RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Additional Practice Questions for Chapter 7C:

Maclaurin Series (Solutions)

1 **(a)** Find the fifth term in the expansion of $\left(x^3-\dfrac{1}{2x}\right)^6$ in descending powers of x. **[2]**

(b) Find the term independent of x in the expansion of $\left(9x-\dfrac{1}{3\sqrt{x}}\right)^{18}$. **[3]**

(c) Find the coefficient of x^{-18} in the expansion of $\left(3x+\dfrac{1}{x^2}\right)^{12}$. **[3]**

(d) Expand $(2+3x+4x^2)^{10}$, up to the term in x^3. **[3]**

(e) The coefficient of x^2 in the expansion of $(2x+k)^6$ is equal to the coefficient of x^5 in the expansion of $(2+kx)^8$. Find k. **[3]**

(f) The expansion of $(1+kx)^n$ up to and including the term in x^3 are $1+36x+66k^2x^2+hx^3$, where n is a positive integer. Find the values of n, k and h. **[4]**

(a) $\left(x^3-\dfrac{1}{2x}\right)^6=\left[x^3\left(1-\dfrac{1}{2x^4}\right)\right]^6=x^{18}\left(1-\dfrac{1}{2x^4}\right)^6$

Fifth term in the expansion of $\left(x^3-\dfrac{1}{2x}\right)^6$ in descending powers of x is

$$x^{18}\binom{6}{4}\left(-\frac{1}{2x^4}\right)^4=\frac{15}{16}x^2$$

(b) General term $=\dbinom{18}{r}(9x)^{18-r}\left(-\dfrac{1}{3\sqrt{x}}\right)^r$

$=\dbinom{18}{r}9^{18-r}\left(-\dfrac{1}{3}\right)^r x^{18-r-\frac{1}{2}r}$

For term independent of x, $18-\dfrac{3}{2}r=0\Rightarrow \quad r=12$

$\therefore$ Term independent of $x=\dbinom{18}{12}9^6\left(-\dfrac{1}{3}\right)^{12}$

$=18\ 564$

(c) General term $=\dbinom{12}{r}(3x)^{12-r}\left(\dfrac{1}{x^2}\right)^r$

$$=\binom{12}{r}3^{12-r}x^{12-r-2r}$$

For term with x^{-18}, $12-3r=-18$

$r=10$

$\therefore$ Coefficient of $x^{-18}=\binom{12}{10}3^{2}$

$= 594$

(d) $\left(2+3x+4x^2\right)^{10}$

$=2^{10}+\binom{10}{1}2^9\left(3x+4x^2\right)+\binom{10}{2}2^8\left(3x+4x^2\right)^2+\binom{10}{3}2^7\left(3x+4x^2\right)^3$

$+\cdots+$ term in ^{20}

$=2^{10}+15360x+20480x^2+11520\left(9x^2+24x^3\right)+15360\left(27x^3\right)$

$+\cdots+$ term in ^{20}

$=1024+15360x+124160x^2+691200x^3+\cdots+$ term in ^{20}

(e) The x^2 term in the expansion of $(2x+k)^6$ is $\binom{6}{4}(2x)^2k^4$.

The x^5 term in the expansion of $(2+kx)^8$ is $\binom{8}{5}2^3(kx)^5$.

So $\binom{6}{4}(2)^2k^4=\binom{8}{5}2^3k^5$

$60k^4=448k^5 \quad\Rightarrow\quad k=\dfrac{15}{112}$

(f) $(1+kx)^n=1+\binom{n}{1}kx+\binom{n}{2}(kx)^2+\binom{n}{3}(kx)^3+\cdots+(kx)^n$

So $1+nkx+\dfrac{n(n-1)}{2}k^2x^2+\dfrac{n(n-1)(n-2)}{3!}k^3x^3=1+36x+66k^2x^2+hx^3$

$nk=36 \quad --- \quad (1)$

$\dfrac{n(n-1)}{2}=66 \quad --- \quad (2)$

$\dfrac{n(n-1)(n-2)}{6}k^3=h \quad --- \quad (3)$

From (2), $n^2-n-132=0 \quad\Rightarrow\quad (n-12)(n+11)=0 \quad\Rightarrow\quad n=12$ since $n>0$.

From (1), $12k=36 \quad\Rightarrow\quad k=3$

From (3), $h=\binom{n}{3}k^3=\binom{12}{3}3^3=5940$

2 Express $\dfrac{2+5x+15x^2}{(2-x)(1+2x^2)}$ in partial fractions. **[2]**

Hence or otherwise, obtain the expansion of $\dfrac{2+5x+15x^2}{(2-x)(1+2x^2)}$ in ascending powers of x, giving the first five terms in the expansion. **[3]**

Find the exact set of values of x for which the expansion is valid. **[2]**

$$\frac{2+5x+15x^2}{(2-x)(1+2x^2)}=\frac{A}{2-x}+\frac{Bx+C}{1+2x^2}$$

$$2+5x+15x^2=A(1+2x^2)+(Bx+C)(2-x)$$

When $x=2$, $72=9A \Rightarrow A=8$.

Comparing coefficients of x^2, $15=2A-B \Rightarrow B=1$.

Comparing constants, $2=A+2C \Rightarrow C=-3$.

$$\therefore \frac{2+5x+15x^2}{(2-x)(1+2x^2)}=\frac{8}{2-x}+\frac{x-3}{1+2x^2}$$

$$=\frac{8}{2}\left(1-\frac{x}{2}\right)^{-1}+(x-3)(1+2x^2)^{-1}$$

$$=4\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+\cdots\right)$$

$$+(x-3)\left(1-2x^2+(2x^2)^2-(2x^2)^3+\cdots\right)$$

$$=1+3x+7x^2-\frac{3}{2}x^3-\frac{47}{4}x^4+\cdots$$

Expansion is valid for

$$\left|\frac{x}{2}\right|<1 \quad \text{and} \quad \left|2x^2\right|<1$$

$$-2<x<2 \qquad -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$$

The set of values of x for which the expansion to be valid is $\left(-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$.

