of normal at point T is $y-\dfrac{at^2}{t+1}=-\dfrac{1}{t(t+2)}\left[x-\dfrac{at}{t+1}\right]$ $t(t+2)y-\dfrac{at^2}{t+1}\times t(t+2)=-x+\dfrac{at}{t+1}$ $t(t+1)(t+2)y-at^3(t+2)=-(t+1)x+at$ $t(t+1)(t+2)y+(t+1)x=at(t^3+2t^2+1)$ (shown)
(ii)	When $t=-\dfrac{1}{2}$, $x=-a$, $y=\dfrac{1}{2}a$ and $\dfrac{dy}{dx}=-\dfrac{3}{4}$ Equation of normal at P is $y-\dfrac{1}{2}a=\dfrac{4}{3}(x+a) \Rightarrow y=\dfrac{4}{3}x+\dfrac{11}{6}a$ OR : At P $\left(-a,\dfrac{1}{2}a\right)$, $\dfrac{at}{t+1}=-a \Rightarrow t=-\dfrac{1}{2}$. Using result in **(i)**, Equation of normal at P is $-\dfrac{3}{8}y+\dfrac{1}{2}x=-\dfrac{11}{16}a$ $\Rightarrow -6y+8x=-11a$ Given that this normal meets the curve again at Q, $-6\left(\dfrac{at^2}{t+1}\right)+8\left(\dfrac{at}{t+1}\right)=-11a$ $\Rightarrow 6t^2-8t=11(t+1)$ $\Rightarrow 6t^2-19t-11=0$ $\Rightarrow (2t+1)(3t-11)=0$ $\Rightarrow t=-\dfrac{1}{2}$ or $\dfrac{11}{3}$ $\quad$ ($t=-\dfrac{1}{2}$ is rejected as this corresponds to point P)

	When $t=\frac{11}{3}$, $x=\frac{a\left(\frac{11}{3}\right)}{\left(\frac{11}{3}\right)+1}=\frac{11}{14}a$, $y=\frac{a\left(\frac{11}{3}\right)^2}{\left(\frac{11}{3}\right)+1}=\frac{121}{42}a$ $\therefore$ The coordinates of Q are $\left(\frac{11}{14}a,\frac{121}{42}a\right)$
(iii)	Mid-point of PQ, M is $\left(-\frac{3}{28}a,\frac{71}{42}a\right)$ $x=-\frac{3}{28}a$ and $y=\frac{71}{42}a$ $\Rightarrow y=-\frac{142}{9}x, x\neq 0$ since a is non-zero

3	**ACJC Prelim 9758/2021/01/Q1 (modified)**

(a) A function is defined as $\mathrm{f}(x)=\frac{1-\mathrm{e}^x}{\mathrm{e}^{x-2}}$.

Describe a sequence of transformations that transforms the graph of $y=\mathrm{e}^x$ onto the graph of $y=\mathrm{f}(x)$. [5]

(b) A function is defined as $\mathrm{g}(x)=\cos 2x$.

(i) Describe a sequence of transformations that transforms the graph of $y=\sin 2x$ onto the graph of $y=\mathrm{g}(x)$. [2]

(ii) Describe a sequence of transformations that transforms the graph of $y=\sin^2 x$ onto the graph of $y=\mathrm{g}(x)$. [3]

Solution:

(a)

$$\mathrm{f}(x)=\frac{1-\mathrm{e}^x}{\mathrm{e}^{x-2}}$$
$$=\frac{1}{\mathrm{e}^{x-2}}-\frac{\mathrm{e}^x}{\mathrm{e}^{x-2}}$$
$$=\mathrm{e}^{-x+2}-\mathrm{e}^2$$

1. Translation of 2 units in the negative x-axis direction.

2. Reflection in the y-axis.
3. Translation of e^2 units in the negative y-axis direction.

Alternative 1:

1. Reflection in the y-axis.
2. Translation of 2 units in the positive x-axis direction.
3. Translation of e^2 units in the negative y-axis direction.

Alternative 2:

$\mathrm{f}(x)=\mathrm{e}^{-x+2}-\mathrm{e}^2=\mathrm{e}^2\cdot\mathrm{e}^{-x}-\mathrm{e}^2$

1. Reflection in the y-axis.
2. Scaling parallel to the y-axis by the scale factor e^2.
3. Translation of e^2 units in the negative y-axis direction.

