# Raffles Institution
# H2 Mathematics (9758)
# Solution for 2020 A-Level Paper 1

**Question 1**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(i)** | $\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\-5\\-2\end{pmatrix}=\begin{pmatrix}-2\\2\\-6\end{pmatrix}=-2\begin{pmatrix}1\\-1\\3\end{pmatrix}$ <br> A vector normal to $\pi_1$ is $\begin{pmatrix}1\\-1\\3\end{pmatrix}$. | |
| **(ii)** | $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}1\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}4\\5\\-6\end{pmatrix}\right\rvert}{\sqrt{11}\sqrt{77}}=\dfrac{19}{\sqrt{11}\sqrt{77}}$ <br> Required angle, $\theta=49.2^{\circ}$ (1dp) | |

**Question 2**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| | $\dfrac{x^2}{1+x^2}+\dfrac{y^2}{1+y^2}=x^3y^5$ <br> $\dfrac{2x\left(1+x^2\right)-2x\left(x^2\right)}{\left(1+x^2\right)^2}+\dfrac{2y\dfrac{dy}{dx}\left(1+y^2\right)-2y\dfrac{dy}{dx}\left(y^2\right)}{\left(1+y^2\right)^2}=5x^3y^4\dfrac{dy}{dx}+3x^2y^5$ <br> At $(1,1)$, <br> $\dfrac{2(2)-2}{4}+\dfrac{4\dfrac{dy}{dx}-2\dfrac{dy}{dx}}{4}=5\dfrac{dy}{dx}+3$ <br> $\dfrac{1}{2}+\dfrac{1}{2}\dfrac{dy}{dx}=5\dfrac{dy}{dx}+3\Rightarrow\dfrac{dy}{dx}=-\dfrac{5}{9}$ <br> Equation of tangent at $(1,1)$ is $y-1=-\dfrac{5}{9}(x-1)\Rightarrow 5x+9y=14$ | Note that question requires answer in the form $ax + by = c$, where $a$, $b$ and $c$ are integers |

**Question 3**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(i)** | $\mathrm{f}(x)=\ln(1+\sin 3x)$<br>$\mathrm{f}'(x)=\dfrac{3\cos 3x}{1+\sin 3x}$<br>$\mathrm{f}''(x)=\dfrac{-9\sin 3x\left(1+\sin 3x\right)-9\cos^2 3x}{\left(1+\sin 3x\right)^2}$<br>$=\dfrac{-9\sin 3x-9}{\left(1+\sin 3x\right)^2}$<br>$=\dfrac{-9}{1+\sin 3x}$<br>Thus, $k=-9$ | $\sin^2 x+\cos^2 x=1$ |
| **(ii)** | $\mathrm{f}(0)=\ln(1)=0$<br>$\mathrm{f}'(0)=3$<br>$\mathrm{f}''(0)=-9$<br>$\mathrm{f}'''(x)=9\left(1+\sin 3x\right)^{-2}\left(3\cos 3x\right)$<br>$\mathrm{f}'''(0)=27$<br>$\mathrm{f}(x)=3x-\dfrac{9}{2}x^2+\dfrac{9}{2}x^3+\ldots$ | |

**Question 4**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| (i) | $z_1 = 1+\sqrt{3}\mathrm{i} = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}, z_2 = 1-\mathrm{i} = \sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}, z_3 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$<br>$\frac{z_1}{z_2 z_3} = \frac{2\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}}{\left(\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\right)\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)} = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}+\frac{\pi}{4}-\frac{\pi}{6}\right)} = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}} = \frac{1}{\sqrt{2}}\left(\cos\frac{5\pi}{12}+\mathrm{i}\sin\frac{5\pi}{12}\right)$ | Note that detailed working and exact values are required as the question states "Do not use a calculator in answering this question". |
| (ii) | $\frac{z_1 z_4}{z_2 z_3} = a\mathrm{i}$, where $a \in \mathbb{R}$<br>$\left\lvert\frac{z_1 z_4}{z_2 z_3}\right\rvert = 1 \Rightarrow a = \pm 1$, that is $\frac{z_1 z_4}{z_2 z_3} = \pm\mathrm{i}$<br><br>$\frac{z_1}{z_2 z_3} = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}} \Rightarrow \frac{z_2 z_3}{z_1} = \sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}}$<br>$z_4 = \mathrm{i}\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}} = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{12}} = \sqrt{2}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$<br>OR $z_4 = -\mathrm{i}\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}} = \sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{11\pi}{12}}$<br>$= \sqrt{2}\left(\cos\left(-\frac{11\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{11\pi}{12}\right)\right)$ | Note that there are 2 answers for $z_4$. |