3 MI Promo 9758/2020/PU2/P1/Q6

(a) Using standard series from the List of Formulae (MF26), expand $\mathrm{e}^{3x}\ln(1+ax)$ as far as the term in x^3, where a is a non-zero constant. Given that there is no term in x^2, determine the coefficient of x^3. [5]

(b) Find the expansion of $\dfrac{1+3x}{\sqrt{9-x^2}}$ in ascending powers of x, up to and including the term in x^3. State the set of values of x for which the expansion is valid. [4]

Qn	Solution
(a)	$\mathrm{e}^{3x}\ln(1+ax)$ $=\left(1+3x+\frac{9x^2}{2}+\frac{27x^3}{6}+\ldots\right)\left(ax-\frac{a^2x^2}{2}+\frac{a^3x^3}{3}-\ldots\right)$ $=ax-\frac{a^2x^2}{2}+\frac{a^3x^3}{3}+3ax^2-\frac{3a^2x^3}{2}+\frac{9ax^3}{2}+\ldots$ $=ax+\left(3a-\frac{a^2}{2}\right)x^2+\left(\frac{a^3}{3}-\frac{3a^2}{2}+\frac{9a}{2}\right)x^3+\ldots$ Since there is no term in x^2, $3a-\frac{a^2}{2}=0$ $a^2-6a=0$ $a=0$ (rejected $\because$ a is non-zero) or 6. Therefore, the coefficient of $x^3=\left(\frac{6^3}{3}-\frac{3\times 6^2}{2}+\frac{9\times 6}{2}\right)$ =45
(b)	$\frac{1+3x}{\sqrt{9-x^2}}=(1+3x)\left(9-x^2\right)^{-\frac{1}{2}}$ $=(1+3x)\left[9\left(1-\frac{x^2}{9}\right)\right]^{-\frac{1}{2}}$ $=(1+3x)\left[9^{-\frac{1}{2}}\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}\right]$ $=\frac{1}{3}(1+3x)\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}$

Qn	Solution
	$=\frac{1}{3}(1+3x)\left[1+\left(-\frac{1}{2}\right)\left(-\frac{1}{9}x^2\right)+...\right]$ $=\frac{1}{3}(1+3x)\left(1+\frac{1}{18}x^2+...\right)$ $=\frac{1}{3}\left(1+3x+\frac{1}{18}x^2+\frac{1}{6}x^3+...\right)$ $=\frac{1}{3}+x+\frac{1}{54}x^2+\frac{1}{18}x^3+...$ For expansion to be valid, $\left\|-\frac{x^2}{9}\right\|<1$ $\left\|x^2\right\|<9$ $\therefore -3<x<3$

4 **9740/2015/01/Q6**

(i) Write down the first three non-zero terms in the Maclaurin series for $\ln(1+2x)$, where $-\frac{1}{2}<x\le\frac{1}{2}$, simplifying the coefficients. [2]

(ii) It is given that the three terms found in part **(i)** are equal to the first three terms in the series expansion of $ax(1+bx)^c$ for small x. Find the exact values of the constants a, b and c and use these values to find the coefficient of x^4 in the expansion of $ax(1+bx)^c$, giving your answer as a simplified rational number. [6]

(i) $\ln(1+2x)=2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}+...=2x-2x^2+\frac{8}{3}x^3+...$

(ii) $ax(1+bx)^c=ax\left[1+bcx+\frac{c(c-1)}{2}(bx)^2+...\right]=ax+(abc)x^2+\frac{ab^2c(c-1)}{2}x^3+...$

By comparing coefficients, $a=2$

$abc=-2\Rightarrow bc=-1$

$\frac{ab^2c(c-1)}{2}=\frac{8}{3}\Rightarrow(-1)(-1-b)=\frac{8}{3}\Rightarrow b=\frac{5}{3},c=-\frac{3}{5}$

$$2x\left(1+\frac{5}{3}x\right)^{-\frac{3}{5}}=2x\left[...+\frac{-\frac{3}{5}(-\frac{3}{5}-1)(-\frac{3}{5}-2)}{3!}\left(\frac{5}{3}x\right)^3+...\right]$$

Coefficient of $x^4=-\frac{104}{27}$

5 Given that $y = \dfrac{1}{\sqrt{1+2x}+\sqrt{1+x}}$ where $x > -0.5$. Show that, provided x is non-zero, then $y = \dfrac{1}{x}\left(\sqrt{1+2x}-\sqrt{1+x}\right)$. **[1]**

Hence

(i) express y as a series of ascending powers of x up to and including the term in x^2 **[4]**

(ii) show that $\dfrac{10}{\sqrt{102}+\sqrt{101}} \approx \dfrac{79407}{160000}$. **[2]**

$$y = \frac{1}{\sqrt{1+2x}+\sqrt{1+x}} \times \frac{\sqrt{1+2x}-\sqrt{1+x}}{\sqrt{1+2x}-\sqrt{1+x}}$$

$$= \frac{\sqrt{1+2x}-\sqrt{1+x}}{(1+2x)-(1+x)} = \frac{1}{x}\left(\sqrt{1+2x}-\sqrt{1+x}\right)$$

