2021 A-Level H3 Physics Suggested Solutions

1 **(a)**

$s = ut + \frac{1}{2}at^2$

$100 = \frac{1}{2}(9.81)t^2$ [1]

$t = 4.5152 \text{ s} \approx 4.5 \text{ s}$ (shown)

(b) **(i)**

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(9.81)\left[(2)(4.5152)\right]^2$ [1]

$= 400 \text{ m}$

(ii) Objects can be launched upwards from the bottom (with a speed of 44.3 m s^{-1}) instead of being dropped at the top from rest. [1]

(c) **(i)**

$v^2 = u^2 + 2as$

Before safety net,

$v^2 = 0 + 2(9.81)x_1$ [1]

During safety net,

$0 = v^2 - 2\left[(4)(9.81)\right]x_2$

$2(9.81)x_1 - 2\left[(4)(9.81)\right]x_2 = 0$ [1]

$x_1 - 4x_2 = 0 \quad \cdots\cdots\cdots\cdots \quad (1)$

$x_1 + x_2 = 100 \quad \cdots\cdots\cdots\cdots \quad (2)$ [1]

Solving simultaneous equations,

$x_1 = 80$

$\therefore 80 = \frac{1}{2}(9.81)t^2$ [1]

$t = 4.0386 \text{ s} \approx 4.0 \text{ s}$

4.0 s is about 0.5 s less than the previous time of 4.5 s. (shown)

(ii) From previous part's simultaneous equations,

$x_2 = 20 \text{ m}$ [1]

2 **(a)** By conservation of momentum,

$$4mv = 4mv_1 + mv_2$$

$$4v = 4v_1 + v_2 \quad \cdots\cdots\cdots\cdots \quad (1)$$ [1]

Since collision is elastic,

$$v = v_2 - v_1 \quad \cdots\cdots\cdots\cdots \quad (2)$$

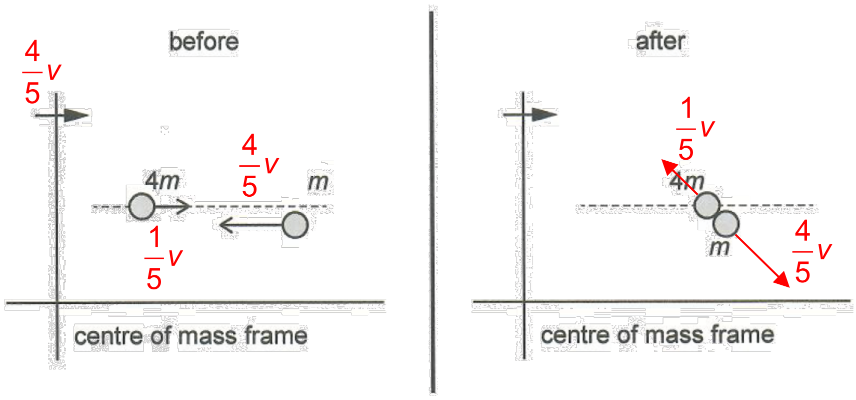
Solving simultaneous equations,

$$v_2 = \frac{8}{5}v$$ [1]

Fraction of KE transferred

$$= \frac{\frac{1}{2}m\left(\frac{8}{5}v\right)^2}{\frac{1}{2}(4m)v^2} = 0.64 \equiv 64\%$$ [1]

(b) **(i)** [3]



$$4mv = 5mv_C$$

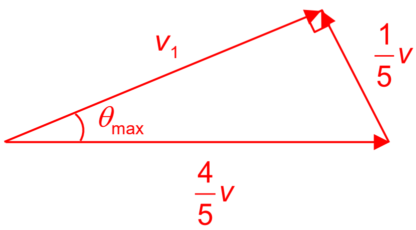
$$v_C = \frac{4}{5}v$$

$$u_{4m,C} = v - \frac{4}{5}v = \frac{1}{5}v$$

$$u_{m,C} = 0 - \frac{4}{5}v = -\frac{4}{5}v$$

The magnitudes of the velocities of each mass in the COM frame do not change before and after the collision, as dictated by kinetic energy conservation (elastic collision).