**Question 5**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(a)** | Given $\underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{b}\times\underset{\sim}{a}$<br>But we know $\underset{\sim}{a}\times\underset{\sim}{b}=-\underset{\sim}{b}\times\underset{\sim}{a}$<br>$\Rightarrow \underset{\sim}{a}\times\underset{\sim}{b}=\underset{\sim}{0}$<br>$\Rightarrow \underset{\sim}{a}=k\underset{\sim}{b},\ k\in\mathbb{R}\setminus\{0\}$<br>That is, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel. | If $k=0$, then $\underset{\sim}{a}=0\underset{\sim}{b}=\underset{\sim}{0}$.<br>But the question stated that $\underset{\sim}{a}$ is a nonzero vector.<br>Thus, $k\neq 0$ . |
| **(b)(i)** | $\left(\underset{\sim}{r}-\underset{\sim}{p}\right)\times\underset{\sim}{q}=\underset{\sim}{0}$<br>$\Rightarrow \underset{\sim}{r}-\underset{\sim}{p}=k\underset{\sim}{q},\ k\in\mathbb{R}$<br>$\therefore \underset{\sim}{r}=\underset{\sim}{p}+k\underset{\sim}{q},\ k\in\mathbb{R}$<br>The set of all possible positions of point $R$ form a line which passes through point $P$ and parallel to vector $\underset{\sim}{q}$. | Clear description is expected. |
| **(ii)** | $\left(\underset{\sim}{r}-\underset{\sim}{p}\right)\bullet\underset{\sim}{q}=0$<br>$\begin{pmatrix}x+1\\y-2\\z-4\end{pmatrix}\bullet\begin{pmatrix}3\\-5\\2\end{pmatrix}=0$<br>$\Rightarrow 3x+3-5y+10+2z-8=0$<br>$3x-5y+2z=-5$<br>$R$ represents all possible positions on plane which passes through $(-1, 2, 4)$ and has normal vector $\begin{pmatrix}3\\-5\\2\end{pmatrix}$. | |

**Question 6**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| | $z^2(2+\mathrm{i})-8\mathrm{i}z+t=0$<br>$(k+k\mathrm{i})^2(2+\mathrm{i})-8\mathrm{i}(k+k\mathrm{i})+t=0$<br>$k^2(1+\mathrm{i})^2(2+\mathrm{i})-8k\mathrm{i}(1+\mathrm{i})+t=0$<br>$k^2(2\mathrm{i})(2+\mathrm{i})-8k(-1+\mathrm{i})+t=0$<br>$k^2(-2+4\mathrm{i})-8k(-1+\mathrm{i})+t=0$<br>$(-2k^2+8k+t)+(4k^2-8k)\mathrm{i}=0$<br>Comparing imaginary part: $4k^2-8k=0$<br>$\Rightarrow k=2$ or $k=0$(rejected as $k\neq 0$)<br>Comparing real part: $-8+16+t=0\Rightarrow t=-8$<br><br>$z^2(2+\mathrm{i})-8\mathrm{i}z-8=0$<br>$z=\dfrac{8\mathrm{i}\pm\sqrt{(-8\mathrm{i})^2-4(2+\mathrm{i})(-8)}}{2(2+\mathrm{i})}$<br>$=\dfrac{8\mathrm{i}\pm\sqrt{32\mathrm{i}}}{2(2+\mathrm{i})}$<br>$=2+2\mathrm{i}$ or $-\dfrac{2}{5}+\dfrac{6}{5}\mathrm{i}$ | Note that you can use GC for this question where appropriate such as evaluating $\dfrac{8\mathrm{i}\pm\sqrt{32\mathrm{i}}}{2(2+\mathrm{i})}$<br><br>Note that coefficients of the equation are not real, so roots do not come in conjugate pair. |

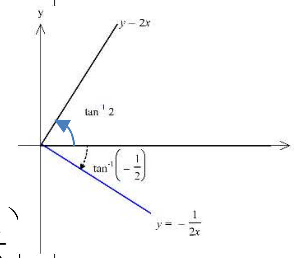
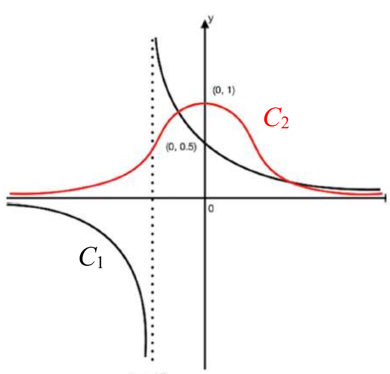
**Question 7**