(i)

$$y = \frac{1}{x}\left[(1+2x)^{1/2}-(1+x)^{1/2}\right]$$

$$= \frac{1}{x}\left[\begin{array}{l}\left(1+\frac{1}{2}(2x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(2x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(2x)^3+\ldots\right) \\ -\left(1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\ldots.\right)\end{array}\right]$$

$$= \frac{1}{x}\left[\frac{1}{2}x-\frac{3}{8}x^2+\frac{7}{16}x^3+\ldots\right]$$

$$= \frac{1}{2}-\frac{3}{8}x+\frac{7}{16}x^2+\ldots$$

(ii) Let $x = \dfrac{1}{100}$.

$$\sqrt{1+2x} = \sqrt{\frac{102}{100}} \quad \text{and} \quad \sqrt{1+x} = \sqrt{\frac{101}{100}}$$

$$y = \frac{10}{\sqrt{102}+\sqrt{101}}$$

$$y \approx \frac{1}{2}-\frac{3}{8}\left(\frac{1}{100}\right)+\frac{7}{16}\left(\frac{1}{100}\right)^2 = \frac{79407}{160000}$$

6 Expand $\dfrac{\sqrt{1-x}}{1+2x}$ in ascending powers of x up to and including the term in x^2. **[4]**

(i) In an attempt to estimate $\sqrt{2}$, a student substituted $x=-1$ in the above expansion. Explain why this does not give a good estimate. **[1]**

(ii) By putting $x=\dfrac{1}{9}$ in the expansion, show that $\sqrt{2}\approx\dfrac{p}{1296}$, where p is a positive integer to be determined. **[2]**

$$\frac{\sqrt{1-x}}{1+2x}=(1-x)^{\frac{1}{2}}(1+2x)^{-1}$$

$$=\left(1-\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(-x)^2+\cdots\right)\left(1-2x+(2x)^2+\cdots\right)$$

$$=\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\cdots\right)\left(1-2x+4x^2+\cdots\right)$$

$$=1+\left(\frac{-1}{2}-2\right)x+\left(\frac{-1}{8}+4+2\left(\frac{1}{2}\right)\right)x^2+\cdots=1-\frac{5}{2}x+\frac{39}{8}x^2+\cdots$$

(i) Expansion is valid for

$|x|<1$ and $|2x|<1$

$-1<x<1$ and $\dfrac{-1}{2}<x<\dfrac{1}{2}$

The set of values of x for which the expansion is valid is $\left(\dfrac{-1}{2},\dfrac{1}{2}\right)$.

Since $x=-1$ does not lie in the set of values of x for which the expansion is valid, this does not give a good estimate.

(ii) When $x=\dfrac{1}{9}$, $\dfrac{\sqrt{1-x}}{1+2x}=\dfrac{\sqrt{\frac{8}{9}}}{\frac{11}{9}}=\dfrac{2\sqrt{2}/3}{11/9}=\dfrac{6\sqrt{2}}{11}$

$$\frac{6\sqrt{2}}{11}\approx1-\frac{5}{2}\left(\frac{1}{9}\right)+\frac{39}{8}\left(\frac{1}{9}\right)^2$$

$$\sqrt{2}\approx\frac{11}{6}\left(\frac{169}{216}\right)=\frac{1859}{1296}$$

Therefore $p=1859$

7 For $n > 0$, the expansion of $(1+mx)^{-n}$, in ascending powers of x, is $1+8x+48x^2+.....$.

(i) Find the constants m and n. **[5]**

(ii) Show that the coefficient of x^{400} is in the form $a(4)^k$ where a and k are constants to be determined. **[2]**

(i) $(1+mx)^{-n} = 1 - mnx + \dfrac{-n(-n-1)}{2}(mx)^2 + ...$

By comparing coefficients,

$-mn = 8$ and $\dfrac{m^2 n(n+1)}{2} = 48$

Since $\dfrac{m^2 n(n+1)}{2} = \dfrac{(mn)^2 + m(mn)}{2}$, $\dfrac{64-8m}{2} = 48$

$\therefore m = -4, n = 2$

(ii) Coefficient of x^{400}

$= \dfrac{-2(-3)(-4)...(-401)}{400!}(-4)^{400}$

$= 401(-1)^{400}(-1)^{400}(4)^{400}$

$= 401(4)^{400}$

$a = 401,\ k = 400.$

8 **RIPromo9740/2015/Q8**

(a) **(i)** Expand $f(x) = \dfrac{2}{2-x} - \dfrac{1}{(1+x)^2}$ as a series in ascending powers of x up to and including the term in x^2. **[3]**

(ii) State the equation of the tangent to the curve $y = f(x)$ at the origin. **[1]**

(b) Using the standard series given in the List of Formulae (MF26) or otherwise, show that the first three non-zero terms in the Maclaurin series for $e^{\sqrt{1+x}}$ can be expressed as $e(1+px+qx^3+...)$ where p and q are constants to be determined. **[6]**

(a)(i)

$$f(x) = \frac{2}{2-x} - \frac{1}{(1+x)^2} = \left(1-\frac{x}{2}\right)^{-1} - (1+x)^{-2}$$

$$= \left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+...\right) - \left(1-2x+3x^2+...\right)$$

$$= \frac{5x}{2} - \frac{11}{4}x^2 + ...$$

(a)(ii) Equation of the tangent to curve at origin is $y = \dfrac{5x}{2}$.

(b)

$$\sqrt{1+x}=1+\frac{1}{2}x+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!}+\ ...$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ ...$$

$$\mathrm{e}^{\sqrt{1+x}}=\mathrm{e}^{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+...}$$

$$=\mathrm{e}^1\mathrm{e}^{\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+...}$$

$$=\mathrm{e}\left(\begin{array}{l}1+\left(\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+...\right)\\+\frac{1}{2!}\left(\frac{1}{2}x-\frac{1}{8}x^2+...\right)^2\\+\frac{1}{3!}\left(\frac{1}{2}x+...\right)^3+...\end{array}\right)$$

$$=\mathrm{e}\left(\begin{array}{l}1+\frac{1}{2}x+\left(-\frac{1}{8}+\frac{1}{2}\left(\frac{1}{2}\right)^2\right)x^2\\+\left(\frac{1}{16}+\frac{1}{2}\left(-\frac{1}{8}\right)+\frac{1}{48}\right)x^3+...\end{array}\right)$$

$$=\mathrm{e}\left(1+\frac{1}{2}x+\frac{1}{48}x^3+...\right)$$

where $p=\frac{1}{2},q=\frac{1}{48}$

9 RVHS Promo 9758/2020/Q3

It is given that $\tan^{-1}(y)=1-e^{-x}$.