(ii)



The maximum angle is when the COM velocity, the velocity of the mass in the COM frame, and the velocity of the mass in the lab frame form a right-angled triangle as shown. [1]

$$\theta_{max} = \sin^{-1}\left(\frac{\frac{1}{5}v}{\frac{4}{5}v}\right) = \sin^{-1}\left(\frac{1}{4}\right) = 14.5^\circ \text{ (3 s.f.)}$$ [2]

3. (a)

KE of mass m is $= \frac{1}{2}mv^2 = \frac{1}{2}\frac{p^2}{m}$

Since angular momentum $L = pr \Rightarrow p^2 = \frac{L^2}{r^2}$ [1]

$\therefore \text{KE} = \frac{1}{2}\frac{p^2}{m} = \frac{L^2}{2mr^2}$ [1]

The total energy of the system is $E_T = \text{GPE} + \text{KE} = -\frac{GMm}{r} + \frac{L^2}{2mr^2}$ [1]

(b) (i) For a body moving in an elliptical orbit, its total energy is given by

$$E_T = \frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right) - \frac{GMm}{r}$$

$$\therefore \quad \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 = E_T - \frac{1}{2}m\left(r\frac{d\theta}{dt}\right)^2 + \frac{GMm}{r}$$

$$= E_T - \frac{L^2}{2mr^2} + \frac{GMm}{r} = E_T - \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right) = E - U_{\text{eff}}$$ [1]

At the apses, $\frac{dr}{dt} = 0$,

furthermore, both total energy E_T and angular momentum L are constant

$$\therefore E_T = -\frac{GMm}{r_p} + \frac{L^2}{2mr_p^2} = -\frac{GMm}{r_a} + \frac{L^2}{2mr_a^2}$$ [1]

$$\therefore \frac{L^2}{2mr_p^2} - \frac{L^2}{2mr_a^2} = \frac{GMm}{r_p} - \frac{GMm}{r_a} \Rightarrow \frac{L^2}{2m}\left(\frac{1}{r_p^2} - \frac{1}{r_a^2}\right) = GMm\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$\Rightarrow \frac{L^2}{2m}\left(\frac{1}{r_p} - \frac{1}{r_a}\right)\left(\frac{1}{r_p} + \frac{1}{r_a}\right) = GMm\left(\frac{1}{r_p} - \frac{1}{r_a}\right) \Rightarrow \frac{L^2}{2m} = \frac{GMm}{\left(\frac{1}{r_p} + \frac{1}{r_a}\right)} = \frac{GMmr_pr_a}{r_a + r_p}$$ [1]

$$\therefore \text{The total energy } E_T = -\frac{GMm}{r_p} + \frac{L^2}{2mr_p^2} = -\frac{GMm}{r_p} + \frac{GMmr_pr_a}{(r_a + r_p)r_p^2}$$

$$= -\frac{GMm}{r_p}\left(1 - \frac{r_a}{(r_a + r_p)}\right) = -\frac{GMm}{r_p}\left(\frac{r_p}{(r_a + r_p)}\right)$$

$$= -\frac{GMm}{(r_a + r_p)} = -\frac{GMm}{2a}$$ [1]

(ii) Total energy of the system is $E_T = \text{GPE} + \text{KE}$

$$\therefore -\frac{GMm}{r} + \frac{1}{2}mv^2 = -\frac{GMm}{2a}$$ [1]

$$\therefore v = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{2a}\right)}$$

$$= \sqrt{2 \times 6.67 \times 10^{-11} \times 5.4 \times 10^{32}\left(\frac{1}{2.8 \times 10^{12}} - \frac{1}{2 \times 3.8 \times 10^{12}}\right)}$$ [1]

$$= 130 \text{ km s}^{-1}$$ [1]

4 **(a)**

$$B = \frac{\mu_o NI}{L} = k\frac{Vd}{\rho l}$$

$$k = \frac{\rho l}{Vd} \times \frac{\mu_o NI}{L}$$ [1]

Substitute $L = Nd$

and $I = \frac{V}{R} = \frac{VA}{\rho l} = \frac{V\pi d^2}{4\rho l}$ into the above equation: [1]

$$k = \frac{\rho l}{Vd} \times \frac{\mu_o}{d} \times \frac{V\pi d^2}{4\rho l}$$ [1]

$$= \frac{\mu_o \pi}{4}$$ [1]

$$= 9.87 \times 10^{-7}$$ [1]

The unit of k is the same as that of μ_o, which is T m A^{-1}.

(b) **(i)**

$$B = k\frac{Vd}{\rho l}$$

$$= 9.87 \times 10^{-7} \times \frac{3.0 \times 0.25 \times 10^{-3}}{1.7 \times 10^{-8} \times 2.9}$$ [1]

$$= \mathbf{0.015\ T}$$ [1]

(ii) The formula $B = \frac{\mu_o NI}{L}$ is obtained by assuming that the solenoid is infinitely long and hence has infinite number of turns. Real solenoids are of finite length and finite number of turns. Fewer turns mean weaker magnetic field. [1]

5 **(a)** Heating a gas at constant volume:
Using the first law of thermodynamics,

$$\Delta U = W + Q = p\Delta V + Q$$

$$\therefore Q = \Delta U, \text{ since } \Delta V = 0$$ [1]

$$C_V = \frac{Q}{\Delta T} = \frac{\Delta U}{\Delta T} = \frac{\frac{3}{2}Nk\Delta T}{\Delta T} = \frac{3}{2}Nk$$ [1]