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(i)</td><td>$\int 2-\sin 4x \ \mathrm{d}x = 2x+\frac{1}{4}\cos 4x + c$</td><td></td></tr>
<tr><td>(ii)</td><td>Let $u = x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 2-\sin 4x$<br><br>$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = 2x+\frac{1}{4}\cos 4x$ from part (i)<br><br>$\int_0^{\frac{\pi}{2}} x\left(2-\sin 4x\right) \ \mathrm{d}x$<br>$= \left[x\left(2x+\frac{1}{4}\cos 4x\right)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left(2x+\frac{1}{4}\cos 4x\right) \ \mathrm{d}x$<br>$= 2\left(\frac{\pi^2}{4}\right)+\frac{\pi}{8}-\left[x^2+\frac{1}{16}\sin 4x\right]_0^{\frac{\pi}{2}}$<br>$= \frac{\pi^2}{2}+\frac{\pi}{8}-\left[\frac{\pi^2}{4}\right]$<br>$= \frac{\pi^2}{4}+\frac{\pi}{8}$</td><td>Clear working is required since exact value is required.<br><br>Note that to check your answer, you can use GC to evaluate $\int_0^{\frac{\pi}{2}} x\left(2-\sin 4x\right) \ \mathrm{d}x$ and compare with the numerical value of your answer, in this case $\frac{\pi^2}{4}+\frac{\pi}{8}$</td></tr>
<tr><td>(iii)</td><td>$\int_0^{\frac{\pi}{2}} \left(2-\sin 4x\right)^2 \ \mathrm{d}x$<br>$= \int_0^{\frac{\pi}{2}} \left(4-4\sin 4x+\sin^2 4x\right) \ \mathrm{d}x$<br>$= \left[4x\right]_0^{\frac{\pi}{2}} + \left[\cos 4x\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1-\cos 8x \ \right)\mathrm{d}x$<br>$= 2\pi + 0 + \frac{1}{2}\left[x-\frac{1}{8}\sin 8x\right]_0^{\frac{\pi}{2}}$<br>$= 2\pi+\frac{\pi}{4} = \frac{9\pi}{4}$</td><td></td></tr>
</table>

**Question 8**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(a)** **(i)** | $u_1 = 4$<br>$u_5 = 4 + 4d = 10 \Rightarrow d = 1.5$<br>$u_{30} = 4 + 29(1.5) = 47.5$ | |
| **(ii)** | $u_{21} + u_{22} + ... + u_{50} = \sum_{r=21}^{50}\left(4 + 1.5(r-1)\right) = 1672.5$ | Using GC |
| **(b)** **(i)** | $4r^4 = 1.6384 \Rightarrow r = 0.8$<br>$S_\infty = \frac{4}{1-0.8} = 20$ | |
| **(ii)** | $\frac{4\left(1-0.8^n\right)}{1-0.8} > 19.6$<br>$\frac{4\left(1-0.8^n\right)}{0.2} > 19.6$<br>$1 - 0.8^n > 0.98$<br>$0.8^n < 0.02$<br>$n \ln 0.8 < \ln 0.02$<br>$n > \frac{\ln 0.02}{\ln 0.8} = 17.531$<br>smallest possible $n = 18$ | Note that<br>ln 0.8<0 |

**Question 9**

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(i)</td><td>Consider $y=2x$ and $y=-\frac{1}{2}x$<br>They are perpendicular to each other since product of gradients is $-1$.<br>Angles lines made with positive $x$-axis are $\tan^{-1}2$ and $\tan^{-1}\left(-\frac{1}{2}\right)$ respectively. Thus $\tan^{-1}2-\tan^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{2}$</td><td></td></tr>
<tr><td>(ii)</td><td>$\frac{1}{x^2+1}=\frac{k}{3x+4}$<br>$3x+4=kx^2+k$<br>$kx^2-3x+k-4=0...(1)$<br>Intersect $\Rightarrow$ (1) has real roots<br>$k>0$ and $9-4k(k-4)\geq 0$<br>$4k^2-16k-9\leq 0$<br>$(2k+1)(2k-9)\leq 0$<br>$-\frac{1}{2}\leq k\leq\frac{9}{2}$<br>$\therefore 0<k\leq\frac{9}{2}$<br>set of values of $k$ is $\left(0,\frac{9}{2}\right]$</td><td>Read question carefully. It is given $k>0$.</td></tr>
<tr><td>(iii)</td><td></td><td>Good to label the intersection of the 2 curves if time permits: (–0.5, 0.8), (2, 0.2)</td></tr>
</table>

| | | |
|---|---|---|
| **(iv)** | $\int_{-\frac{1}{2}}^{2}\left(\frac{1}{x^2+1}-\frac{2}{3x+4}\right)\,\mathrm{d}x$<br>$=\left[\tan^{-1}x\right]_{-\frac{1}{2}}^{2}-\frac{2}{3}\left[\ln(3x+4)\right]_{-\frac{1}{2}}^{2}$<br>$=\tan^{-1}2-\tan^{-1}\left(-\frac{1}{2}\right)-\frac{2}{3}\left(\ln 10-\ln\frac{5}{2}\right)$<br>$=\frac{\pi}{2}-\frac{2}{3}\ln 4$<br>$=\frac{\pi}{2}-\frac{4}{3}\ln 2$ | Recall from (i), $\tan^{-1}2-\tan^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{2}$ |

