Complex Numbers

1. ACJC/2016/I/8 (part of)

The complex number z is given by $z = k + \mathrm{i}$ where k is a non-zero real number.

(i) Find the possible values of k if $z = k + \mathrm{i}$ satisfies the equation $z^3 - \mathrm{i}z^2 - 2z - 4\mathrm{i} = 0$. [3]

2. ACJC Prelim 9758/2023/01/Q2

Do not use a calculator in answering this question.

The complex numbers z and w satisfy the following equations.

$$w + z^* = -2 + 4\mathrm{i}$$
$$z + 2 = 3\mathrm{i}w$$

Find z and w, giving your answers in the form $a + \mathrm{i}b$, where a and b are real numbers. [4]

3. CJC Prelim 9758/2023/01/Q1

The complex numbers z and w satisfy the following equations.

$$2z + 1 = |w|$$

$$2w - z = 4 + 24\mathrm{i}$$

Find z and w, giving your answers in the form $a + \mathrm{i}b$ where a and b are real numbers. [4]

4. CJC/2021/II/3 (part of)

Do not use a calculator in answering this question.

It is given that z_1 and z_2 are the roots of the equation $z^2 - 2\mathrm{i}z - 2 = 0$, where $\arg(z_1) < \arg(z_2)$.

(i) Show that $z_1 = 1 + \mathrm{i}$. [3]

(ii) Given that $x = z_1$, find x^2, x^3 and x^4 in cartesian form. Given also that $x = z_1$ is a root of the equation $x^4 - 6x^3 + sx^2 - 18x + 10 = 0$, where s is real, find the value of s and the other roots of the equation. [6]

5. CJC/2016/I/3

The cubic equation $x^3 + ax^2 + bx + c = 0$, where a, b and c are constants, has roots $3 + \mathrm{i}$ and 2.

(i) One JC2 student remarked that the third root is 3 − i. State a necessary assumption the student made in order that the remark is true. [1]

(ii) Given that the assumption in part **(i)** holds, find the values of a, b and c. [4]

6. EJC Prelim 9758/2025/P1/Q8

Do not use a calculator in answering this question.

The complex number w is such that $w^2 = 2\text{i}$ and $0 \le \arg(w) \le \frac{\pi}{2}$.

(a) Find w. [3]

(b) Given that one of the roots of the equation $z^3 - z^2 + (1-\text{i})z + s = 0$ is w, find the other roots of the equation and the value of s. [5]

Hence, find the roots of the equation $-\text{i}v^3 + v^2 + (1+\text{i})v + s = 0$. [2]

7. DHS/2016/I/2

The complex number w is such that $kw^2 + kww^* + \text{i}w - \text{i}w^* - 1 = 0$, where w^* is the complex conjugate of w and k is a real and non-zero constant.

(i) For $w = a + b\text{i}$ where a and b are real numbers, obtain an expression for b in terms of a and k. Explain why w is either purely real or purely imaginary. [4]

(ii) Using your result in part (i), or otherwise, find the real roots of the equation $2w^2 + 2ww^* + \text{i}w - \text{i}w^* - 1 = 0$ [2]

8. CJC/2009/I/8(a)

(a) One of the roots of the equation $z^4 - z^3 + 4z^2 + 3z + p = 0$ is 1 – 2i, where p is a constant. A student says that "One of the other roots must be 1 + 2i." Explain why this statement is not entirely correct. [1]

(i) Determine the value of p. [1]

(ii) Find the exact values of all the other roots of the equation. [3]

9. IJC/2016/I/8(b) (modified)

(b) (i) If $z = \cos\theta + \text{i}\sin\theta$, where $0 \le \theta \le \frac{\pi}{2}$, show that $1 - z^2 = 2\sin\theta(\sin\theta - \text{i}\cos\theta)$. [2]

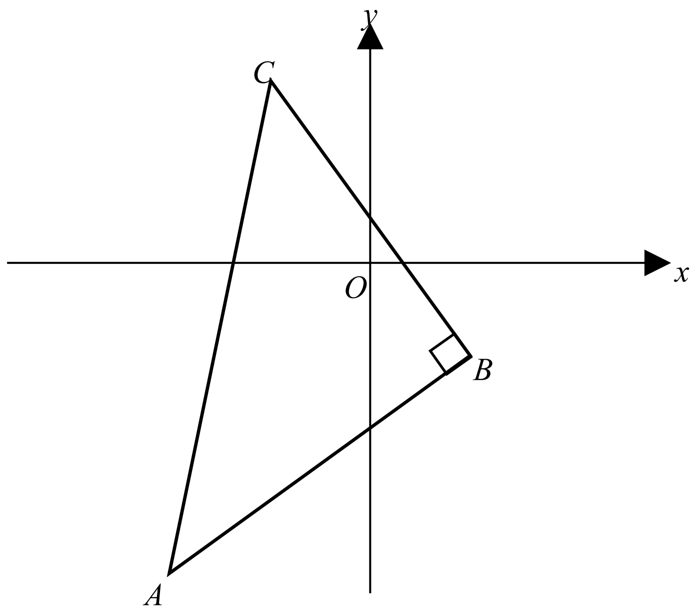
(ii) Hence find $\left|1 - z^2\right|$ in terms of θ. [3]

