Revision – CRV, Uniform Distribution, Exponential Distributions (21 marks, 40mins)

1. The duration, T minutes, of telephone calls to a particular destination can be modelled by the probability density function $f(t)=\begin{cases}\alpha e^{-\alpha t} & t\geq 0,\\ 0 & \text{otherwise,}\end{cases}$ where α is a positive constant.

A particular company charges each call at the rate of 15 cents per minute or **part** of a minute. Let the cost of such a call be X cents. Show that

$$P(X=15r)=e^{-r\alpha}(e^{\alpha}-1),\qquad r=1,2,3,\ldots$$

and that the mean cost per call in cents is $\dfrac{15}{1-e^{-\alpha}}$. **[6]**

[You may assume that $\sum_{r=1}^{\infty} rx^r=\dfrac{x}{(1-x)^2}$, $|x|<1$.]

When a caller telephones the particular company, there is a probability of $\frac{1}{2}$ that he will be asked to hold the line. When he is asked to hold the line and decides to do so, the total time taken for the call has the above exponential distribution with $\alpha=\frac{1}{6}$; if he decides not to do he incurs a cost of 15 cents for the wasted call. When he is not asked to hold the line the total time for the call has the exponential distribution with $\alpha=\frac{1}{2}$.

Calculate the expected cost if he rings the company and is asked to hold the line and he

(i) holds and complete the call. **[1]**

(ii) hangs up and calls again later, and he completes the second call regardless of whether he is asked to hold the line or not. **[3]**

2. The random variable Θ follows a uniform distribution on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Find the probability density function of the random variable $R=A\sin\Theta$, where A is a positive constant. **[5]**

3. The continuous random variable X has probability density function f given by

$$f(x)=\begin{cases}\frac{1}{2}k & -\pi\leq x\leq -\frac{\pi}{2},\\ k & -\frac{\pi}{2}<x<\frac{\pi}{2},\\ \frac{1}{2}k & \frac{\pi}{2}\leq x\leq \pi,\\ 0 & \text{otherwise.}\end{cases}$$

Show that $k=\dfrac{2}{3\pi}$. **[2]**

The random variable Y is defined by $Y=\tan X$.

Show that $P(|Y|<y)=\dfrac{2}{\pi}\tan^{-1}y$, for $y>0$. **[4]**

Q1 Exponential Distribution

$$
\begin{aligned}
\mathrm{P}\left(X=15r\right) &= \mathrm{P}\left(r-1<T\le r\right) \\
&= \int_{r-1}^{r} \alpha \mathrm{e}^{-\alpha t}\ \mathrm{d}t \\
&= \left[-\mathrm{e}^{-\alpha t}\right]_{r-1}^{r} \\
&= \mathrm{e}^{-(r-1)\alpha} - \mathrm{e}^{-r\alpha} \\
&= \mathrm{e}^{-r\alpha}(\mathrm{e}^{\alpha}-1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{r=1}^{\infty} 15r\mathrm{e}^{-r\alpha}(\mathrm{e}^{\alpha}-1) \\
&= 15(\mathrm{e}^{\alpha}-1)\sum_{r=1}^{\infty} r\mathrm{e}^{-r\alpha} \\
&= 15(\mathrm{e}^{\alpha}-1)\sum_{r=1}^{\infty} r\left(\mathrm{e}^{-\alpha}\right)^{r} \\
&= 15(\mathrm{e}^{\alpha}-1)\frac{\mathrm{e}^{-\alpha}}{(1-\mathrm{e}^{-\alpha})^2} \\
&= 15\frac{1-\mathrm{e}^{-\alpha}}{(1-\mathrm{e}^{-\alpha})^2} \\
&= \frac{15}{1-\mathrm{e}^{-\alpha}} \quad \text{(shown)}
\end{aligned}
$$

(i) $\mathrm{E}(X) = \dfrac{15}{1-\mathrm{e}^{-\frac{1}{6}}} = 97.708 = 97.7 \text{ cents}$

(ii) On the second call, if he is asked to hold the line, $\mathrm{E}(X) = 97.708 \text{ cents}$

if he is not asked to hold the line, $\mathrm{E}(X) = \dfrac{15}{1-\mathrm{e}^{-\frac{1}{2}}}$ = 38.122 cents

Hence expected cost = 15 + $\frac{1}{2}\left(97.708\right) + \frac{1}{2}\left(38.122\right)$ = 82.915 = 82.9 cents

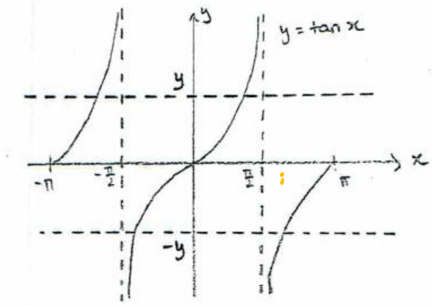
Q2 Uniform

1	Since $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, thus $R \in \left(-A, A\right)$. $F_{\Theta}(\theta) = \dfrac{\theta + \frac{\pi}{2}}{\pi}$. For $-A < r < A$,	B1: obtain cdf of Θ B1: Support of R

	$P(R \le r) = P(A\sin\theta \le r)$ $= P\left(\sin\theta \le \frac{r}{A}\right)$ $= P\left(\theta \le \sin^{-1}\left(\frac{r}{A}\right)\right)$ $= \frac{1}{\pi}\left[\sin^{-1}\left(\frac{r}{A}\right) + \frac{\pi}{2}\right]$ $f_R(r) = \frac{d}{dr}[F_R(r)]$ $= \frac{1}{\pi}\frac{d}{dr}\left[\sin^{-1}\left(\frac{r}{A}\right) + \frac{\pi}{2}\right]$ $= \frac{1}{\pi\sqrt{A^2 - r^2}}$ Thus $f_R(r) = \begin{cases} \frac{1}{\pi\sqrt{A^2 - r^2}}, & -A < r < A, \\ 0, & \text{otherwise.} \end{cases}$	M1: Rewrite inequality, making θ the subject M1: Obtain cdf of R A1: Differentiate to get pdf of R

Q3 CRV (2003/VJC/FM/Prelim)

By considering area of rectangles,



$$P(|Y| < y) = P(-y < \tan X < y)$$

$$= P(-\pi < X < -\pi + \tan^{-1} y) + P(-\tan^{-1} y < X < \tan^{-1} y)$$

$$+ P(\pi - \tan^{-1} y < X < \pi)$$

$$= \frac{1}{3\pi}\left[\tan^{-1} y\right] + \frac{2}{3\pi}\left[2\tan^{-1} y\right] + \frac{1}{3\pi}\left[\tan^{-1} y\right]$$

$$= \frac{2}{\pi}\tan^{-1} y$$