Topic: Algebraic manipulation, surds and indices (including exponential equations)

Question 1 [2008 EOY, Q1]	marks
Express $\dfrac{1}{3-2\sqrt{2}}$ in the form $a+b\sqrt{2}$, where *a* and *b* are real numbers. **Solution:** $\dfrac{1}{3-2\sqrt{2}}\times\dfrac{3+2\sqrt{2}}{3+2\sqrt{2}}=\dfrac{3+2\sqrt{2}}{9-8}=3+2\sqrt{2}$	[2]
Question 2 [2008 EOY, Q2]	**marks**
Evaluate $25^{-1.5}\div 32^{0.4}$ without a calculator, leaving your answer as a fraction in its simplest form. **Solution:** $25^{-1.5}\div 32^{0.4}=5^{-3}\div 2^{2}=\dfrac{1}{500}$	[2]
Question 3 [2009 EOY, Q2]	**marks**
Express $\dfrac{2\sqrt{5}-3}{2\sqrt{5}+4}+\dfrac{2}{\sqrt{3}}$ in the form $a+b\sqrt{3}+c\sqrt{5}$ where *a*, *b* and *c* are rational numbers. **Solution:** $\dfrac{2\sqrt{5}-3}{2\sqrt{5}+4}+\dfrac{2}{\sqrt{3}}=\left(\dfrac{2\sqrt{5}-3}{2\sqrt{5}+4}\times\dfrac{2\sqrt{5}-4}{2\sqrt{5}-4}\right)+\left(\dfrac{2}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}\right)$ $=\dfrac{32-14\sqrt{5}}{4}+\dfrac{2\sqrt{3}}{3}$ $=8+\dfrac{2}{3}\sqrt{3}-\dfrac{7}{2}\sqrt{5}$	[3]

Question 4 [2010 EOY, Q1]	marks
(i) Simplify $(\sqrt[3]{3}+\sqrt[3]{2})(\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4})$. **<u>Solution:</u>** $(\sqrt[3]{3}+\sqrt[3]{2})(\sqrt[3]{9}-\sqrt[3]{6}+\sqrt[3]{4})=\sqrt[3]{27}-\sqrt[3]{18}+\sqrt[3]{12}+\sqrt[3]{18}-\sqrt[3]{12}+\sqrt[3]{8}$ [M1] $=\sqrt[3]{27}+\sqrt[3]{8}$ [A1] $=5$	[2]
(ii) A triangle has an area of $(5\sqrt{6}+18)$ units² and the length of its base is $(2\sqrt{3}-3)$ unit. Find its height in the form $(a\sqrt{2}+b\sqrt{3}+c\sqrt{6}+d)$ units, where *a*, *b*, *c* and *d* are integers. **<u>Solution:</u>** Height $=\dfrac{5\sqrt{6}+18}{2\sqrt{3}-3}\times 2$ units $=\dfrac{(10\sqrt{6}+36)(2\sqrt{3}+3)}{(2\sqrt{3}-3)(2\sqrt{3}+3)}$ units $=\dfrac{20\sqrt{18}+30\sqrt{6}+72\sqrt{3}+108}{12-9}$ units $=\dfrac{60\sqrt{2}+30\sqrt{6}+72\sqrt{3}+108}{3}$ units $=20\sqrt{2}+24\sqrt{3}+10\sqrt{6}+36$ units	[3]
Question 5 [2010 EOY, Q2]	marks
Solve for *x* when $\frac{1}{2}\sqrt{2x-3}=\sqrt{\frac{3x+5}{2}}$. **<u>Solution:</u>** $\frac{1}{2}\sqrt{2x-3}=\sqrt{\frac{3x+5}{2}}\Rightarrow\frac{2x-3}{4}=\frac{3x+5}{2}$ $\Rightarrow 4x-6=12x+20$ $\Rightarrow x=-\frac{13}{4}$ However, $x=-\frac{13}{4}$ is rejected as $\sqrt{2x-3}$ will be undefined. So no solution.	[3]