10. The complex numbers a and b are given by $a = -\left(1+\sqrt{3}\mathrm{i}\right)$ and $b = \frac{1}{2}(1-\mathrm{i})$.

(i) Without using a calculator, find the value of $a^2 b$ in the form $x + \mathrm{i}y$. [2]

(ii) Find the modulus and argument of $a^2 b$. [3]

(ii) The diagram below shows an isosceles right triangle ABC, where the points A, B and C represent the complex numbers a, b and c respectively. Find the exact value of c. [2]



11. MI/2016/I/12 (modified)

(i) For real a and b, one root of the equation $az^3 - 9z^2 + bz - 5 = 0$ is $z = 2 - \mathrm{i}$. Find the values of a and b. [4]

(ii) Hence find all the roots of the equation in part **(i)** in exact form. [4]

12. AJC/2011/I/13

(i) The complex numbers p and q satisfy the simultaneous equations

$$p^* + 10\mathrm{i} = q\mathrm{i} + 5$$

$$|p|^2 - q - 5 + 2\mathrm{i} = 0.$$

Given that Im(p) < 0, find p in the Cartesian form. [3]

Hence find the values of n for which p^{2n} is purely imaginary. [2]

(ii) The complex number w is such that $\arg\left(\frac{w}{p} - p^*\right) = -\frac{\pi}{2}$ and $w + w^* = -2$.

Find w. [4]

13. MJC/2016/I/10

(a) Solve the simultaneous equations

$$z = w + 2\mathrm{i} - 1 \quad \text{and} \quad z^2 - \mathrm{i}w + \frac{5}{2} = 0,$$

giving z and w in the form $x + y\mathrm{i}$ where x and y are real. [5]

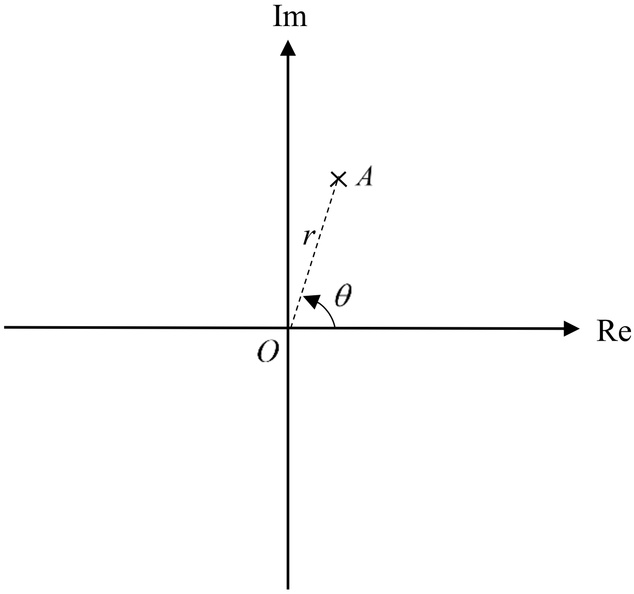
(b) Given that $z = w - \dfrac{1}{w}$ where $w = 2(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \le \pi$, express the real and imaginary parts of z in terms of θ. [3]

14. RI Promo 9758/2025/Q3

The point A on the Argand diagram below represents the complex number w_1 with modulus r and argument θ. It is given that the complex numbers w_2 and w_3 satisfy the equations

$$w_2 = -w_1 \quad \text{and} \quad w_3 = \mathrm{i}w_2.$$

Let B, C and D represent the complex numbers w_2, w_3 and $w_2 + w_3$ respectively.