Alternative 3:

$\mathrm{f}(x)=\mathrm{e}^{-x+2}-\mathrm{e}^2=\mathrm{e}^2\left(\mathrm{e}^{-x}-1\right)$

1. Reflection in the y-axis.
2. Translation of 1 unit in the negative y-axis direction.
3. Scaling parallel to the y-axis by the scale factor e^2.

(b)(i)

We note that $\sin 2\left(x+\frac{\pi}{4}\right)=\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$.

Hence translation of $\frac{\pi}{4}$ units in the negative x-axis direction.

(ii)

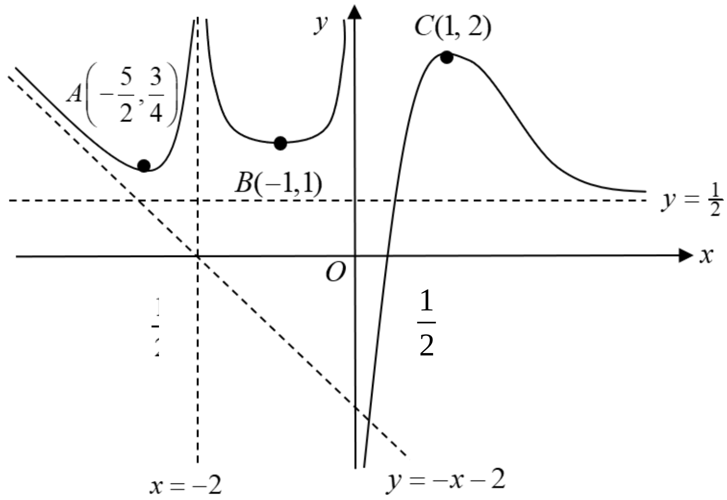
We use the double angle formula to note that $\mathrm{g}(x)=\cos 2x=1-2\sin^2 x$.

Hence
1. Reflection in the x-axis.
2. Scaling parallel to the y-axis by the scale factor 2.
3. Translation of 1 unit in the positive y-axis direction.

[Note that 1 and 2 are interchangeable]

4 DHS Prelim 9758/2022/01/Q2

The diagram below shows the graph of $y = \mathrm{f}(x)$. The curve crosses the x-axis at $x = \frac{1}{2}$ and has turning points at $A\left(-\frac{5}{2}, \frac{3}{4}\right)$, $B(-1,1)$ and $C(1, 2)$. It has asymptotes $y = -x - 2$, $y = \frac{1}{2}$, $x = 0$ and $x = -2$.



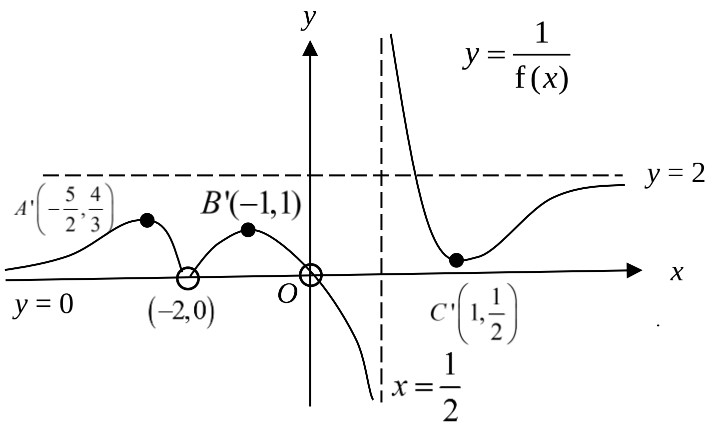
On separate diagrams, sketch the following graphs. In each case, show clearly the equations of any asymptotes, coordinates of any points of intersection with both axes and the points corresponding to A, B and C.

