2022 HCI Prelim Paper 1_Modified [96 marks]

1 Consider the function $\mathrm{f}(x,y)=\sqrt{x^2+ky}$, where k is a non-zero constant.

(a) Find, in terms of k, the linearization of $\mathrm{f}(x,y)$ at the point $(2,2)$. [3]

(b) By using the answer in part **(a)** and a suitable integer value of k, find the approximate value of $\sqrt{2.1^2+3.9}$, giving your answer in the form $\frac{p}{q}\sqrt{b}$, where p, q and b are integers to be determined. [2]

2 Consider the following system of linear equations, where $a, b \in \mathbb{R}$.

$$x+y+az=0$$
$$x-2y+az=0$$
$$x+y+z=b$$

Let S be the solution set of this system of linear equations. By performing suitable row operations, state the possible values of a and b such that

(i) $S=\{\mathbf{v}\}$, where $\mathbf{v}$ is a nonzero vector, [2]

(ii) S is not a subspace of $\mathbb{R}^3$. Find the possible sets of S in this case. [3]

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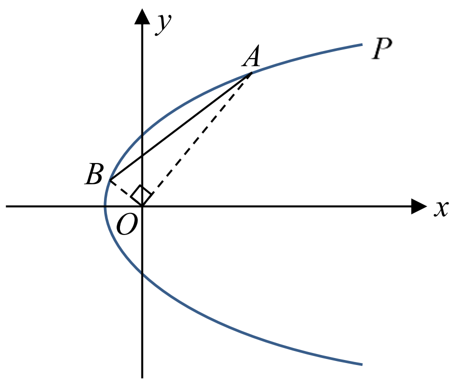


In the diagram, the origin O is the focus of a parabola P with equation $y^2=4h(x+h)$, where h is a positive constant.

(i) By taking the pole to be at O, find a polar equation of P. [2]

(ii) A and B are points on P such that OA is perpendicular to OB. By using differentiation, find the exact minimum area of the triangle OAB in terms of h. [You are not required to prove the area found is a minimum.] [5]

4 **(a)** A fixed point of a function $g(x)$ is a real number α such that $\alpha = g(\alpha)$. An approximate value for a fixed point α is to be found by using the fixed point iteration method of the form $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$, with an initial estimate of $x_0 = 1.5$. For each of the following expressions for $g(x)$, explain with reason(s) or with the aid of suitable diagrams, whether it is suitable for fixed point iteration. If the $g(x)$ is found suitable for fixed point iteration, obtain an estimate for α correct to 4 decimal places.

(i) $g(x) = x - x^3 - 4x^2 + 6$ [2]

(ii) $g(x) = \left(\dfrac{6}{x} - 4x\right)^{\frac{1}{2}}$ [1]

(iii) $g(x) = \left(\dfrac{6}{4+x}\right)^{\frac{1}{2}}$ [3]

(b) The function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$ is the normal density function with a mean value of zero and standard deviation σ. The probability of a randomly chosen value x as described by this function lying in the interval $[a,b]$, is given by $\int_a^b f(x)\,dx$.

Use Simpson's rule with 5 ordinates to find an approximation to the probability that a randomly chosen value x will lie in the interval $[-\sigma, \sigma]$. Give your answer correct to 3 decimal places. [4]

5 **Do not use a calculator in answering this question.**

(i) Show algebraically that

$$\cos\left(\frac{\pi}{10}\right)+\cos\left(\frac{3\pi}{10}\right)+\cos\left(\frac{5\pi}{10}\right)+\cos\left(\frac{7\pi}{10}\right)+\cos\left(\frac{9\pi}{10}\right)=0.$$ [2]

(ii) The equation $z^{10}+(13z-1)^{10}=0$ has 10 complex roots denoted by z_i and z_i^*, where z_i^* are the complex conjugates of z_i, $i=1,2,3,4,5$, respectively. By expressing these roots as $\frac{1}{z}=13-\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi<\theta\leq\pi$, or otherwise, evaluate

$$\frac{1}{z_1 z_1^*}+\frac{1}{z_2 z_2^*}+\frac{1}{z_3 z_3^*}+\frac{1}{z_4 z_4^*}+\frac{1}{z_5 z_5^*}.$$ [8]

6 **(i)** Let $I_n=\int_0^t x^n(1+x^2)^{\frac{1}{2}}\,\mathrm{d}x$. Show that

$$(n+2)I_n=t^{n-1}(1+t^2)^{\frac{3}{2}}-(n-1)I_{n-2}, \text{ where } n\geq 2.$$ [3]

(ii) By considering the derivative of $\ln(x+\sqrt{1+x^2})$, find the exact value of $\int_0^{\frac{4}{3}}\sqrt{1+x^2}\,\mathrm{d}x$ in the form $m+n\ln 3$, where m, n are constants to be determined. [4]

(iii) A curve C has parametric equations

$$x=\tan\theta,\quad y=\frac{1}{1+\cos 2\theta}\quad \text{for } -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

A part of C bounded by $x=0$ and $x=\frac{4}{3}$ is rotated through 2π radians about the x-axis. By converting the equations to cartesian form, or otherwise, find the exact value of the surface area of revolution thus formed. [4]

7 Taking the origin as the pole, the equation of the curve G in polar coordinates $r=\mathrm{f}(\theta)$ satisfies the differential equation

$$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2}+4r=5\sin 3\theta \quad \text{for } \frac{\pi}{5}\leq\theta\leq\frac{3\pi}{5}.$$

It is given that when $\theta=\frac{\pi}{2}$, $r=1$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta}=-2$.