(i) Show that $e^x\dfrac{dy}{dx}=1+y^2$. [1]

(ii) By further differentiation of the expression in part (i), find the Maclaurin series for y, up to and including the term in x^2. [4]

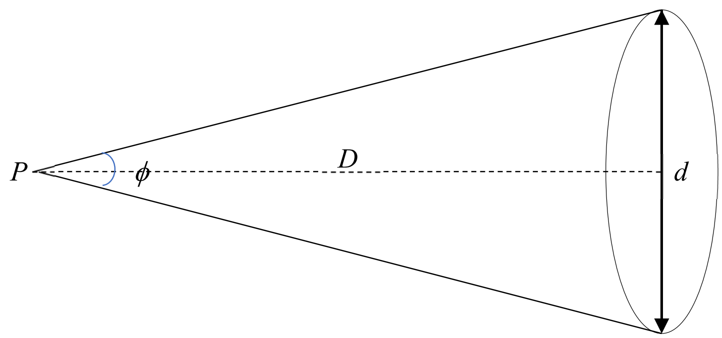
(iii) Hence, or otherwise, find the Maclaurin series for $\dfrac{1+y^2}{e^x}$, up to and including the term in x. [1]

(i)	**Method 1:** $\tan^{-1}(y)=1-e^{-x}$ Differentiate implicitly w.r.t. x, $\dfrac{1}{1+y^2}\left(\dfrac{dy}{dx}\right)=e^{-x}$ $e^x\dfrac{dy}{dx}=1+y^2$. (Shown) **Or** **Method 2:** $\tan^{-1}(y)=1-e^{-x}$ $y=\tan\left(1-e^{-x}\right)$ Differentiate w.r.t. x, $\dfrac{dy}{dx}=e^{-x}\sec^2\left(1-e^{-x}\right)$ $e^x\dfrac{dy}{dx}=1+\tan^2\left(1-e^{-x}\right)$ $e^x\dfrac{dy}{dx}=1+y^2$. (Shown)
(ii)	$e^x\dfrac{dy}{dx}=1+y^2$ (1) Differentiate implicitly w.r.t. x, $e^x\dfrac{dy}{dx}+e^x\dfrac{d^2y}{dx^2}=2y\dfrac{dy}{dx}$ (2) When $x=0, \tan^{-1}(y)=1-e^{-0}=0 \Rightarrow y=\tan(0)=0$, From (1): $e^0\dfrac{dy}{dx}=1+0^2 \Rightarrow \dfrac{dy}{dx}=1$, From (2):

	$1+0^2+e^0\dfrac{d^2y}{dx^2}=2(0)(1)\Rightarrow\dfrac{d^2y}{dx^2}=-1$, Hence, $y=0+(1)x+\left(\dfrac{-1}{2!}\right)x^2+\ldots$ $=x-\dfrac{1}{2}x^2+\ldots$
(iii)	$e^x\dfrac{dy}{dx}=1+y^2$ $\dfrac{dy}{dx}=\dfrac{1+y^2}{e^x}$. Hence, $\dfrac{1+y^2}{e^x}=\dfrac{dy}{dx}$ $=\dfrac{d}{dx}\left(x-\dfrac{1}{2}x^2+\ldots\right)$ $=1-x+\ldots$

10 EJC Prelim 9758/2020/01/Q11

(a) The angular diameter of an object is the angle the object makes (subtends) as seen by an observer.
As shown in the diagram below, ϕ denotes the angular diameter (measured in radians) of a circle whose plane is perpendicular to the line between the point of view (point P) and the centre of said circle. D denotes the distance from point P to the centre of the circle and d denotes the diameter of the circle.

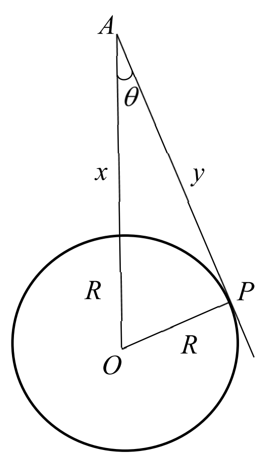


(i) Show that, if ϕ is sufficiently small, $\phi D = d$. [2]

The equation in **(i)** is often used in astronomy to estimate the diameters of stars from the angular diameter, assuming their shapes to be approximately circular.

(ii) If the angular diameter of a star is measured to be 0.00873 rad and the distance of the star from the earth is 9.46×10^{12} km, estimate the diameter of the star. [1]

(b) An astronaut A is at a large distance x km from the surface of the earth. The radius of the earth is assumed to be a constant R km. The furthest point on the earth's surface that the astronaut can see is a point P such that $AP = y$ km and the angle $OAP = \theta$, where O is the centre of the earth (see diagram).