(a) On the copy of the Argand diagram in the Printed Answer Book, plot the points B, C and D, indicating clearly the modulus and argument of w_2 and w_3. [3]

(b) State, in radians, the angle BOC. [1]

(c) By considering the quadrilateral $OBDC$, find $|w_2 + w_3|$ in terms of r and $\arg(w_2 + w_3)$ in terms of θ. [2]

15. NYJC/2011/I/8

A complex number $z = x + \mathrm{i}y$ is represented by the point P in an Argand diagram. If the complex number w, where $w = \dfrac{z - 8\mathrm{i}}{z + 6}$, ($z \ne -6$), has its real part zero, show that the locus of P in the Argand diagram is a circle, and find the radius and the coordinates of the centre of this circle. If, however, w is real, find the locus of P in this case. [6]

16. PJC/2011/I/9(modified)

The polynomial $\mathrm{P}(z)$ has real coefficients. The equation $\mathrm{P}(z)=0$ has a root $x+\mathrm{i}y$, where x and y are real numbers.

(i) Write down a second root in terms of x and y, and hence show that a quadratic factor of $\mathrm{P}(z)$ is $z^2-2xz+x^2+y^2$. [3]

(ii) z_1 and z_2 are roots of the equation $\mathrm{P}(z)=0$, where $z_1=1+\sqrt{3}\mathrm{i}$ and $z_2=\mathrm{i}z_1$. Write down the exact modulus and argument of z_2. State the geometrical relationship between z_1 and z_2 and illustrate this relationship clearly on an Argand diagram. [3]

(iii) Given further that $\mathrm{P}(z)$ is of degree four, express $\mathrm{P}(z)$ as a product of two quadratic factors with real coefficients, giving each factor in exact form. [3]

17. ACJC/2014/I/4

(i) Given that $1+\mathrm{i}$ is a root of the equation $2w^3+aw^2+bw-2=0$, find the values of the real numbers a and b. [4]

(ii) For these values of a and b, solve the equation in part **(ii)**, without the use of a calculator. [3]

18. SRJC/2016/II/3(a)

(a) The complex number w is such that $w = a + \mathrm{i}b$, where a and b are non-zero real numbers. The complex conjugate of w is denoted by w*. Given that $\dfrac{(w^*)^2}{w}=3-\mathrm{i}b$, solve for a and b and hence write down the possible values of w. [3]

19. TJC/2016/I/12(a)

(a) The complex numbers z_1 and z_2 satisfy the following simultaneous equations

$$2z_1+\mathrm{i}\,z_2^{\,*}=7-6\mathrm{i},$$
$$z_1-\mathrm{i}z_2=6-6\mathrm{i}.$$

Find z_1 and z_2 in the form $x+y\mathrm{i}$, where x and y are real. [4]

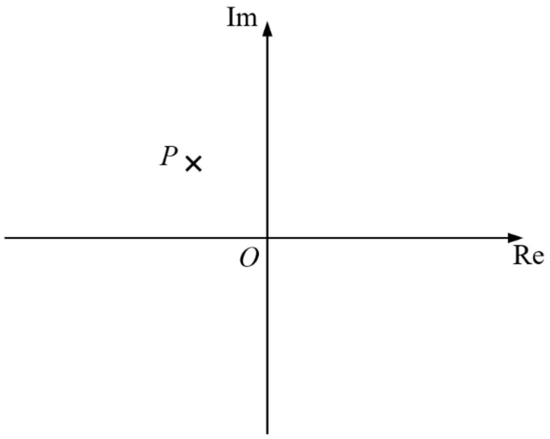
20. VJC/2016/II/3

The points A, B, C and D represent the complex numbers $-2+5\mathrm{i}$, z_1 , $4+\mathrm{i}$ and z_2 respectively. Given that $ABCD$ is a square, labelled in an anti-clockwise direction, show that $z_1=-1$. Find z_2. [4]

21. VJC Prelim 9758/2025/P2/Q4

Do not use a calculator in answering this question.

The complex number z has modulus 1 and argument θ, where $\frac{\pi}{2} < \theta < \pi$, and the complex number w is given by $w = \mathrm{i}\sqrt{3}z$. The point P on the Argand diagram represents z.



(a) On the copy of the Argand diagram with origin O in the Printed Answer Booklet, plot the points Q and R to represent w and $z - w$ respectively. Show clearly the geometrical relationship between the points P, Q and R. [3]

(b) Find the area of the quadrilateral $ORPQ$. [1]

It is given that $\theta = \frac{3\pi}{4}$.

(c) Find z in the form $x + y\mathrm{i}$, where x and y are real numbers. [2]

(d) Show that $z - w = k\left[\left(\sqrt{3}-1\right)+\left(\sqrt{3}+1\right)\mathrm{i}\right]$, where k is a constant to be determined. [2]

(e) Hence show that $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. [1]

22. CJC/2015/I/1

A calculator is not to be used in answering this question.