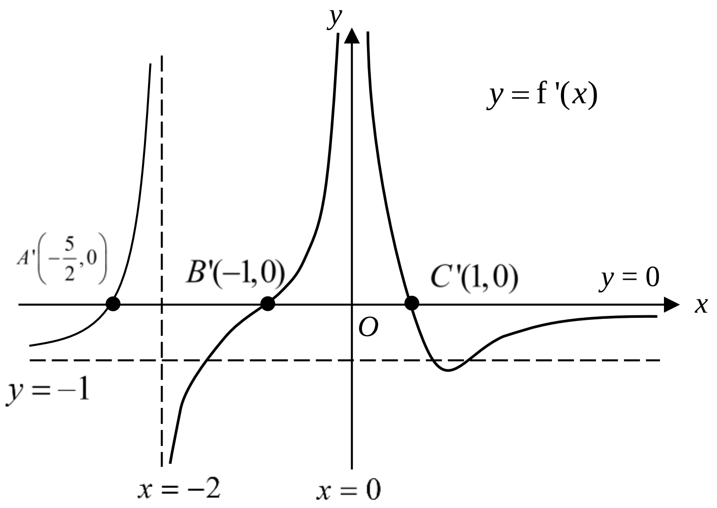
(a) $y = \dfrac{1}{\mathrm{f}(x)}$ [3]

(b) $y = \mathrm{f}'(x)$ [3]

Solution:

(a)



(b)	

5	**ACJC Prelim 9758/2021/01/Q4(modified)**

The curve C has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x(x+a)}{x-a}$ and a is a positive real constant.

(i) Find algebraically, in terms of a, the set of values that y can take. [4]

(ii) Sketch C, indicating clearly the equations of any asymptotes and the coordinates of the turning points and points where the curve crosses the axes. [4]

(iii) By adding a suitable graph to your same diagram in **(ii)** and labelling it clearly, solve the inequality $\dfrac{3x(x+a)}{x-a} < |x|$. [3]

Solution:

(i) Consider $y = k$, where k is a constant

$$\frac{x(x+a)}{x-a} = k$$

$$x^2 + ax = xk - ak$$

$$x^2 + (a-k)x + ak = 0 \qquad (*)$$

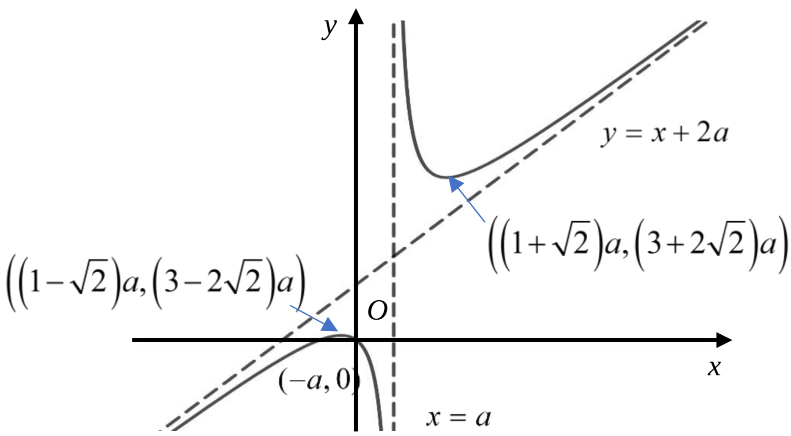
For the range of values that y can take, the line $y = k$ and the curve C would have point(s) of intersection.

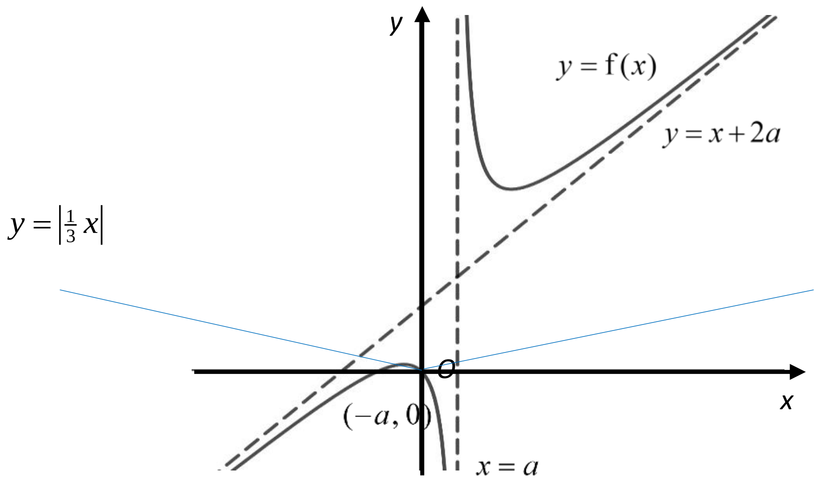
$$\therefore \text{Discriminant} \geq 0$$