Heating a gas at constant pressure:
Using the first law of thermodynamics,

$$\Delta U = W + Q$$

$$\frac{3}{2}Nk\Delta T = -p\Delta V + C_P\Delta T$$ since the gas expands, W has a negative value. [1]

$$\frac{3}{2}Nk = -\frac{p\Delta V}{\Delta T} + C_P$$

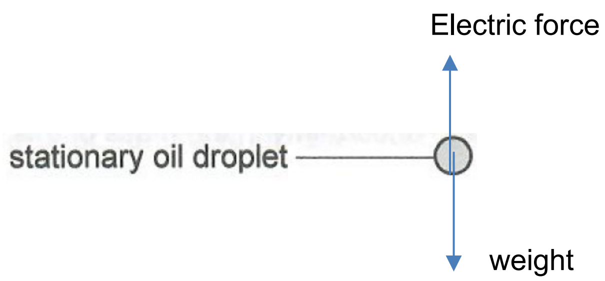
From ideal gas law, $pV = NkT$ [1]

$$\therefore p\Delta V = Nk\Delta T \Rightarrow \frac{p\Delta V}{\Delta T} = Nk$$

$$\therefore C_P = Nk + \frac{3}{2}Nk = \frac{5}{2}Nk$$ [1]

(c) In the first scenario, the heat supplied to the gas only increases its internal energy and there is no work done by the gas.
However, in the second scenario, the heat supplied increases its internal energy by the same amount (for the same increase in temperature as before) and enables the gas to do work against the external pressure as well. Hence the heat supplied in the second scenario is greater and this explains why C_p is greater than C_V. [1]

6 **(a)** **(i)**



[1]

(ii) As the electric force is upwards, which is opposite in direction as the downward electric field, the oil droplet has a negative charge. Hence, it gained electrons. [1]

(iii)

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi\left(\frac{D}{2}\right)^3$$

$$= \frac{1}{6}\pi D^3$$

$$\therefore m = \rho V$$

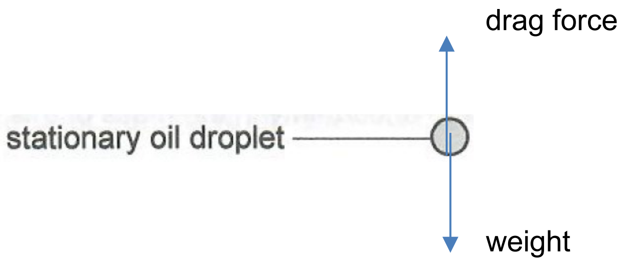
$$= \frac{1}{6}\pi\rho D^3 \quad \text{(shown)}$$ [1]

(iv)

$$m = \frac{1}{6}\pi\rho D^3$$
$$= \frac{1}{6}\pi(960)\left(0.5\times10^{-6}\right)^3$$
$$= 6.2831\times10^{-17}\text{ kg}$$ [1]

$$\frac{\Delta m}{m} = \frac{\Delta\rho}{\rho} + 3\left(\frac{\Delta D}{D}\right)$$
$$\Delta m = \left(\frac{\Delta\rho}{\rho} + 3\left(\frac{\Delta D}{D}\right)\right)m$$
$$= \left(\frac{1}{100} + 3\left(\frac{0.1}{0.5}\right)\right)\left(6.2831\times10^{-17}\right)$$
$$= 3.8327\times10^{-17}$$
$$= 4\times10^{-17}\text{ kg (to 1 s.f.)}$$ [1]
$$\therefore m = (6\pm4)\times10^{-17}\text{ kg}$$ [1]

(b) (i)



[1]

(ii) At terminal velocity, drag force = weight

$$F_D = 3\pi\eta D v = mg$$
$$v = \frac{F_D}{3\pi\eta D}$$
$$B = \frac{\rho g D^2}{\eta v}$$
$$= \frac{\rho g D^2}{\eta}\left(\frac{3\pi\eta D}{F_D}\right)$$
$$= \frac{3\pi\eta\rho g D^3}{\eta m g}$$
$$= \frac{3\pi\eta\rho g D^3}{\eta g\left(\frac{1}{6}\pi\rho D^3\right)}$$
$$= 18$$ [1]

