# Anglo-Chinese School

(Independent)

## FINAL EXAMINATION 2023
## YEAR 5 IB DIPLOMA PROGRAMME

**MATHEMATICS : ANALYSIS AND APPROACHES**

**HIGHER LEVEL**

**PAPER 2**

Monday 25 September 2023

2 hours

**Candidate Session Number**

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| 0 | 0 | 2 | 3 | 2 | 9 | | | | |

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**INSTRUCTIONS TO CANDIDATES**

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions on the writing paper provided. Fill in your session number on each answer sheet, and attach them to this examination paper using the string provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics Analysis and Approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**

| **Section A (55 Marks)** | | **Section B (55 Marks)** | |
|---|---|---|---|
| **Question** | **Marks** | **Question** | **Marks** |
| **1** | | **9** | |
| **2** | | | |
| **3** | | | |
| **4** | | **10** | |
| **5** | | | |
| **6** | | | |
| **7** | | **11** | |
| **8** | | | |
| **Subtotal** | | **Subtotal** | |
| **TOTAL** | **/ 110** | | |

**This question paper consists of 11 printed pages including this cover page.**

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 7]

Consider the geometric sequence 2,8,32,…

(a) Write down the common ratio of the sequence. [1]

(b) Find the largest term of the sequence that has only four digits. [3]

(c) The sum of $n$ terms of this geometric series is less than 100000. Find the largest possible value of $n$. [3]

**2.** [Maximum mark: 7]

(a) Prove that $\log_{\sqrt{a}} m = \log_a m^2$. [3]

(b) Hence, solve the equation $\log_{\sqrt{2}} x + 3\log_x 2 = 5$, giving your answer in exact form. [4]

**3.** [Maximum mark: 8]

A quadratic function $f(x) = a(x - \lambda)(x - 2)$ has axis of symmetry $x = 4.5$ and $y$-intercept at $(0, -4)$.

(a) Find the value of $\lambda$. [2]

(b) Find the value of $a$. [2]

(c) Find the possible values of $k$, for which the line $y = kx + 9$ is a tangent to the graph of $y = f(x)$. [4]

**4.** [Maximum mark: 4]

The equation $2x^2 - 3x - 6 = 0$ has roots $\alpha$ and $\beta$. The equation $x^2 + px + q = 0$ has roots $2\alpha$ and $2\beta$. Find $p$ and $q$.

**5.** [Maximum mark: 4]

Prove that $\arcsin\left(\frac{1}{x}\right) \equiv \operatorname{arccosec}(x)$. Hence or otherwise, solve $\sin\left(\operatorname{arccosec}(2y)\right) = e^{-1}$, leaving your answer in terms of $e$.

**6.** [Maximum mark: 9]

Consider the following functions

$$f(t) = 2\sin(4t) + t \qquad 0 \le t \le 2\pi$$
$$g(t) = t + k, \qquad k \in \mathbb{R}$$

(a) For $k = -2$, find the value(s) of $t$, where $f(t) = g(t),\ 0 \le t \le 2\pi$. [4]

(b) Determine the range of $k$ such that the graph of $y = g(t)$ intersects the graph of $y = f(t)$. [2]

(c) Find the value of $k$ where the graph of $y = g(t)$ has the most number of intersections with the graph of $y = f(t)$. State the total number of intersections. [3]

**7.** [Maximum mark: 7]

Consider a complex number $z = x + iy$, such that $\arg(z - 1 + i) = \frac{\pi}{5}$.

(a) Find the equation of the straight line formed, in terms of its gradient and its intercept on the $y$-axis. [5]

(b) The same line is rotated anti-clockwise by $60^0$ about $(1,-1)$ and results in an equation $\arg(z - 1 + i) = \beta$. Find $\beta$. [2]

**8.** [Maximum mark: 9]

Consider two points A and B with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ and $|\boldsymbol{b}| = 10$.

(a) Determine the vector equation of a line passing through A that is perpendicular to both $\boldsymbol{a}$ and to the line $\boldsymbol{r} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$. [4]

(b) Deduce that the minimum value of $|\boldsymbol{a} + \boldsymbol{b}|$ is 7 and find the maximum value of $|\boldsymbol{a} + \boldsymbol{b}|$. [3]

(c) Given that $|\boldsymbol{a} + \boldsymbol{b}|$ is a minimum, find $\boldsymbol{b}$. [2]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer sheets provided. Please start each question on a new page.

**9.** [Maximum Mark: 25]

Consider the functions: $f(x) = 2^x + \frac{1}{2^x}, x \in \mathbb{R}.$

$$g(x) = \frac{1}{x^2 - 1},\ x \in \mathbb{R},\ x \neq 1,\ x \neq -1.$$

(a) Show that $f$ is an even function. [2]
(b) Determine whether $f$ has an inverse, stating your reasons clearly. [2]
(c) By sketching the graphs, solve the inequality $f(x) > g(x)$. [5]
(d) Find $g \circ g$. [3]
(e) Describe the sequence of transformations that maps the graph of $y = g(x)$ to the graph of $y = \frac{1}{-2x^2 + 2} + 5$. [4]
(f) Given that the domain of $g$ is restricted to $x < -1$, find the value of $g^{-1}(2)$. [3]

A function $h$ is defined by $h(x) = \frac{x^2 + 2x}{x + 1},$ where $x \in \mathbb{R},\ x \neq -1.$

(g) Sketch the graph of $y = h(x),$ showing clearly any asymptote(s), intercept(s) and stationary point(s). [4]
(h) Given a straight line $y = m(x + 1),$ determine the range of $m$ such that there would be at least one intersection with the graph of $y = h(x)$. [2]

**10.** [Maximum Mark: 11]

Consider the complex numbers $z = 2\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$ and $w = 8\left(\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\right),$

where $k \in \mathbb{Z}^+$.

(a) Find the modulus of $zw$. [1]
(b) Find the argument of $zw$ in terms of $k$. [2]

Suppose that $zw$ is purely real,

(c) Find the least value of $k$ for $k \in \mathbb{Z}^+$. [3]
(d) Given $p^4 = 8z,$ use de Moivre's theorem to find all the values of $p$ in the form $re^{i\theta},$ where $-\pi < \theta \leq \pi$. [5]

**11.** [Maximum Mark: 19]

Consider the 3 planes

$$\begin{aligned} \prod_1 : &\quad 6x+3y-z &&= -1 \\ \prod_2 : &\quad 2x-7y+5z &&= 1 \\ \prod_3 : &\quad 12x+6y-2z &&= -10 \end{aligned}$$

(a) Show that the 3 planes do not intersect at a common point. [2]

(b) Describe the geometrical relationship of the 3 planes represented by this system of equations. [2]

(c) Find the vector equation of $L$, the line of intersection of $\prod_1$ and $\prod_2$ in the form

$$\boldsymbol{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix},\ \lambda \in \mathbb{R}.$$ [5]

(d) Find the distance between $L$ and $\prod_3$. [3]

(e) Find the coordinates of the foot of the perpendicular from the origin to $\prod_3$. [4]

(f) Point $P$ is the reflection of the origin in the plane $\prod_3$. Hence, find the coordinates of $P$. [3]

**END OF PAPER**