(i) Show that $y = x\left(1+\dfrac{2R}{x}\right)^{\frac{1}{2}}$. [3]

(ii) It is given that R is small compared to x. Show that, if $\alpha = \dfrac{R}{x}$,
$\tan\theta \approx \alpha - \alpha^2 + 1.5\alpha^3$. [4]

(iii) It is also given that $\theta = 0.0345$ rad and the astronaut A is 180,000 km from the surface of the earth, find α and hence estimate the radius of the earth. Leave your answer to the nearest km. [3]

(a)(i) $\tan\left(\dfrac{\phi}{2}\right) = \dfrac{d}{2D}$ Since ϕ is small, $\dfrac{\phi}{2} \approx \dfrac{d}{2D} \Rightarrow \phi D = d$
(ii) $d = (0.00873)\left(9.46\times10^{12}\right) = 8.26\times10^{10}$ km
(b)(i) $R^2 + y^2 = \left(x+R\right)^2$ $R^2 + y^2 = x^2 + 2xR + R^2$ $y^2 = x^2 + 2xR$ $y^2 = x^2\left(1+\dfrac{2R}{x}\right) \Rightarrow y = x\left(1+\dfrac{2R}{x}\right)^{\frac{1}{2}}$
(ii) $\tan\theta = \dfrac{R}{y} = \dfrac{R}{x}\left(1+\dfrac{2R}{x}\right)^{-\frac{1}{2}} = \alpha\left(1+2\alpha\right)^{-\frac{1}{2}}$ Since R is small relative to x, then $\alpha = \dfrac{R}{x}$ is small $\tan\theta = \alpha\left(1+2\alpha\right)^{-\frac{1}{2}}$ $= \alpha\left(1 + \left(-\dfrac{1}{2}\right)(2\alpha) + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(2\alpha)^2 + \ldots\right)$ $\approx \alpha\left(1 - \alpha + 1.5\alpha^2\right)$ $= \alpha - \alpha^2 + 1.5\alpha^3$
(iii) $\theta = 0.0345$ $\tan\left(0.0345\right) = \alpha - \alpha^2 + 1.5\alpha^3$ From GC, we have $\alpha = 0.0357$ $\dfrac{R}{x} = 0.0357$ $R = 0.0357\left(180000\right) = 6426$ km

11 If $y = \ln(\cos 2x)$, prove that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -4$.

Hence, by repeated differentiation of this result, or otherwise, obtain the Maclaurin expansion of y in ascending powers of x up to and including the term in x^4.

Using $x = \dfrac{\pi}{6}$, show that $\ln 2 \approx \dfrac{\pi^2}{36}\left(2 + \dfrac{\pi^2}{27}\right)$. **[10]**

$$y = \ln(\cos 2x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin 2x}{\cos 2x} = -2\tan 2x$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -4\sec^2 2x = -4\left(\tan^2 2x + 1\right) = -4\tan^2 2x - 4$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\tan^2 2x = -4 \quad \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -4 \quad \text{(shown)}$$

Differentiate w.r.t x, $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = 0$

Differentiate w.r.t x, $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}\right) + 2\left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2 = 0$

At $x = 0$, $y = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -4$, $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 0$, $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} = -32$

The Maclaurin series is $y = -\dfrac{4}{2!}x^2 - \dfrac{32}{4!}x^4 + \ldots = -2x^2 - \dfrac{4}{3}x^4 + \ldots$

When $x = \dfrac{\pi}{6}$, $\ln\left(\cos\dfrac{\pi}{3}\right) \approx -2\left(\dfrac{\pi}{6}\right)^2 - \dfrac{4}{3}\left(\dfrac{\pi}{6}\right)^4$

$$\ln\left(\frac{1}{2}\right) \approx \frac{\pi^2}{36}\left(-2 - \frac{4}{3}\cdot\frac{\pi^2}{36}\right)$$

$$\ln 2 \approx \frac{\pi^2}{36}\left(2 + \frac{\pi^2}{27}\right) \quad \text{(shown)}$$

12 Given that $y=\dfrac{2}{\sqrt{1+\sin 2x}}$, wherever it is defined, show that

$$(1+\sin 2x)\frac{\mathrm{d}y}{\mathrm{d}x}+y\cos 2x=0.$$

Hence, by further differentiation of this result, obtain the Maclaurin expansion of y in ascending powers of x, up to the term in x^3.

By putting $x=\dfrac{\pi}{12}$, find an approximate value of $\sqrt{\dfrac{2}{3}}$, correct to 2 decimal places. **[10]**

$y=\dfrac{2}{\sqrt{1+\sin 2x}}$

$\Rightarrow y^2\left(1+\sin 2x\right)=4$

Differentiate w.r.t x, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(1+\sin 2x\right)+y^2\left(2\cos 2x\right)=0$

Since $y\neq 0$, dividng by $2y$, $(1+\sin 2x)\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\cos 2x=0$ (shown)

Differentiate w.r.t x, $(1+\sin 2x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(3\cos 2x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}-2y\sin 2x=0$

Differentiate w.r.t x,

$(1+\sin 2x)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(5\cos 2x\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-\left(8\sin 2x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}-4y\cos 2x=0$

At $x=0$, $y=2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-2$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=6$, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-22$

The Maclaurin series is $y=2-2x+\dfrac{6}{2!}x^2-\dfrac{22}{3!}x^3+\ldots=2-2x+3x^2-\dfrac{11}{3}x^3+\ldots$

Take $x=\dfrac{\pi}{12}$, $\dfrac{2}{\sqrt{1+\sin\left(\pi/6\right)}}\approx 2-2\left(\dfrac{\pi}{12}\right)+3\left(\dfrac{\pi}{12}\right)^2-\dfrac{11}{3}\left(\dfrac{\pi}{12}\right)^3$

$$\frac{2}{\sqrt{3/2}}\approx 2-\frac{\pi}{6}+\frac{\pi^2}{48}-\frac{11\pi^3}{5184}$$

$$\sqrt{\frac{2}{3}}\approx\frac{1}{2}\left(2-\frac{\pi}{6}+\frac{\pi^2}{48}-\frac{11\pi^3}{5184}\right)=0.81\text{ (2 d.p.)}$$

13 RI Prelim 9758/2020/02/Q4

Given that $y = \mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}$, show that $\left(4+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$. [2]

(i) By repeated differentiation of the above result, find the Maclaurin series for $\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}$ up to and including the term in x^3. [5]

(ii) Hence find the Maclaurin series for $\dfrac{\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}}{\left(1+x\right)^2}$ up to and including the term in x^2.