$$(a-k)^2 - 4ak \geq 0$$

$$a^2 - 2ak + k^2 - 4ak \geq 0$$

$$a^2 - 6ak + k^2 \geq 0$$

	Consider $k^2 - 6ak + a^2 = 0$ $k = \dfrac{6a \pm \sqrt{36a^2 - 4a^2}}{2} = \left(3 \pm 2\sqrt{2}\right)a$ $\therefore k \geq \left(3+2\sqrt{2}\right)a$ or $k \leq \left(3-2\sqrt{2}\right)a$ Hence, set of values of y is $\left(-\infty, \left(3-2\sqrt{2}\right)a\right] \cup \left[\left(3+2\sqrt{2}\right)a, \infty\right)$.
(ii)	We have $\mathrm{f}(x) = \dfrac{x(x+a)}{x-a} = x + 2a + \dfrac{2a^2}{x-a}$ and thus the oblique asymptote has equation $y = x + 2a$. When $y = 0$, $x = -a$ or 0. When $y = \left(3 \pm 2\sqrt{2}\right)a$, $x = \left(1 \pm \sqrt{2}\right)a$ (solve via quadratic equation). Otherwise, $\mathrm{f}'(x) = 1 - \dfrac{2a^2}{(x-a)^2} = 0 \Leftrightarrow x = a \pm \sqrt{2}a = (1 \pm \sqrt{2})a$. Or we note from (*) that when y takes on these values (*) has only repeated real roots, and sum of roots = –coefficient of x = $k - a$ which implies that $x = \dfrac{k-a}{2} = \dfrac{\left(3 \pm 2\sqrt{2}\right)a - a}{2} = \left(1 \pm \sqrt{2}\right)a$ 

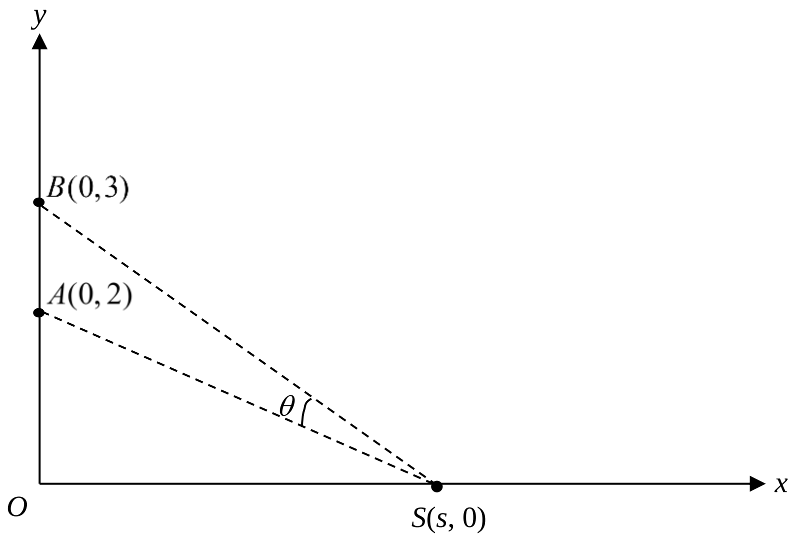
(iii)	 Consider $\dfrac{3x(x+a)}{x-a}=-x$ $3x^2+3ax=-x^2+ax$ $4x^2+2ax=0$ $x(2x+a)=0$ $x=0$ or $x=-\frac{a}{2}$ Hence, $y=\left\lvert\frac{1}{3}x\right\rvert$ intersect the graph at $x=0$ and $x=-\dfrac{a}{2}$. From the graph, $x<-\dfrac{a}{2}$ or $0<x<a$. Note : $y=\frac{1}{3}x$ does not intersect the graph at a positive value of x since the gradient is less steep than that of the oblique asymptote.