Question 6 [2010 EOY, Q3]	marks
(i) Make h the subject of the formula $\frac{g}{2r}=\frac{1}{y}\sqrt{\frac{h^2-1}{2-h^2}}$. **Solution:** $$\frac{g}{2r}=\frac{1}{y}\sqrt{\frac{h^2-1}{2-h^2}} \Rightarrow \frac{g^2}{4r^2}=\frac{1}{y^2}\left(\frac{h^2-1}{2-h^2}\right)$$ $$\Rightarrow g^2y^2(2-h^2)=4r^2(h^2-1)$$ $$\Rightarrow 4r^2h^2+h^2g^2y^2=2g^2y^2+4r^2$$ $$\Rightarrow h^2=\frac{2g^2y^2+4r^2}{4r^2+g^2y^2}$$ $$\Rightarrow h=\pm\sqrt{\frac{2g^2y^2+4r^2}{4r^2+g^2y^2}}$$	[2]
(ii) Simplify $\frac{x^2-3x+2}{2x^2-x-3}\div\frac{2x^2+7x+6}{8x^2-18}\times\frac{x^2+x-2}{x^2-2x+1}$. **Solution:** $$\frac{x^2-3x+2}{2x^2-x-3}\div\frac{2x^2+7x+6}{8x^2-18}\times\frac{x^2+x-2}{x^2-2x+1}$$ $$=\frac{(x-1)(x-2)}{(2x-3)(x+1)}\times\frac{2(2x-3)(2x+3)}{(2x+3)(x+2)}\times\frac{(x+2)(x-1)}{(x-1)^2}$$ $$=\frac{2(x-2)}{x+1} \text{ or } \frac{2x-4}{x+1}$$	[3]

Question 7 [2011 EOY, Q1]	marks
Without using a calculator, simplify $\dfrac{6}{1-\sqrt{3}}+\dfrac{9}{\sqrt{3}}$. **Solution:** $\left(\dfrac{6}{1-\sqrt{3}}+\dfrac{9}{\sqrt{3}}\right)=\dfrac{6\left(\sqrt{3}\right)+9\left(1-\sqrt{3}\right)}{\sqrt{3}\left(1-\sqrt{3}\right)}$ $=\dfrac{9-3\sqrt{3}}{\sqrt{3}-3}\bullet\left[\dfrac{\sqrt{3}+3}{\sqrt{3}+3}\right]$ $=\dfrac{9\sqrt{3}+27-9-9\sqrt{3}}{3-9}$ $=\dfrac{18}{-6}=-3$ OR $\left(\dfrac{6}{1-\sqrt{3}}+\dfrac{9}{\sqrt{3}}\right)=\dfrac{6\left(1+\sqrt{3}\right)}{\left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)}+\dfrac{9\sqrt{3}}{\left(\sqrt{3}\right)^2}$ $=-3\left(1+\sqrt{3}\right)+3\sqrt{3}\quad M1$ $=-3-3\sqrt{3}+3\sqrt{3}=-3$	[3]

Question 8 [2011 EOY, Q3]	marks
Show that $(2x+y)\left(4x^2-2xy+y^2\right)=8x^3+y^3$. **Solution:** $(2x+y)\left(4x^2-2xy+y^2\right)$ $=8x^3-4x^2y+2xy^2+4yx^2-2xy^2\ +y^3$ $=8x^3+y^3\ (shown)$	[1]