[3]

	$y = \mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)} \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2}$ $= \frac{2\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}}{4+x^2}$ $\left(4+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}$ $\left(4+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$ (Shown)
(i) **[5]**	$\left(4+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$ ---------- (1) $\left(4+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 2\frac{\mathrm{d}y}{\mathrm{d}x}$ ---------- (2) $\left(4+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $\left(4+x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ---------- (3) When $x=0$, $y=1$. From equations (1), (2), (3) we get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{4}$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\frac{1}{8}$. By Maclaurin's Theorem, $\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)} = 1 + \frac{1}{2}x + \frac{1}{4}\left(\frac{x^2}{2!}\right) - \frac{1}{8}\left(\frac{x^3}{3!}\right)$ $= 1 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \ldots$

(ii) **[3]**	$\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}=1+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{48}x^3+\ldots$ $\frac{\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}}{(1+x)^2}=\left(1+\frac{1}{2}x+\frac{1}{8}x^2-\ldots\right)(1+x)^{-2}$ $=\left(1+\frac{1}{2}x+\frac{1}{8}x^2-\ldots\right)\left(1-2x+3x^2-\ldots\right)$ $=1+\left(\frac{1}{2}-2\right)x+\left(\frac{1}{8}-1+3\right)x^2+\ldots$ $\frac{\mathrm{e}^{\tan^{-1}\left(\frac{x}{2}\right)}}{(1+x)^2}=1-\frac{3}{2}x+\frac{17}{8}x^2+\ldots$

14 ACJCPrelims9740/2006/01/Q16OR

Let $y = \ln\left(1 + \tan x\right)$

(i) Show that $\dfrac{dy}{dx} = e^{y} + 2e^{-y} - 2$. **[3]**

(ii) By differentiating this result repeatedly, show that

$$\frac{d^3 y}{dx^3} = \left(e^{y} - 2e^{-y}\right)\frac{d^2 y}{dx^2} + \left(e^{y} + 2e^{-y}\right)\left(\frac{dy}{dx}\right)^2 .$$ **[4]**

(iii) Find the series expansion of y in ascending powers of x up to and including the term in x^3. **[3]**

(iv) Show that, when x is small enough for powers above the third to be neglected,

$$\ln\left(\frac{1+\tan x}{1+x}\right) \approx \frac{1}{3}x^3 .$$ **[2]**

(i)	**Method 1:** $y = \ln(1+\tan x)$ $e^{y} = 1+\tan x$ Differentiate w.r.t x, $e^{y}\dfrac{dy}{dx} = \sec^2 x$ $= 1+\tan^2 x$ $= 1+(e^{y}-1)^2$ $= e^{2y} - 2e^{y} + 2$ $\therefore \dfrac{dy}{dx} = e^{y} + 2e^{-y} - 2$ **Method 2:** $y = \ln(1+\tan x)$ $\dfrac{dy}{dx} = \dfrac{\sec^2 x}{1+\tan x} = \dfrac{1+\tan^2 x}{1+\tan x}$ Since $\tan x = e^{y} - 1$, $\dfrac{dy}{dx} = \dfrac{1+\left(e^{y}-1\right)^2}{1+\left(e^{y}-1\right)}$ $= \dfrac{e^{2y} - 2e^{y} + 2}{e^{y}}$ $= e^{y} + 2e^{-y} - 2$ (shown)
(ii)	Differentiating w.r.t. x, $\dfrac{d^2 y}{dx^2} = \left(e^{y} - 2e^{-y}\right)\dfrac{dy}{dx}$ Differentiating w.r.t. x, $\dfrac{d^3 y}{dx^3} = \left(e^{y} - 2e^{-y}\right)\dfrac{d^2 y}{dx^2} + \left(e^{y} + 2e^{-y}\right)\left(\dfrac{dy}{dx}\right)^2$
(iii)	At $x = 0$, $y = 0$, $\dfrac{dy}{dx} = 1$, $\dfrac{d^2 y}{dx^2} = -1$, $\dfrac{d^3 y}{dx^3} = 4$ $\therefore \ln(1+\tan x) = 0 + 1x + \dfrac{-1}{2}x^2 + \dfrac{4}{6}x^3 + \ldots = x - \dfrac{1}{2}x^2 + \dfrac{2}{3}x^3 + \ldots$
(iv)	$\ln\left(\dfrac{1+\tan x}{1+x}\right) = \ln\left(1+\tan x\right) - \ln(1+x)$ $= (x - \dfrac{1}{2}x^2 + \dfrac{2}{3}x^3 + \ldots) - (x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \ldots) \approx \dfrac{x^3}{3}$ (shown)

15 NJCPrelims9740/2006/01/Q6

Given that a curve is defined by

$$\left(\ln y+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\tan^{-1}\left(2x\right)=1$$

and passes through (0,1).