6	**HCI Prelim 9758/2021/01/Q9**
	Given that $y=(\tan x+\sec x)^2$, show that $\cos x\dfrac{dy}{dx}=2y$. [2] **(i)** By repeat differentiation, find the Maclaurin series for y up to and including the term in x^3. [4] **(ii)** Using the result obtained in **(i)** estimate the value of $\tan 1^\circ+\sec 1^\circ$ to 4 decimal places. [2] **(iii)** By expressing y in terms of sine and cosine, use the standard series in MF27 to find the series expansion for y up to and including the term in x^3. [3] **(iv)** Comment on the results obtained from part **(i)** and part **(iii)**. [1]

Solution:	
	$y=(\tan x+\sec x)^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=2(\tan x+\sec x)(\sec^2 x+\sec x\tan x)$ $\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=2(\tan x+\sec x)(\sec x+\tan x)$ $\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=2y$ (shown)
(i)	Differentiate w.r.t x, $\cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\sin x\frac{\mathrm{d}y}{\mathrm{d}x}=2\frac{\mathrm{d}y}{\mathrm{d}x}$ Differentiate w.r.t x, $\cos x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-\sin x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\sin x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ When $x=0$, $y=1, \frac{\mathrm{d}y}{\mathrm{d}x}=2, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4, \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=10$ $\therefore y=\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2}x^2+\frac{\mathrm{f}'''(0)}{3!}x^3+\dots$ $=1+2x+2x^2+\frac{5}{3}x^3+\dots$
(ii)	Note that x in (i) is in radian. $1^{\mathrm{o}}=\frac{\pi}{180}$ $\tan 1^{\mathrm{o}}+\sec 1^{\mathrm{o}}\approx\sqrt{1+2(\frac{\pi}{180})+2(\frac{\pi}{180})^2+\frac{5}{3}(\frac{\pi}{180})^3}$ $=\sqrt{1.03552468}$ $=1.0176$(4d.p.)

(iii)	$y=\left(\tan x+\sec x\right)^2$ $=\left(\dfrac{\sin x+1}{\cos x}\right)^2$ $=(1+\sin x)^2(\cos x)^{-2}$ $\approx\left(1+x-\dfrac{x^3}{6}\right)^2\left(1-\dfrac{x^2}{2}\right)^{-2}$ $=\left[1+2\left(x-\dfrac{x^3}{6}\right)+\left(x-\dfrac{x^3}{6}\right)^2\right](1+x^2+...)$ $\approx(1+2x+x^2-\dfrac{1}{3}x^3)(1+x^2+...)$ $=1+2x+2x^2+\dfrac{5}{3}x^3+...$
(iv)	We expect the answers from both parts to be the same and the results have proven so.

7 In the diagram below, a soccer player, represented by the point $S(s,0)$, is running with a ball along the touchline, represented by the x-axis. The goalposts are represented by the points $A(0,2)$ and $B(0,3)$ and $\theta=\angle ASB$. It is assumed that $s>0$.



(i) By applying the cosine rule, show that $\cos^2\theta=\dfrac{(s^2+6)^2}{(s^2+4)(s^2+9)}$. [2]

(ii) Let $f(s)=\dfrac{(s^2+6)^2}{(s^2+4)(s^2+9)}$. Express $f(s)$ in the form $1+\dfrac{P}{s^2+4}+\dfrac{Q}{s^2+9}$, where P and Q are constants to be determined. Hence find $f'(s)$. [4]

	(iii)	The soccer player decides to shoot when θ is maximum. Using your result from **(ii),** determine the exact value of s at which the soccer player should shoot, justifying your answer. [4]

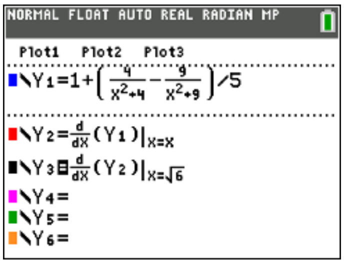
Solution:

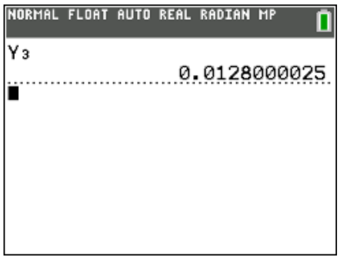
7 (i)	Applying the cosine rule to ΔASB, we have $(3-2)^2=(\sqrt{2^2+s^2})^2+(\sqrt{3^2+s^2})^2-2\sqrt{2^2+s^2}\sqrt{3^2+s^2}\cos\theta.$ Rearranging we have $\cos\theta=\dfrac{4+s^2+9+s^2-1}{2\sqrt{4+s^2}\sqrt{9+s^2}}=\dfrac{s^2+6}{\sqrt{4+s^2}\sqrt{9+s^2}},$ which upon squaring both sides gives the desired result.
(ii)	We have $\dfrac{(s^2+6)^2}{(s^2+4)(s^2+9)}=1+\dfrac{P}{s^2+4}+\dfrac{Q}{s^2+9},$ so $(s^2+6)^2=(s^2+4)(s^2+9)+P(s^2+9)+Q(s^2+4).$ Comparing coefficients gives $9P+4Q=0$, $P+Q=-1,$ so $P=\dfrac{4}{5}\text{ and }Q=-\dfrac{9}{5}.$ Thus $\mathrm{f}(s)=1+\dfrac{1}{5}\left(\dfrac{4}{s^2+4}-\dfrac{9}{s^2+9}\right)$ and $\mathrm{f}'(s)=\dfrac{2s}{5}\left(\dfrac{9}{(s^2+9)^2}-\dfrac{4}{(s^2+4)^2}\right).$
(iii)	When $\mathrm{f}'(s)=0$, we have $\dfrac{2s}{5}\left(\dfrac{9}{(s^2+9)^2}-\dfrac{4}{(s^2+4)^2}\right)=0$ $\dfrac{9}{(s^2+9)^2}-\dfrac{4}{(s^2+4)^2}=0$ (as $s>0$) $\dfrac{9}{(s^2+9)^2}=\dfrac{4}{(s^2+4)^2}$ $\dfrac{3}{s^2+9}=\dfrac{2}{s^2+4}$ (as all quantities are positive) Thus $s^2=6$, giving $s=\sqrt{6}$ as $s>0$.

	$\mathrm{f}'(s)=\dfrac{2s}{5}\left(\dfrac{9}{(s^2+9)^2}-\dfrac{4}{(s^2+4)^2}\right)=\dfrac{2s}{5}\left(\dfrac{3^2(s^2+4)^2-2^2(s^2+9)^2}{(s^2+9)^2(s^2+4)^2}\right)$ $=\dfrac{2s}{5}\left(\dfrac{(5s^2+30)(s^2-6)}{(s^2+9)^2(s^2+4)^2}\right)=\dfrac{2s(s^2+6)(s+\sqrt{6})(s-\sqrt{6})}{(s^2+9)^2(s^2+4)^2}$ This implies that

s	$\sqrt{6}^-$	$\sqrt{6}$	$\sqrt{6}^+$
$(s-\sqrt{6})$	<0	0	>0
$\mathrm{f}'(s)$	<0	0	>0
$\mathrm{f}(s)$	$\searrow$	—	$\nearrow$

Alternatively, $\mathrm{f}''(s)=0.0128>0$ therefore $\cos^2\theta$ is minimum when $s=\sqrt{6}$.

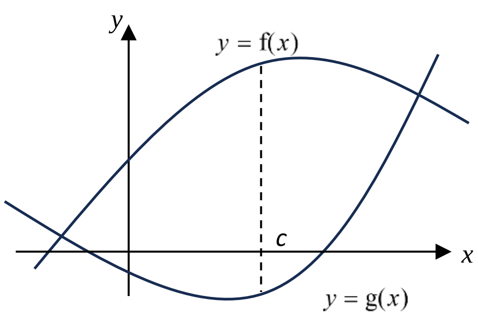


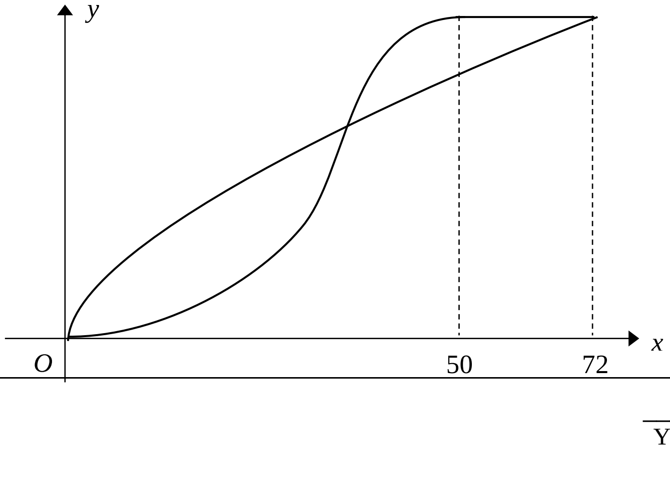


Since $\cos^2\theta$ is decreasing for $\theta\in\left[0,\dfrac{\pi}{2}\right]$, when f is minimum, $\cos^2\theta$ is minimum and this implies θ is maximum. Hence the soccer player should shoot when $s=\sqrt{6}$.