**Question 10**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(i)** | $\frac{\mathrm{d}P}{\mathrm{d}t}=-0.03P$ | |
| **(ii)** | $\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}=-0.03$<br>$\ln P=-0.03t+C,\ P>0$ and $C$ is an arbitrary constant<br>$P=Ae^{-0.03t},\quad A=e^{C}>0$<br>Note that as $t\to\infty,\ e^{-0.03t}\to 0$ and thus $P\to 0$<br>So the number of sheep will approach 0 if this situation continues many years. | Note that no further information were given in the question, so we can't find value of $A$.<br>Explain in context of the question |
| **(iii)** | $\frac{\mathrm{d}P}{\mathrm{d}t}=-0.03P+n$ | |
| **(iv)** | $\frac{1}{n-0.03P}\frac{\mathrm{d}P}{\mathrm{d}t}=1$<br>$-\frac{100}{3}\ln\lvert n-0.03P\rvert=t+b,$ where $b$ is an arbitrary constant<br>$\ln\lvert n-0.03P\rvert=-0.03t-0.03b$<br>$n-0.03P=Be^{-0.03t},\quad B=\pm e^{-0.03b}$<br>$P=\frac{100}{3}\left(n-Be^{-0.03t}\right)$ | |
| **(v)** | $t\to\infty,\ e^{-0.03t}\to 0$ and given $P\to 500$<br>$\frac{100}{3}n=500$<br>$\therefore n=15$ | |

**Question 11**

| No. | Suggested Solution | Remarks for Student |
|---|---|---|
| **(i)** | $\theta = \angle DKA - \angle CKA$<br>$\tan\theta = \tan\left(\angle DKA - \angle CKA\right)$<br>$= \dfrac{\tan\angle DKA - \tan\angle CKA}{1+\left(\tan\angle DKA\right)\left(\tan\angle CKA\right)}$<br>$= \dfrac{\frac{a+4}{x} - \frac{a}{x}}{1+\left(\frac{a+4}{x}\right)\left(\frac{a}{x}\right)}$<br>$= \dfrac{a+4-a}{x+a\left(\frac{a+4}{x}\right)}$<br>$= \dfrac{4x}{x^2+a^2+4a}$ | |
| **(ii)** | Let $y = \tan\theta = \dfrac{4x}{x^2+4a+a^2}$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4\left(x^2+4a+a^2\right)-2x\left(4x\right)}{\left(x^2+4a+a^2\right)^2} = 0$<br>$4x^2+16a+4a^2-8x^2=0$<br>$4x^2-16a-4a^2=0$<br>$x^2 = 4a+a^2$<br>$\therefore x = \sqrt{4a+a^2}$ since $a>0$<br>$\tan\theta = \dfrac{4\sqrt{4a+a^2}}{\left(4a+a^2\right)+4a+a^2} = \dfrac{2}{\sqrt{4a+a^2}}$ | |
| **(iii)** | In reality, there are other factors such as wind direction, and skill and strength of the player kicking the ball. | |
| **(iv)** | $\tan\angle KDA = \dfrac{x}{4+a}$<br>$= \dfrac{\sqrt{4a+a^2}}{4+a}$<br>$= \dfrac{\sqrt{a\left(4+a\right)}}{4+a}$<br>$= \sqrt{\dfrac{a}{4+a}}$<br>$\approx 1$ if $a$ is much greater than 4<br>$\angle KDA \approx \dfrac{\pi}{4}$ or $45^\circ$ if $a$ is much greater than 4 | |

| | | |
|---|---|---|
| **(v)** | $XY = 50 \Rightarrow XC = 23$<br>$\Rightarrow 0 \le a \le 23$<br>$\Rightarrow 0 \le 4a + a^2 \le 621$<br>$\Rightarrow 0 \le \sqrt{4a + a^2} \le \sqrt{621}$<br>$\Rightarrow \dfrac{1}{\sqrt{4a + a^2}} \ge \dfrac{1}{\sqrt{621}}$<br>$\Rightarrow \tan\theta = \dfrac{2}{\sqrt{4a + a^2}} \ge \dfrac{2}{\sqrt{621}}$<br>$\therefore \tan^{-1}\dfrac{2}{\sqrt{621}} \le \theta < 90^\circ$, ie, $4.6^\circ \le \theta < 90^\circ$<br>$\left(\text{Answer in radian: } 0.0801 \le \theta < \dfrac{\pi}{2}\right)$ | |