Find the Maclaurin Series for y, up to and including the term in x^3. **[5]**

Deduce the equation of tangent to the curve at $x=0$. **[1]**

$\left(\ln y+1\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}+\tan^{-1}\left(2x\right)=1$

Differentiate w.r.t. x, $\left(\ln y+1\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{1}{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\dfrac{2}{1+4x^2}=0$

Differentiate w.r.t. x,

$\left(\ln y+1\right)\left(\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)+\dfrac{3}{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-\dfrac{1}{y^2}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^3-\dfrac{16x}{\left(1+4x^2\right)^2}=0$

When $x=0,\quad y=1,\quad \dfrac{\mathrm{d}y}{\mathrm{d}x}=1,\quad \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-3,\quad \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=10$

The Maclaurin series is $y=1+\dfrac{1}{1!}x+\dfrac{\left(-3\right)}{2!}x^2+\dfrac{10}{3!}x^3+\ldots\quad =1+x-\dfrac{3}{2}x^2+\dfrac{5}{3}x^3+\ldots$

Equation of tangent to the curve at $x=0$ is $y=1+x$

16 ACJCPrelims9233/2005/01/Q14

If $y = \mathrm{e}^{\cos^{-1} x}$, where $\cos^{-1} x$ denotes the principal value, show that

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

By further differentiation, obtain the expansion of y in ascending powers of x up to and including the term in x^3 in terms of $\mathrm{e}^{\frac{\pi}{2}}$. Hence show that for small values of x,

$$\mathrm{e}^{\cos^{-1} x} \approx \mathrm{e}^{\frac{\pi}{2} - x} + kx^3,$$

where the value of k is to be found. **[9]**

$y = \mathrm{e}^{\cos^{-1} x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{\sqrt{1-x^2}}\mathrm{e}^{\cos^{-1} x} \Rightarrow \sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{\cos^{-1} x}$

$\sqrt{1-x^2}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{2x}{2\sqrt{1-x^2}}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}\mathrm{e}^{\cos^{-1} x}$

$\left(1-x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\cos^{-1} x} \Rightarrow \left(1-x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$

Differentiating wrt x, $\left(1-x^2\right)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

When $x = 0$, $y = \mathrm{e}^{\frac{\pi}{2}}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{\frac{\pi}{2}}$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{e}^{\frac{\pi}{2}}$, $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -2\mathrm{e}^{\frac{\pi}{2}}$

By Maclaurin Theorem,

$y = \mathrm{e}^{\cos^{-1} x} = \mathrm{e}^{\frac{\pi}{2}}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \ldots\right)$

$\mathrm{e}^{\frac{\pi}{2} - x} = \mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^{-x} = \mathrm{e}^{\frac{\pi}{2}}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right)$

$\mathrm{e}^{\cos^{-1} x} \approx \mathrm{e}^{\frac{\pi}{2}}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right) - \frac{1}{6}\mathrm{e}^{\frac{\pi}{2}}x^3$

$= \mathrm{e}^{\frac{\pi}{2} - x} - \frac{1}{6}\mathrm{e}^{\frac{\pi}{2}}x^3$

Hence, $k = -\frac{1}{6}\mathrm{e}^{\frac{\pi}{2}}$

17 TJCPrelims9740/2006/01/Q1 (modified)

The equation $4\cos 2x + 10\sin 3x + 3 = 0$ has a root α close to zero. By using suitable approximations, find an approximate value for α. **[4]**

$4\cos 2\alpha + 10\sin 3\alpha + 3 = 0$

Since α is small, $4\left[1 - \frac{1}{2}(2\alpha)^2\right] + 10(3\alpha) + 3 \approx 0$

$8\alpha^2 - 30\alpha - 7 \approx 0$

$\alpha \approx -0.220$ or $\alpha \approx 3.97$

Since the root α is close to zero, $\alpha \approx -0.220$ (3 s.f.)

18 NYJCPrelims9740/2006/01/Q1

Show that, if x is so small that terms in x^3 and higher powers may be neglected, then

$$\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right) \approx \frac{1}{2} - x^2. \quad \textbf{[3]}$$

$$\begin{aligned}
&\sin\left(\frac{\pi}{4}+x\right)\sin\left(\frac{\pi}{4}-x\right)\\
&=\left(\sin\frac{\pi}{4}\cos x+\cos\frac{\pi}{4}\sin x\right)\left(\sin\frac{\pi}{4}\cos x-\cos\frac{\pi}{4}\sin x\right)\\
&=\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)\\
&=\frac{1}{2}\left(\cos^2 x-\sin^2 x\right)\\
&=\frac{1}{2}\cos 2x\\
&\approx\frac{1}{2}\left(1-\frac{(2x)^2}{2}\right) \text{ since } x \text{ is small}\\
&=\frac{1}{2}-x^2 \quad \text{(shown)}
\end{aligned}$$

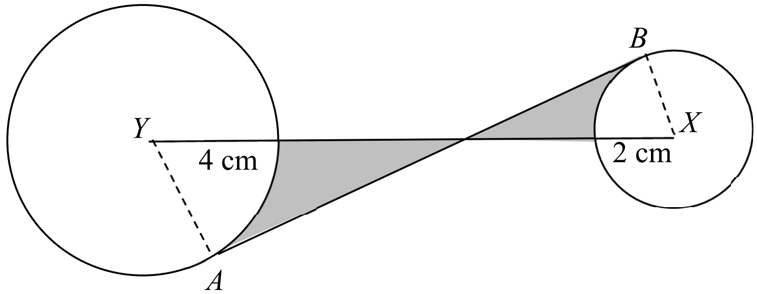
19 **NJCPrelims9740/2004/01/Q4**

The diagram below shows two circles of radius 2 cm and 4 cm, with centre X and Y respectively. The line AB is tangent to both circles and angle AYX is $\left(\frac{\pi}{3}-x\right)$ radians.

Show that the area of the shaded region is $10\left(\tan\left(\frac{\pi}{3}-x\right)-\left(\frac{\pi}{3}-x\right)\right)$.

If x is sufficiently small for x^3 and higher powers of x to be neglected, find the approximate area of the shaded region in terms of x. **[6]**

[You may use the result: $\left(1+\sqrt{3}\,x\right)^{-1} \approx 1-\sqrt{3}\,x+3x^2$].