By considering $w = z^2 - z$ or otherwise, solve

$$z^4 - 2z^3 - 2z^2 + 3z - 10 = 0 \quad \text{where} \quad z \in \mathbb{C},$$

leaving the roots in exact form. [4]

23. ACJC/2011/II/3

A complex number z is represented by the point P on an Argand diagram. Given that $z = (1+\mathrm{i})(t-2) + \frac{1-\mathrm{i}}{t(1+\mathrm{i})}$ where t is real,

(i) find the imaginary part of z, [2]

(ii) find the Cartesian equation of the locus of point P, [1]

(iii) sketch the locus of point P in an Argand diagram, stating clearly any intercepts and asymptotes exactly. [2]

24. RJC/2014/I/8

One root of the equation $z^2 + az^* + b = 0$, where a and b are real, is w. Show that w^* is also a root of this equation. [2]

Solve the equation $z^2 + 6z^* + 9 = 0$, giving your answers in the form $x + \mathrm{i}y$. [4]

25. DHS/2018/1/7(b)

(b) When the polynomial $ax^4 + bx^3 + cx^2 + 24x - 44$, where a, b and c $\in \mathbb{R}$, is divided by $(x-1)$, $(x+1)$ and $(x-2)$, the remainders are -18, -54 and 0 respectively.

(i) Find the values of a, b and c. [3]

(ii) The equation $ax^4 + bx^3 + cx^2 + 24x - 44 = 0$, with the values of a, b and c found in part **(i)**, has a root $3-(\sqrt{2})\mathrm{i}$. Find the other roots of the equation, showing your working clearly. [3]

26. DHS/2019/II/3

The complex number z is such that $az^2 + bz + a = 0$ where a and b are real constants. It is given that $z = z_0$ is a solution to this equation where $\mathrm{Im}(z_0) \neq 0$.

(i) Verify that $z = \dfrac{1}{z_0}$ is the other solution. Hence show that $|z_0| = 1$. [4]

Take $\mathrm{Im}(z_0) = \frac{1}{2}$ for the rest of the question.

(ii) Find the possible complex numbers for z_0. [2]

(iii) If $\mathrm{Re}(z_0) > 0$, find b in terms a. [2]

27. ACJC Prelim/2022/02/Q3

(i) Find the roots of the equation $\mathrm{i}z^2 - (5+\mathrm{i})z + 2 - 6\mathrm{i} = 0$, giving your answers in cartesian form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$. [2]

(ii) Hence find the roots of the equation $-\mathrm{i}w^2 - (1-5\mathrm{i})w + 2 - 6\mathrm{i} = 0$, giving your answers in cartesian form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$. [2]

(iii) Given that the roots found in part (i) are also roots of the equation $P(z) = 0$, where $P(z)$ is a polynomial of degree 4 with real coefficients, find $P(z)$. [3]

28. VJC Prelim 9758/2023/02/Q4(b)

(b) Given that the modulus of the complex number $\dfrac{w-2\mathrm{i}}{3+2\mathrm{i}w}$ is $\dfrac{1}{\sqrt{3}}$, find the value of $|w|$, leaving your answer in exact form. [4]

(You may assume the fact that $\left|\dfrac{u}{v}\right| = \dfrac{|u|}{|v|}$.)

29. DHS Prelim 9758/2024/01/Q7

The points A, B and C represent the complex numbers a, b and c respectively, such that $a = 0$, $b = 3$ and $c = -2 + \mathrm{i}$. The three complex numbers are roots to the equation $\mathrm{f}(z) = 0$ where $\mathrm{f}(z)$ is a quartic polynomial with real coefficients and z is a complex variable.

(a) Express $\mathrm{f}(z)$ as a product of two quadratic factors with real coefficients. [3]

(b) Sketch an Argand diagram showing the roots of the equation $\mathrm{f}(z) = 0$. [2]

(c) The point W represents the complex number w, such that $c = \mathrm{i}\,w$. Find the value of $\overrightarrow{AC} \bullet \overrightarrow{AW}$ and the area of the triangle APW where P represents the complex number $c - b$. [4]

30. TJC Prelim 9758/2024/02/Q5(a)

(a) The complex number $z = x + \mathrm{i}y$ is represented by the point $P(x, y)$ in an Argand diagram and satisfies the equation

$$zz^* = (2 + 3\mathrm{i})z^* + (2 - 3\mathrm{i})z + 12 \quad .$$

(i) Show that P is a point on a circle, and state the centre and radius of the circle. [3]

(ii) The point Q represents the complex number $-4 - 5\mathrm{i}$. Find the smallest possible length PQ. [2]