8	**ACJC Prelim 9758/2023/01/Q9**

(a) The following diagram shows the graphs of two differentiable functions $y=\mathrm{f}(x)$ and $y=\mathrm{g}(x)$. At $x=c$, the distance between the two functions is a maximum. Show that $\mathrm{f}'(c)=\mathrm{g}'(c)$ where $\mathrm{f}'(x)$ and $\mathrm{g}'(x)$ are derivatives of $\mathrm{f}(x)$ and $\mathrm{g}(x)$ respectively. Using the second derivative, explain clearly why the distance is a maximum. [3]



(b)	Two runners Andy and Ben, with different strategies and abilities, participate in a 10000 m race. The distance in metres ran in x minutes by Andy and Ben are given by the functions A(x) and B(x) respectively as defined below. $$\mathrm{A}(x)=\begin{cases}-0.16x^3+12x^2, & 0\le x\le 50\\ 10000, & x>50\end{cases}.$$ $$\mathrm{B}(x)=5000\log_3(x+9)-10000,\ \ 0\le x\le 72\ .$$ **(i)** Sketch the graphs of $y=\mathrm{A}(x)$ and $y=\mathrm{B}(x)$ on the same diagram for $0\le x\le 72$. [2] **(ii)** Andy and Ben are furthest apart at $x=c$. By using **(a)** or otherwise, show that $x=c$ satisfies the equation $Px^2+Qx=\dfrac{R}{x+9}$, where P, Q and R are constants to be determined, and find the value(s) of c. [4] **(iii)** Hence or otherwise, find the furthest distance between Andy and Ben and state who was leading at that moment. [2]
Solution:	
8(a)	Let $D=\mathrm{f}(x)-\mathrm{g}(x)$. $\dfrac{\mathrm{d}D}{\mathrm{d}x}=\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{f}(x)-\mathrm{g}(x)\big)=\mathrm{f}'(x)-\mathrm{g}'(x)$ At $x=c$, distance is maximum, i.e. $\left.\dfrac{\mathrm{d}D}{\mathrm{d}x}\right\|_{x=c}=0\Rightarrow \mathrm{f}'(c)-\mathrm{g}'(c)=0\Rightarrow \mathrm{f}'(c)=\mathrm{g}'(c)$ (shown) From the diagram, at $x=c$, $y=\mathrm{f}(x)$ is concave downwards, i.e. $\mathrm{f}''(c)<0$ $y=\mathrm{g}(x)$ is concave upwards, i.e. $\mathrm{g}''(c)>0$ Therefore at $x=c$, $\dfrac{\mathrm{d}^2D}{\mathrm{d}x^2}=\mathrm{f}''(c)-\mathrm{g}''(c)<0$. Hence the distance is a maximum.
(b) (i)	

(b) (ii)	From (a), we know that the maximum will occur when $\text{A}'(c) = \text{B}'(c)$, and c will satisfy the equation $\text{A}'(x) = \text{B}'(x)$. $\frac{\text{d}}{\text{d}x}\left(-0.16x^3 + 12x^2\right) = \frac{\text{d}}{\text{d}x}\left(5000\log_3(x+9) - 10\,000\right)$ $-0.48x^2 + 24x = \frac{5000}{(x+9)\ln 3}$ Therefore $P = -0.48, Q = 24, R = \frac{5000}{\ln 3}$. Using GC to solve the equation gives $x = 11.906\ (5\text{ s.f.}) = 11.9\ (3\text{ s.f.})$ $x = 46.296\ (5\text{ s.f.}) = 46.3\ (3\text{ s.f.})$
(b) (iii)	$\text{A}(11.906) - \text{B}(11.906) = -2404.8$ $\text{A}(46.296) - \text{B}(46.296) = 1580.9$ Therefore the furthest distance between them is 2404.8m (or 2400m) and Ben is in front of Andy.