(iii) In equilibrium, weight = electric force

$$W = F_E$$

$$mg = neE \quad \text{(where n is the number of electrons)} \qquad [1]$$

$$= ne\left(\frac{V}{d}\right) \quad \text{(where d is the distance between plates)}$$

$$neV = mgd$$

$$k = \frac{neV}{v^{\frac{3}{2}}}$$

$$= \frac{neV}{\left(\frac{\rho g D^2}{\eta B}\right)^{\frac{3}{2}}}$$

$$= \frac{neV(\eta B)^{\frac{3}{2}}}{\left(\rho g D^2\right)^{\frac{3}{2}}}$$

$$= \frac{mgd(\eta B)^{\frac{3}{2}}}{\left(\rho g D^2\right)^{\frac{3}{2}}}$$

$$= \frac{\pi \rho D^3 g d(\eta B)^{\frac{3}{2}}}{6\left(\rho g D^2\right)^{\frac{3}{2}}}$$

$$= \frac{\pi d(\eta B)^{\frac{3}{2}}}{6(\rho g)^{\frac{1}{2}}} \qquad [1]$$

$$= \frac{\pi\left(6.0\times10^{-3}\right)\left(1.8\times10^{-5}\right)^{\frac{3}{2}}(18)^{\frac{3}{2}}}{6(960)^{\frac{1}{2}}(9.81)^{\frac{1}{2}}} \qquad [1]$$

$$= 1.8879\times10^{-10}\ \text{kg m}^{\frac{1}{2}}\ \text{s}^{-\frac{1}{2}}$$

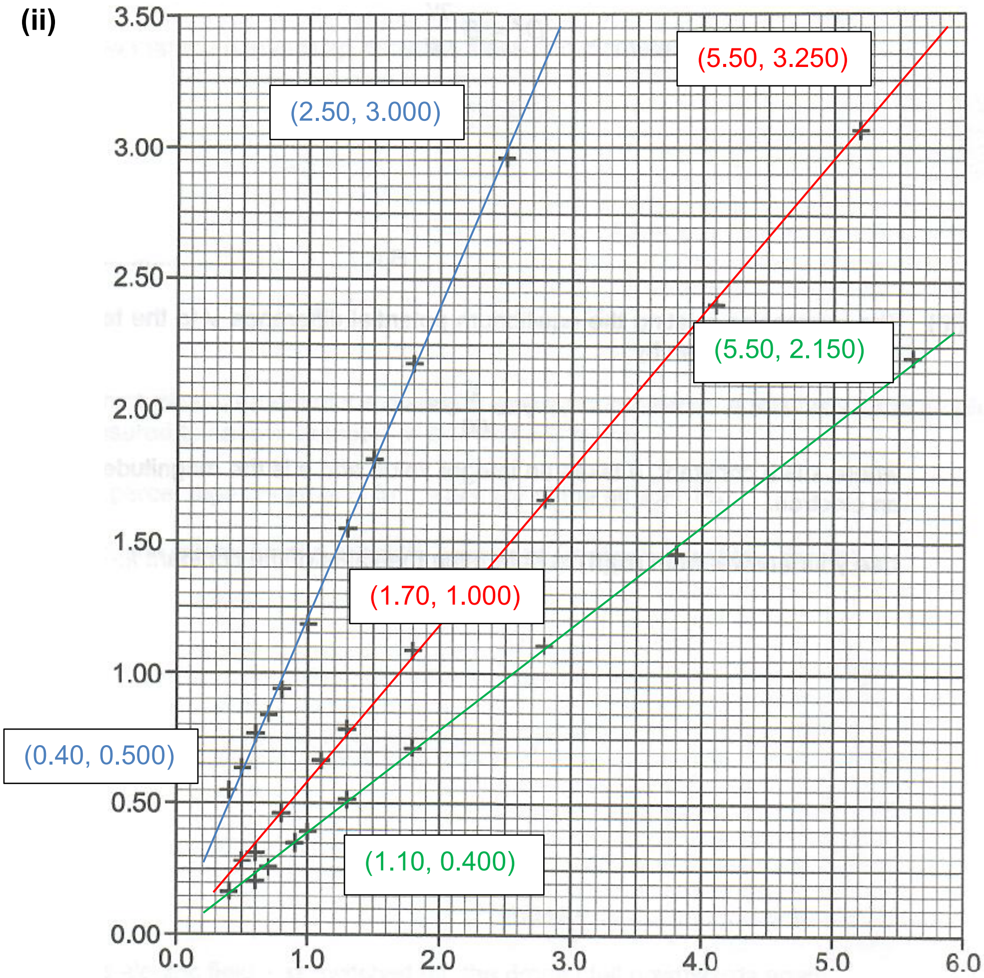
$$= 1.89\times10^{-10}\ \text{kg m}^{\frac{1}{2}}\ \text{s}^{-\frac{1}{2}}\ \text{(to 3 s.f.)}$$

(c) (i) Since $V = \frac{k}{ne}v^{\frac{3}{2}}$, when a graph of V is plotted against $v^{1.5}$, the gradient of the graph is $\frac{k}{ne}$, [1]

which varies with n, the number of electrons in the oil droplet. [1]