(i) Find the polar equation of G. [7]

(ii) Sketch G, indicating clearly the point with polar coordinates $(1,\frac{\pi}{2})$ on G. [1]

(iii) Show that the area enclosed by G is given by $a\pi+b(\sin\frac{\pi}{5}-\sin\frac{2\pi}{5})$ square units, where a, b are exact rational numbers to be determined. [5]

8 In a habitat, the number of rabbits R_n (in thousands) and wildcats W_n (in thousands) in n years can be modelled by

$$R_n = \frac{3}{2}R_{n-1} - \frac{1}{2}W_{n-1} ,$$

$$W_n = \frac{3}{4}R_{n-1} + \frac{1}{4}W_{n-1} ,$$

where $n \geq 1$, and R_0 and W_0 are the initial number of rabbits and wildcats respectively.

(i) Find an expression for R_n in terms of R_{n-1} and R_{n-2}. [2]

(ii) Show that the 2×1 matrix $\begin{pmatrix} R_n \\ W_n \end{pmatrix}$ can be expressed as $\mathbf{PDP}^{-1}\begin{pmatrix} R_0 \\ W_0 \end{pmatrix}$, where $\mathbf{P}$ is a 2×2 invertible matrix and $\mathbf{D}$ is a 2×2 diagonal matrix. [4]

(iii) Using the result in part **(ii)**, find R_n and W_n in terms of n, R_0 and W_0. [3]

(iv) Given that the population of both rabbits and wildcats decrease and approach a non-zero equilibrium population, find a relationship between R_0 and W_0. [3]

9 The sequence of integers $\{u_1, u_2, u_3, ...\}$ is defined by the second-order recurrence relation

$$u_1 = a \ , \ u_2 = ka \quad \text{and} \quad u_{n+2} = 2u_{n+1} - 2u_n \quad \text{for } n \geq 1,$$

where a, k are positive constants.

(i) Show that $u_{n+4} = -4u_n$ for all $n \geq 1$. [2]

It is given that u_{4r-2} is a multiple of ka for all positive integers r.

(iii) Find $\sum_{r=1}^{4N} u_r$ in terms of a, k and N. [3]

(iv) Find u_n in the form $p^n(A\cos qn + B\sin qn)$, where A, B are constants in terms of a, k and p, q are constants to be determined.

Hence find $\sum_{r=1}^{20} p^r(A\cos qr + B\sin qr)$ in terms of a and k. [5]

10 A *predator-prey model* is a pair of first-order non-linear differential equations used to describe the dynamic interactions between two species in ecological systems, one as predator and the other as prey. A financial analyst adopted the *predator-prey model* to predict the stock market prices of two companies X and Y. The stock prices $\$x$ and $\$y$ of companies X and Y respectively in t months are modelled as follows.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2x - 0.01xy,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -0.05y + 0.004xy.$$

It is given that the initial stock prices of companies X and Y are \$10.00 and \$15.00 respectively.

(i) A 'modified' Euler method with step size h is given by

$$x_2 = x_1 + h\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=x_1,\ y=y_1},$$
$$y_2 = y_1 + h\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{x=x_1,\ y=y_1}.$$

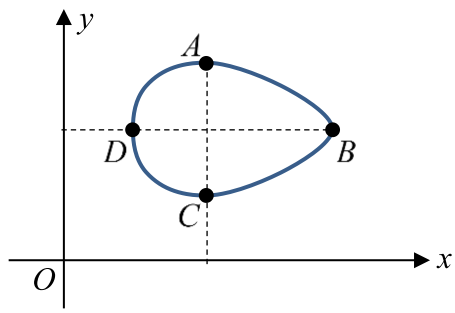
Using this 'modified' Euler method with step size 0.5, estimate the stock prices of companies X and Y after 1 month. Give your answers correct to 2 decimal places. [4]

(ii) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence, using integration, show that

$$y^g \mathrm{e}^{hy} = Kx^m \mathrm{e}^{nx},$$

where g, h, K, m, n are constants to be determined. [5]

(iii) The graph below shows the relationship between x and y.



It is given that the tangents of the curve at the points A and C are parallel to the x-axis, while the tangents of the curve at the points B and D are parallel to the y-axis. By finding the x-coordinate of points A and C, and the y-coordinates of points B and D, determine the coordinates of the points A, B, C and D. [3]

(iv) Using the graph and answers found in part **(iii)**, comment on what the *predator-prey model* indicates about the stock prices of companies X and Y. [1]

End of Paper