(ii)	
Hence simplify $\dfrac{8x^3+y^3}{x^2-xy-6y^2}\times\left(\dfrac{x^2-3xy}{2x+y}\right)\div\sqrt{\dfrac{16x^4}{x^2+4xy+4y^2}}$.	[4]
Solution: $$\frac{(2x)^3+y^3}{(x+2y)[x-3y]}\bullet\frac{x(x-3y)}{(2x+y)}\bullet\left[\frac{(x+2y)}{4x^2}\right]$$ $$=\frac{(2x+y)(4x^2-2xy+y^2)}{(x+2y)(x-3y)}\bullet\frac{x(x-3y)}{2x+y}\bullet\frac{x+2y}{4x^2}$$ $$=\frac{4x^2-2xy+y^2}{4x}$$	
Question 9 [2012 EOY, Q2]	marks
Without using a calculator, simplify $\dfrac{3}{\left(2-\sqrt{5}\right)^2}-\dfrac{2}{\left(2+\sqrt{5}\right)^2}$.	[4]
Solution: $$\frac{3}{\left(2-\sqrt{5}\right)^2}-\frac{2}{\left(2+\sqrt{5}\right)^2}$$ $$=\frac{3\left(2+\sqrt{5}\right)^2-2\left(2-\sqrt{5}\right)^2}{\left[\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)\right]^2}$$ $$=\frac{3\left(9+4\sqrt{5}\right)-2\left(9-4\sqrt{5}\right)}{(1)}$$ $$=9+20\sqrt{5}$$	

Question 10 [2012 EOY, Q10 a, b]	marks
(a) Factorise $81a^4 - 27a^2b^2 - 4b^4$ completely **<u>Solution:</u>** $81a^4 - 27a^2b^2 - 4b^4 = (9a^2 + b^2)(9a^2 - 4b^2)$ $= (9a^2 + b^2)(3a + 2b)(3a - 2b)$	[2]
(b) Given that $2x = 3\sqrt{px^2 + q}$, express x in terms of p and q. **<u>Solution:</u>** $2x = 3\sqrt{px^2 + q}$ $4x^2 = 9px^2 + 9q$ $x^2(4 - 9p) = 9q$ $x^2 = \dfrac{9q}{4 - 9p}$ $x = \pm\sqrt{\dfrac{9q}{4 - 9p}}$	[2]

Question 11 [2013 EOY, Q1 a, b]	marks
(i) Find the product of $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{1}{3}} + y^{\frac{1}{3}})$ and $(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$ and simplify the answer. **<u>Solution:</u>** $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{1}{3}} + y^{\frac{1}{3}}) \times (x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$ $= (x - y)(x + y)$ $= x^2 - y^2$	[2]

(ii) If $y=\dfrac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}$, simplify $y+\dfrac{1}{y}$. **Solution:** $\dfrac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}+\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}$ $=\dfrac{(x+a)-2\sqrt{x+a}\sqrt{x-a}+(x-a)+(x+a)+2\sqrt{x+a}\sqrt{x-a}+(x-a)}{(x+a)-(x-a)}$ $=\dfrac{2x+2a+2x-2a}{2a}$ $=\dfrac{2x}{a}$	[3]
Question 12 [2013 EOY, Q6]	marks
(i) Simplify $\dfrac{a-b}{(b+c)(a+c)}+\dfrac{b-c}{(a+c)(a+b)}+\dfrac{c-a}{(a+b)(b+c)}$. **Solution:** $\dfrac{(a-b)(a+b)+(b-c)(b+c)+(c-a)(c+a)}{(a+b)(b+c)(a+c)}$ $=\dfrac{a^2-b^2+b^2-c^2+c^2-a^2}{(a+b)(b+c)(a+c)}$ $=\dfrac{0}{(a+b)(b+c)(a+c)}=0$	[3]
(ii) Simplify $\dfrac{2x^2-6x+4}{x^2-6x+5}\times\dfrac{x^2-x-20}{3x^2+6x-24}$. **Solution:** $\dfrac{2(x^2-3x+2)}{(x-1)(x-5)}\times\dfrac{(x-5)(x+4)}{3(x^2+2x-8)}$ $=\dfrac{2(x-1)(x-2)}{(x-1)(x-5)}\times\dfrac{(x-5)(x+4)}{3(x-2)(x+4)}$ $=\dfrac{2}{3}$	[3]