Answers

1 (i) $k = \pm\sqrt{3}$

2 $z = -0.5 - 4.5\text{i}$, $w = -1.5 - 0.5\text{i}$

3 $z = 6$, $w = 5 + 12\text{i}$

4 (ii) $s = 15$; $1 - \text{i}$, $2 + \text{i}$ and $2 - \text{i}$

5 (ii) $a = -8$, $b = 22$ and $c = -20$

6 (a) $w = 1 + \text{i}$, (b) $z = 0$ or $z = -\text{i}$, $s = 0$, (c) $v = 1 - \text{i}$ or 0 or -1

7 (i) $b = \dfrac{2ka^2 - 1}{2}$ (ii) $w = -\dfrac{1}{2}$ or $w = \dfrac{1}{2}$

8 (a) The statement is only true if p is real.
(i) Using GC, $p = 5$. (ii) $z = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

9 (bii) $\left|1 - z^2\right| = 2\sin\theta, \arg\left(1 - z^2\right) = \theta - \dfrac{\pi}{2}$

10 (i) $\left(\sqrt{3} - 1\right) + \left(\sqrt{3} + 1\right)\text{i}$ (ii) $2\sqrt{2}$; $\dfrac{5\pi}{12}$ (ii) $\left(1 - \sqrt{3}\right) + \text{i}$

11 (i) $a = 2, b = 14$ (ii) $2 - \text{i}$, $2 + \text{i}$ and $\dfrac{1}{2}$

12 (i) $p = 3 - 3i$, $n = 2k + 1,\ k \in \mathbb{Z}$ (ii) $w = -1 - 19i$

13 (a) $w = \dfrac{3}{2} - \dfrac{1}{2}\text{i}$, $z = \dfrac{1}{2} + \dfrac{3}{2}\text{i}$ or $w = \dfrac{1}{2} - \dfrac{5}{2}\text{i}$, $z = -\dfrac{1}{2} - \dfrac{1}{2}\text{i}$
(b) $\text{Re}(z) = \dfrac{3}{2}\cos\theta$, $\text{Im}(z) = \dfrac{5}{2}\sin\theta$

14 (b) $\dfrac{\pi}{2}$ (c) $\sqrt{2}r$, $-\dfrac{3\pi}{4} + \theta$

15 $(x+3)^2 + (y-4)^2 = 5^2$, locus is a circle of centre (-3,4) and radius 5 ;
locus of P: $3y - 4x = 24$

16 (i) $z = x - \text{i}y$ is another root
(ii) $\left|z_2\right| = 2$, $\arg\left(z_2\right) = \dfrac{5\pi}{6}$, z_2 is an anti-clockwise rotation of z_1 about the origin by $\dfrac{\pi}{2}$.
(iii) $\left(z^2 - 2z + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right)$

17 (i) $a = -5$, $b = 6$ (ii) $w = 1 + \text{i}$, $1 - \text{i}$, $\dfrac{1}{2}$.

18 (a) $a = -3$; $b = \pm 3$; $-3 \pm 3\text{i}$

19 (a) $1 - 4\text{i}$; $2 + 5\text{i}$

20 $z_2 = 3 + 6\text{i}$

21 $\sqrt{3}$; $z=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}$; $k=\frac{1}{\sqrt{2}}$

22 $\frac{1\pm\sqrt{21}}{2}$, $\frac{1\pm\sqrt{7}\mathrm{i}}{2}$

23 (i) $\mathrm{Im}(z)=t-2-\frac{1}{t}$ (ii) $y=x-\frac{1}{x+2}$, $\frac{3\sqrt{29}}{2}$

24 $z=-3$, $3+6\mathrm{i}$, $3-6\mathrm{i}$

25 (b)(i) $a=1,\ b=-6,\ c=7$ (b)(ii) $3+(\sqrt{2})\mathrm{i}$, 2 and -2

26 (ii) $z_0=\frac{\sqrt{3}}{2}+\mathrm{i}\frac{1}{2}$ or $-\frac{\sqrt{3}}{2}+\mathrm{i}\frac{1}{2}$ (iii) $b=-\sqrt{3}a$

27 (i) $1-3\mathrm{i}$ or $-2\mathrm{i}$ (ii) $w=3+\mathrm{i}$ or 2 (iii) $z^4-2z^3+14z^2-8z+40$

28 $\sqrt{3}$

29 (a) $(z^2-3z)(z^2+4z+5)$ (c) 0, area = 5.5 units^2

30 (a)(i) centre (2, 3) and radius 5 (ii) 5