Millikan analyzed several droplets with different number of electrons n in each, hence only for data with the same number of n will yield a linear fit, and not a linear fit for all the data. [1]

(ii)



[1 mark] for drawing the three lines of best-fit

For the top line,

$$\text{Gradient} = \frac{3.000 - 0.500}{2.50 - 0.40}\left(\frac{10^3}{10^{-6}}\right)$$

$$= \frac{2.500}{2.10}\left(\frac{10^3}{10^{-6}}\right)$$

$$= 1.19 \times 10^9$$

For the middle line,

$$\text{Gradient} = \frac{3.250 - 1.000}{5.50 - 1.70}\left(\frac{10^3}{10^{-6}}\right)$$

$$= \frac{2.250}{3.80}\left(\frac{10^3}{10^{-6}}\right)$$

$$= 0.592 \times 10^9$$

For the bottom line,

$$\text{Gradient} = \frac{2.150 - 0.400}{5.50 - 1.10}\left(\frac{10^3}{10^{-6}}\right)$$
$$= \frac{1.750}{4.40}\left(\frac{10^3}{10^{-6}}\right)$$
$$= 0.398 \times 10^9$$

[1 mark] for calculating the values of the three gradients.

$$\frac{\text{Gradient of top line}}{\text{Gradient of middle line}} = \frac{1.19 \times 10^9}{0.592 \times 10^9}$$
$$= 2.0101$$
$$\approx 2$$

$$\frac{\text{Gradient of top line}}{\text{Gradient of bottom line}} = \frac{1.19 \times 10^9}{0.398 \times 10^9}$$
$$= 2.9899$$
$$\approx 3$$

The top line has the highest gradient; hence it corresponds to the smallest value of n, which is 1 (ie., these oil droplets have gained one electron each).

For top line, n = 1
For middle line, n = 2
For bottom line, n = 3
[1 mark] for deducing/stating the three values of n

Using the top line,

$$\text{Gradient} = \frac{k}{ne}$$
$$e = \frac{k}{n(\text{gradient})}$$
$$= \frac{1.8879 \times 10^{-10}}{(1)(1.19 \times 10^9)}$$
$$= 1.5865 \times 10^{-19} \text{ C}$$

Using the middle line,

$$\text{Gradient} = \frac{k}{ne}$$
$$e = \frac{k}{n(\text{gradient})}$$
$$= \frac{1.8879 \times 10^{-10}}{2(0.592 \times 10^9)}$$
$$= 1.5945 \times 10^{-19} \text{ C}$$

Using the bottom line,

$$\text{Gradient} = \frac{k}{ne}$$

$$e = \frac{k}{n(\text{gradient})}$$

$$= \frac{1.8879 \times 10^{-10}}{3(0.398 \times 10^{9})}$$

$$= 1.5811 \times 10^{-19} \text{ C}$$

The mean value of e is =

$$\langle e \rangle = \frac{(1.5865 + 1.5945 + 1.5811) \times 10^{-19}}{3}$$

$$= 1.5874 \times 10^{-19} \text{ C}$$

The uncertainty in e is =

$$\Delta e = \frac{e_{max} - e_{min}}{2}$$

$$= \frac{(1.5945 - 1.5811) \times 10^{-19}}{2}$$ [1]

$$= 7 \times 10^{-22} \text{ C} \quad \text{(to 1 s.f.)}$$

[1]

$$\therefore e = (1.587 \pm 0.007) \times 10^{-19} \text{ C}$$

Section B

7. **(a)** **(i)** [1]

$$g = \frac{GM}{r^2}$$

$$\therefore M = \frac{gr^2}{G} = \frac{1.62 \times (1.74 \times 10^{6})}{6.67 \times 10^{-11}}$$ [1]

$$= 7.35 \times 10^{22} \text{ kg}$$ [1]

(ii) Letting:
L = angular momentum of Moon about Earth,
I = moment of inertia of the system and
ω = angular frequency = 2π/period,

$$L = I\omega = mr^2\omega$$ [1]

$$= 7.3534 \times 10^{22} (3.84 \times 10^{8})^2 \times \frac{2\pi}{28 \times 24 \times 3600} = 2.82 \times 10^{34} \text{ kgm}^2\text{s}^{-1}$$ [1]

(b) Since the gravitational force of the Earth on the Moon provides the centripetal force for the circular motion of the Moon around the Earth,

[1]