Question 13 [2014 EOY, Q1]	marks
(i) Factorise completely $(x-3)^3+(x-1)^3$. **Solution:** $(x-3)^3+(x-1)^3$ $=(x-3+x-1)[(x-3)^2-(x-3)(x-1)+(x-1)^2]$ $=(2x-4)(x^2-6x+9-x^2+4x-3+x^2-2x+1)$ $=2(x-2)(x^2-4x+7)$	[3]
(ii) Simplify $\dfrac{x-\dfrac{x-1}{2}}{2-\dfrac{x-1}{x}}$. **Solution:** $=\dfrac{\dfrac{2x-x+1}{2}}{\dfrac{2x-x+1}{x}}$ $=\dfrac{x}{2}$ or $\dfrac{x-\dfrac{x-1}{2}}{2-\dfrac{x-1}{x}}$ $=\dfrac{2x^2-x(x-1)}{4x-2(x-1)}$ $=\dfrac{x^2+x}{2x+2}$ $=\dfrac{x(x+1)}{2(x+1)}$ $=\dfrac{x}{2}$	[3]

Question 14 [2014 EOY, Q2]	marks
The area of a rectangle $ABCD$ is given by $\left(1+3\sqrt{2}\right)$ m^2, and the length of side AB is $\left(2+\sqrt{8}\right)$ m. Find, without using a calculator, the length of side BC in the form of $(a+b\sqrt{2})$ m, where a and b are real numbers. **<u>Solution:</u>** Length of the side BC $=\dfrac{1+3\sqrt{2}}{2+\sqrt{8}}$ $=\dfrac{(1+3\sqrt{2})(2-2\sqrt{2})}{(2+\sqrt{8})(2-2\sqrt{2})}$ $=\dfrac{2-2\sqrt{2}+6\sqrt{2}-12}{-4}$ $=\dfrac{-10+4\sqrt{2}}{-4}$ $=\left(\dfrac{5}{2}-\sqrt{2}\right)$ m	[4]

Question 15 [2015 EOY, Q3]	marks
(i) Make p the subject of the formula if $s=\dfrac{4(2p^2+1)}{u^2}$. **<u>Solution:</u>** $s=\dfrac{4(2p^2+1)}{u^2}$ $\Rightarrow su^2=8p^2+4$ $\Rightarrow p^2=\dfrac{su^2-4}{8}$ $\Rightarrow p=\pm\sqrt{\dfrac{su^2-4}{8}}$	[3]

(ii) Simplify into a single fraction $\dfrac{x^3+8y^3}{x^2-4y^2}-\dfrac{16xy^2-4x^3}{(4y-2x)^2}$. **<u>Solution:</u>** $\dfrac{x^3+8y^3}{x^2-4y^2}-\dfrac{16xy^2-4x^3}{(4y-2x)^2}$ $=\dfrac{(x+2y)(x^2-2xy+4y^2)}{(x+2y)(x-2y)}-\dfrac{4x(2y-x)(2y+x)}{4(2y-x)^2}$ $=\dfrac{x^2-2xy+4y^2}{x-2y}-\dfrac{2xy+x^2}{2y-x}$ $=\dfrac{2x^2+4y^2}{x-2y}$	[4]
Question 16 [2015 EOY, Q4]	marks
(i) Given that $x^2-4x=y^2-4y$ and that $x \neq y$, find the value of $3\sqrt{x+y}$. **<u>Solution:</u>** $x^2-4x=y^2-4y \Rightarrow (x-y)(x+y)=4(x-y)$ Since $x \neq y$, $x-y \neq 0 \Rightarrow x+y=4$ Hence $3\sqrt{x+y}=3\sqrt{4}$ $=6$	[3]

(ii) Using a suitable substitution, or otherwise, solve for x if $$2(27^x) = 45(3^x) + \frac{243}{3^x}.$$ **Solution:** $2(27^x) = 45(3^x) + \frac{243}{3^x} \Rightarrow 2(81^x) = 45(9^x) + 243$ Let $y = 9^x$, equation becomes $2y^2 = 45y + 243 \Rightarrow 2y^2 - 45y - 243 = 0$ Solving, $y = 27$ ($y = -\frac{9}{2}$ is rejected since 9^x cannot be negative) Hence $9^x = 27 \Rightarrow x = \frac{3}{2}$	[4]