Area of shaded region

$$=\frac{1}{2}(4)\left[4\tan\left(\frac{\pi}{3}-x\right)\right]-\frac{1}{2}\left(4^2\right)\left(\frac{\pi}{3}-x\right)+\frac{1}{2}(2)\left[2\tan\left(\frac{\pi}{3}-x\right)\right]-\frac{1}{2}\left(2^2\right)\left(\frac{\pi}{3}-x\right)$$

$$=8\tan\left(\frac{\pi}{3}-x\right)-8\left(\frac{\pi}{3}-x\right)+2\tan\left(\frac{\pi}{3}-x\right)-2\left(\frac{\pi}{3}-x\right)$$

$$=10\left(\tan\left(\frac{\pi}{3}-x\right)-\left(\frac{\pi}{3}-x\right)\right) \quad \text{(shown)}$$

Method 1: Find Maclaurin series

Let $y=10\left(\tan\left(\frac{\pi}{3}-x\right)-\left(\frac{\pi}{3}-x\right)\right)$

Differentiating w.r.t x, $\frac{dy}{dx}=10\left(-\sec^2\left(\frac{\pi}{3}-x\right)-(0-1)\right)$

Differentiating w.r.t x, $\frac{d^2y}{dx^2}=10\left(-2\sec^2\left(\frac{\pi}{3}-x\right)\tan\left(\frac{\pi}{3}-x\right)(-1)\right)$

When $x=0$, $y=10\left(\sqrt{3}-\frac{\pi}{3}\right)$, $\frac{dy}{dx}=-30$, $\frac{d^2y}{dx^2}=80\sqrt{3}$

Required Area $\approx 10\left(\sqrt{3}-\frac{\pi}{3}-3x+4\sqrt{3}\,x^2\right)$

Method 2: Use standard series of $\sin x$ and $\cos x$

$$\text{Required Area} = 10\left[\frac{\sin\left(\frac{\pi}{3}-x\right)}{\cos\left(\frac{\pi}{3}-x\right)} - \left(\frac{\pi}{3}-x\right)\right]$$

$$= 10\left[\frac{\sqrt{3}\cos x - \sin x}{\cos x + \sqrt{3}\sin x} - \left(\frac{\pi}{3}-x\right)\right]$$

$$\approx 10\left[\frac{\sqrt{3}\left(1-\frac{x^2}{2}\right) - x}{\left(1-\frac{x^2}{2}\right) + \sqrt{3}\,x} - \left(\frac{\pi}{3}-x\right)\right]$$

$$\approx 10\left[\left(\sqrt{3} - x - \frac{\sqrt{3}x^2}{2}\right)\left(1-\sqrt{3}x+\frac{x^2}{2}+\left(-\sqrt{3}x+...\right)^2\right) - \left(\frac{\pi}{3}-x\right)\right]$$

$$\approx 10\left[\left(\sqrt{3} - x - \frac{\sqrt{3}x^2}{2}\right) + \left(-3x+\sqrt{3}x^2\right) + \frac{\sqrt{3}x^2}{2} + 3\sqrt{3}x^2 - \frac{\pi}{3} + x\right]$$

$$\approx 10\left(\sqrt{3} - \frac{\pi}{3} - 3x + 4\sqrt{3}\,x^2\right)$$

20 HCIPrelims9740/2021/01/Q9

Given that $y = (\tan x + \sec x)^2$, show that $\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y$. [2]

(i) By repeat differentiation, find the Maclaurin series for y up to and including the term in x^3. [4]

(ii) Using the result obtained in (i) estimate the value of $\tan 1^{\circ} + \sec 1^{\circ}$ to 4 decimal places. [2]

(iii) By expressing y in terms of sine and cosine, use the standard series in MF26 to find the series expansion for y up to and including the term in x^3. [3]

(iv) Comment on the results obtained from part **(i)** and part **(iii)**. [1]

21	$y = (\tan x + \sec x)^2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2(\tan x + \sec x)(\sec^2 x + \sec x \tan x)$ $\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2(\tan x + \sec x)(\sec x + \tan x)$ $\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y$

(i)	Diff wrt x: $-\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x}+\cos x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\dfrac{\mathrm{d}y}{\mathrm{d}x}$ Diff wrt x: $-\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}-\sin x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\cos x\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}-\sin x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Sub $x=0$ $y=1,\dfrac{\mathrm{d}y}{\mathrm{d}x}=2,\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4,\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=10$ Let $y=\mathrm{f}(x)$ $y=\mathrm{f}(0)+\mathrm{f}\,'(0)x+\dfrac{\mathrm{f}\,''(0)}{2}x^2+\dfrac{\mathrm{f}\,'''(0)}{3!}x^3+...$ $=1+2x+2x^2+\dfrac{5}{3}x^3+...$
(ii)	$1^\circ=\dfrac{\pi}{180}$ Sub $x=\dfrac{\pi}{180}$ $\tan 1^\circ+\sec 1^\circ=\sqrt{1+2(\dfrac{\pi}{180})+2(\dfrac{\pi}{180})^2+\dfrac{5}{3}(\dfrac{\pi}{180})^3}$ $=\sqrt{1.03552468}$ $=1.0176(4\text{d.p.})$
(iii)	$y=(\dfrac{\sin x+1}{\cos x})^2$ $=(1+\sin x)^2(\cos x)^{-2}$ $=(1+x-\dfrac{x^3}{6})^2(1-\dfrac{x^2}{2})^{-2}$ $=[1+2(x-\dfrac{x^3}{6})+(x-\dfrac{x^3}{6})^2](1+x^2+...)$ $=(1+2x+x^2-\dfrac{1}{3}x^3)(1+x^2+...)$ $=1+2x+2x^2+\dfrac{5}{3}x^3+...$
(iv)	Answers from both parts are consistent