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{GM}{r}$$ [1]

$$\text{But, } g \text{ on Earth} = \frac{GM}{R^2}$$

$$\therefore GM = gR^2$$

$$\therefore v^2 = \frac{gR^2}{r} \Rightarrow v = \sqrt{\frac{gR^2}{r}}$$

(c)

$$L_{Earth} = I\omega = \frac{2}{5}MR^2\omega$$ [1]

$$\text{Since } g = \frac{GM}{R^2}$$

$$\therefore M = \frac{gR^2}{G}$$

$$\therefore L_{Earth} = \frac{2}{5}\left(\frac{gR^2}{G}\right)R^2 \times \frac{2\pi}{T} = \frac{4\pi}{5}\frac{gR^4}{GT} \equiv k\frac{gR^\beta}{GT}$$ [1]

$$\Rightarrow k = \frac{4\pi}{5}, \quad \beta = 4$$ [1]

(d) **(i)** The gravitational force of the Moon on the water on the side of the Earth closer to it (the Moon) is stronger than that on the water on the opposite side of the Earth.

The stronger pull on the water on the closer side causes the water there to bulge out towards the Moon and this corresponds to a high tide. [1]

On the other hand, the weaker pull on the water on the side furthest away does not cause it to shift as much towards the Moon as it does on the near side, so it appears as a bulge away from the Moon. This is the second high tide. [1]

(ii)

$$\omega = \frac{2\pi}{T} = 2\pi T^{-1}$$

$$\therefore \frac{d\omega}{dT} = 2\pi(-1)T^{-2} = -\frac{2\pi}{T^2}$$ [1]

$$\text{since } \frac{\Delta\omega}{\Delta T} \approx \frac{d\omega}{dT}$$ [1]

$$\Rightarrow \Delta\omega \approx \frac{2\pi}{T^2}\Delta T \text{ or } \frac{2\pi\Delta T}{T^2}$$

(iii)

$$\Delta\omega \approx \frac{2\pi\Delta T}{T^2} = \frac{2\pi\left(\frac{4\text{ hours} \times 3600\text{ s}}{4 \times 10^6}\right)}{(22\text{ hours} \times 3600\text{ s})^2}$$ [1]

$$= 3.6 \times 10^{-12}\text{ rads}^{-1}$$ [1]

(iv) $L = I\omega \Rightarrow \dfrac{dL}{d\omega} = I$

since $\dfrac{\Delta L}{\Delta\omega} \approx \dfrac{dL}{d\omega} \Rightarrow \Delta L \approx \dfrac{dL}{d\omega}\Delta\omega = I\Delta\omega$ [1]

$\therefore \Delta L = \dfrac{2}{5}\dfrac{R^4 g}{G}\Delta\omega = \dfrac{2}{5}\dfrac{\left(6378\times10^3\right)^4 9.81}{6.67\times10^{-11}}\times 3.6065\times10^{-12}$ [1]

$= 3.5\times10^{26}\ \text{kgm}^2\text{s}^{-1}$

(v) $t = \dfrac{7.1\times10^{33}}{3.5\times10^{26}} = 2.0\times10^5$ years [1]

(vi) Since the angular deceleration α is constant over the past four million years, the angular velocity ω is related to time t by the equation

$$\omega = \omega_0 - \alpha t$$

Since $\omega = \dfrac{2\pi}{T}$,

$$\therefore \frac{2\pi}{T} = \frac{2\pi}{T_0} - \alpha t$$

$$\therefore T = \frac{1}{\dfrac{1}{T_0} - \dfrac{\alpha}{2\pi}t}$$

This means that T and t are **<u>not linearly related</u>**. Hence, it is not valid to assume that T, the rotational period of the Earth increases at a constant rate. [1]

<u>Alternative answer</u>:
For this part, can also just simply say that since ω and t are linearly related, and since $\omega = \dfrac{2\pi}{T}$, T is therefore not linearly related to t, and so it is not valid to assume that the rotational period of the Earth increases at a constant rate. [1]

8 **(a)** **(i)** A capacitor of capacitance of 320 μF will store 320 μC of charge when a potential difference of 1 V is applied across it. [2]

(ii) Applying Kirchhoff's voltage law around the circuit,

$$V_b - \frac{q}{C} - Ir = 0,$$ [1]

where q is the charge stored in the capacitor and I, the current flowing in the circuit.

Differentiating the above equation wrt to time t, we have [1]

$$-\frac{1}{C}\frac{dq}{dt} - r\frac{dI}{dt} = 0,$$

since V_b, C and r are constants.

Therefore,

$$\frac{dI}{dt} = -\frac{1}{rC}\frac{dq}{dt} \quad \text{(shown)}$$

(iii)

$$I = I_0 e^{-t/rC}$$

$$t = rC\ln\left(\frac{I_0}{I}\right), \quad \text{where } I_0 = \frac{V_b}{r}$$

$$t = (0.15)(320\times10^{-6})\ln\left(\frac{1.5/0.15}{0.10}\right)$$ [1]

$$= 2.21\times10^{-4} \text{ s}$$ [1]

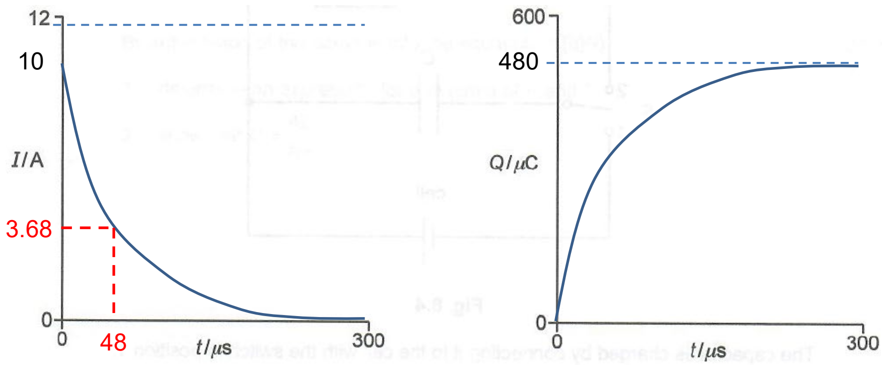
(iv)

Max initial current, $I_0 = \frac{V_b}{r} = \frac{1.5}{0.15} = 10$ A

Max final charge, $Q_0 = C\,V_b = (320\times10^{-6})(1.5) = 480\ \mu\text{C}$ [1]

Time-constant of RC circuit is $\tau = RC$,

$\tau = rC = (0.15)(320\times10^{-6}) = 48\,\mu s$, corr. to $I = (1/e)\,I_0 = 3.68$ A [1]



[1]

[1]

(v) Power dissipated in battery during charging = $I^2 r$

Hence, energy dissipated during charging = $\int_0^\infty I^2 r\,dt$ [1]

[1]

$$= \int_0^\infty I_0^2 r e^{-\frac{2t}{rC}} dt$$ [1]

$$= I_0^2 r \int_0^\infty e^{-\frac{2t}{rC}} dt$$

$$= I_0^2 r \left[-\frac{rC}{2} e^{-\frac{2t}{rC}} \right]_0^\infty$$

$$= -\frac{rC}{2} I_0^2 r [0-1]$$

$$= \frac{1}{2} I_0^2 r^2 C$$

$$= \frac{1}{2} \left(\frac{1.5}{0.15} \right)^2 (0.15)^2 (320 \times 10^{-6})$$

$$= 3.6 \times 10^{-4} \text{ J}$$

(b) (i) When the capacitor is fully charged and the switch S is turned to position 2, the initial energy stored in the capacitor is now distributed across the capacitor, inductor and resistor, such that

$$U_{\text{total}} = U_{\text{inductor}} + U_{\text{resistor}} + U_{\text{capacitor}}$$

$$\Rightarrow \frac{d}{dt} U_{\text{total}} = \frac{d}{dt} U_{\text{inductor}} + \frac{d}{dt} U_{\text{resistor}} + \frac{d}{dt} U_{\text{capacitor}}$$ [1]

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{LI^2}{2} \right) + I^2 R + \frac{d}{dt}\left(\frac{Q^2}{2C} \right)$$ [1]

$$\Rightarrow 0 = LI \frac{dI}{dt} + I^2 R + \frac{Q}{C} \frac{dQ}{dt}$$

$$\Rightarrow 0 = L \frac{dI}{dt} + IR + \frac{Q}{C} \qquad \text{(divide by } I = \frac{dQ}{dt}\text{)}$$

Divide by L and replace I with $\frac{dQ}{dt}$,

$$\Rightarrow \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0 \text{ (shown)}$$

(ii)1

$$L \frac{d^2}{dt^2}\left(Ate^{-\alpha t} \right) + R \frac{d}{dt}\left(Ate^{-\alpha t} \right) + \frac{Ate^{-\alpha t}}{C} = 0$$

$$LC \frac{d}{dt}\left(Ae^{-\alpha t} - A\alpha te^{-\alpha t} \right) + RC\left(Ae^{-\alpha t} - A\alpha te^{-\alpha t} \right) + Ate^{-\alpha t} = 0$$ [1]

$$LC\left(-A\alpha e^{-\alpha t} - A\alpha e^{-\alpha t} + A\alpha^2 te^{-\alpha t} \right) + RC\left(Ae^{-\alpha t} - A\alpha te^{-\alpha t} \right) + Ate^{-\alpha t} = 0$$

[1]

At time $t = 0$, $LC(-A\alpha - A\alpha) + RC(A) = 0$ [1]

$\Rightarrow$ $2LCA\alpha = RCA$

$\Rightarrow$ $\alpha = \dfrac{R}{2L}$

(ii)2

Continuing from above,

$$LC\left(-2A\alpha e^{-\alpha t} + A\alpha^2 te^{-\alpha t}\right) + RC\left(Ae^{-\alpha t} - A\alpha te^{-\alpha t}\right) + Ate^{-\alpha t} = 0$$

Dividing throughout by $LCAe^{-\alpha t}$,

$$\left(-2\alpha + \alpha^2 t\right) + \frac{R}{L}(1 - \alpha t) + \frac{t}{LC} = 0$$ [1]

Subbing $\alpha = \dfrac{R}{2L}$,

$$\left(-\frac{R}{L} + \frac{R^2 t}{4L^2}\right) + \frac{R}{L}\left(1 - \frac{Rt}{2L}\right) + \frac{t}{LC} = 0 \rightarrow \frac{R^2 t}{4L^2} - \frac{R^2 t}{2L^2} + \frac{t}{LC} = 0$$

$$\therefore \frac{R^2}{4L^2} = \frac{1}{LC} \Rightarrow C = \frac{4L}{R^2}$$ [1]

9 (a) (i) In the centre of mass frame, total momentum is zero.

$$u \times (-dm) = (m + dm) \times dv$$ [1]

$$-udm = mdv + dmdv$$

Since $dmdv \approx 0$, [1]

$$dv \approx -\frac{u}{m}dm$$

(ii) Integrating,

$$\Delta v = -\int_{m_0}^{m_f} \frac{u}{m}dm$$ [1]

$$= -u\left[\ln m\right]_{m_0}^{m_f}$$ [1]

$$= -u\ln\left(\frac{m_f}{m_0}\right)$$

$$= u\ln\left(\frac{m_0}{m_f}\right)$$ [1]

(b) (i) Rearranging,

$$\frac{m_f}{m_0} = \exp\left(-\frac{\Delta v}{u}\right) = \exp\left(-\frac{9700}{4500}\right) = 0.116$$ [1]

Therefore, the percentage fuel is 88.4%. [1]

(ii) First stage:

$$\frac{m_1}{m_0} = \exp\left(-\frac{\Delta v_1}{u}\right) = \exp\left(-\frac{5000}{4500}\right) = 0.329$$ [1]

After the first stage is jettisoned, mass of remaining rocket

$$m_2 = m_1 - 0.08m_0 = 0.329m_0 - 0.08m_0 = 0.249m_0$$

Second stage:

$$\frac{m_f}{m_2} = \exp\left(-\frac{\Delta v_2}{u}\right) = \exp\left(-\frac{4700}{4500}\right) = 0.352$$ [1]

So fuel burnt

$$\begin{aligned} m_{fuel} &= m_0 - m_1 + m_2 - m_f \\ &= m_0 - 0.329m_0 + 0.249m_0 - 0.352 \times 0.249m_0 = 0.832m_0 \end{aligned}$$ [1]

Percentage fuel is 83.2%.

(iii) A lower amount of fuel is required for any given load. [1]
or
For the same amount of fuel, a greater load can be delivered to space.

(i) Average thrust

$$\langle F \rangle = u\frac{dm}{dt}$$ [1]

Exhaust velocity

$$u = \langle F \rangle \div \frac{dm}{dt} = 35.1 \times 10^6 \div \frac{2.16 \times 10^6}{152} = 2470 \text{ m s}^{-1}$$ [1]

(ii) Change in velocity

$$\Delta v = u \ln\left(\frac{m_0}{m_f}\right) = 2470 \ln\left(\frac{2.97}{2.97 - 2.16}\right) = 3200 \text{ m s}^{-1}$$ [1]

(iii) The rocket passed through a relatively thin layer of atmosphere compared to the distance it would travel during the initial 152 s. [1]
The speed of the rocket as it passed through the atmosphere is relatively low, compared to its final speed. [1]

(iv) Ratio of final speeds

$$\frac{v}{v'} = \frac{\langle F \rangle - mg'}{\langle F \rangle} = 1 - \frac{mg'}{\langle F \rangle}$$ [1]

Rearranging,

$$g' = \frac{\langle F \rangle}{m}\left[1 - \frac{v}{v'}\right]$$
$$= \frac{35.1 \times 10^6}{2.97 \times 10^6}\left[1 - \frac{2300}{3200}\right] = 3.32 \text{ m s}^{-2}$$ [1]

Note: You may use the average mass of rocket + fuel in your calculation.

(d) **(i)** The lower value of average gravitational acceleration accounts for the gradually decreasing mass of the rocket and fuel. [1]

(ii) Due to Earth's rotation about its axis, rockets launched from the equator has the maximum initial velocity (v=$r\omega$) towards the East. Hence launching rockets close to equator reduces the fuel required for the mission. [1]