2024 NYJC JC2 Common Test Exam 9649/2 Solution

SOLUTIONS:

Qn	Suggested Solution	
1(a)	Equation of line l is $y=\left(\tan\theta\right)x$. Unit direction vector of l is $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$. Projection of $\begin{pmatrix}1\\0\end{pmatrix}$ onto $l=\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right]\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ $=\cos\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ Projection of $\begin{pmatrix}0\\1\end{pmatrix}$ onto $l=\left[\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right]\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ $=\sin\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ $\mathbf{A}=(\mathbf{A}(\mathbf{e}_1))\quad \mathbf{A}(\mathbf{e}_2))$ $\mathbf{A}=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$.	

(b)	$\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$ $\cos^2\theta x+\sin\theta\cos\theta y=0$ $x=-\tan\theta y$ Basis for nullspace for T is $\left\{\begin{pmatrix}-\tan\theta\\ 1\end{pmatrix}\right\}$ [or $\left\{\begin{pmatrix}1\\ -\cot\theta\end{pmatrix}\right\}$] Alternatively, Note that all position vectors that are perpendicular l will be mapped to the zero vector. Gradient of $l=\tan\theta$ Equation of line perpendicular to l: $y=(-\cot\theta)x$. Basis for nullspace for T is $\left\{\begin{pmatrix}1\\ -\cot\theta\end{pmatrix}\right\}$.	
(c)	$\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}1\\ 5\end{pmatrix}=\begin{pmatrix}\frac{3+5\sqrt{3}}{4}\\ \frac{\sqrt{3}+5}{4}\end{pmatrix}$ $\cos^2\theta+5\sin\theta\cos\theta=\frac{3+5\sqrt{3}}{4}$ --- (1) $\sin\theta\cos\theta+5\sin^2\theta=\frac{\sqrt{3}+5}{4}$ --- (2) **Method 1:** $(1)\times5+(2)$: $5+26\sin\theta\cos\theta=\frac{1}{4}(20+26\sqrt{3})$ $13\sin2\theta=\frac{13}{2}(\sqrt{3})$ $\sin2\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{6}$ or $\frac{\pi}{3}$ When $\theta=\frac{\pi}{3}$, $LHS=\cos^2\frac{\pi}{3}+5\sin\frac{\pi}{3}\cos\frac{\pi}{3}=2.42$, $RHS=2.92$ for (1), thus rejected. Thus $\theta=\frac{\pi}{6}$ **Method 2:** $\frac{(2)}{(1)}$: $\frac{\sin\theta(\cos\theta+5\sin\theta)}{\cos\theta(\cos\theta+5\sin\theta)}=\tan\theta=\frac{\left(\frac{\sqrt{3}+5}{4}\right)}{\left(\frac{3+5\sqrt{3}}{4}\right)}=\frac{\sqrt{3}+5}{\sqrt{3}\left(\sqrt{3}+5\right)}=\frac{1}{\sqrt{3}}$ $\theta=\frac{\pi}{6}$	

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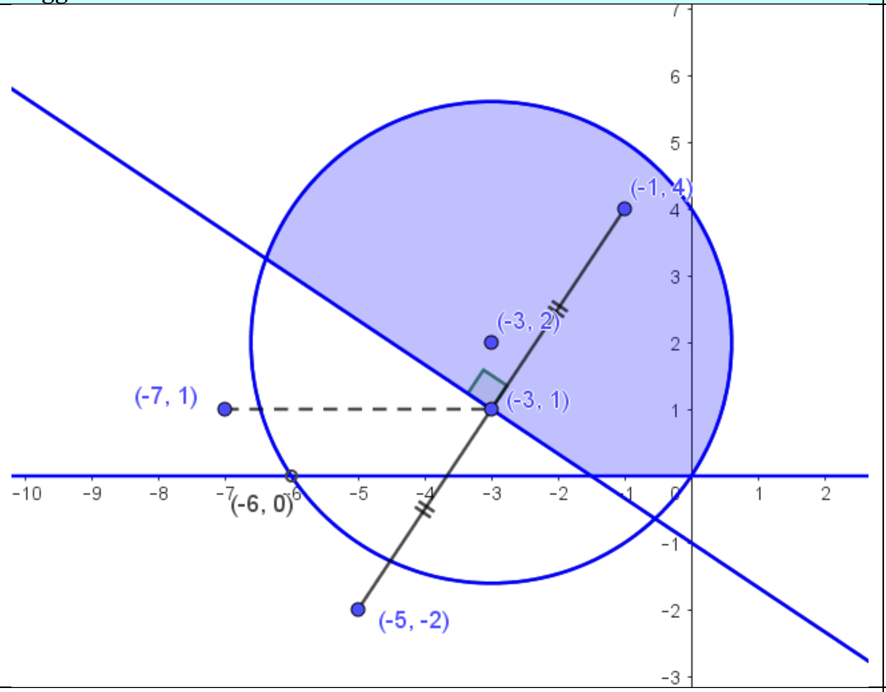
Qn	**Suggested Solution**	
2 (a)	Since $\mathbf{A}\begin{pmatrix}1\\1\\1\end{pmatrix}=\mathbf{0}=0\begin{pmatrix}1\\1\\1\end{pmatrix}$, **A** has eigenvalue 0 and corresponding eigenvector $\begin{pmatrix}1\\1\\1\end{pmatrix}$.	
(b)	Since nullity(T) = 1, rank(T) = 2. Since $\begin{pmatrix}-5\\-2\\-4\end{pmatrix}=-\begin{pmatrix}2\\0\\2\end{pmatrix}-\begin{pmatrix}3\\2\\2\end{pmatrix}$, take 2nd and 3rd columns of **A** as basis of range space of T. Since $\begin{pmatrix}2\\0\\2\end{pmatrix}=2\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}3\\2\\2\end{pmatrix}=2\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}1\\2\\0\end{pmatrix}$, hence $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\0\end{pmatrix}$ form a basis of the range space of T.	Need to link to range space of T. Just saying they are L.I. is insufficient.
(c)	Since $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\0\end{pmatrix}$ are basis of the range space of T, $\mathbf{A}\begin{pmatrix}p\\q\\r\end{pmatrix}=k\begin{pmatrix}1\\0\\1\end{pmatrix}+m\begin{pmatrix}1\\2\\0\end{pmatrix}$ $\begin{pmatrix}-5p+2q+3r\\-2p+2r\\-4p+2q+2r\end{pmatrix}=\begin{pmatrix}k+m\\2m\\k\end{pmatrix}$ Thus $k=-4p+2q+2r$, $m=-p+r$ $\mathbf{A}\begin{pmatrix}p\\q\\r\end{pmatrix}=k\begin{pmatrix}1\\0\\1\end{pmatrix}+m\begin{pmatrix}1\\2\\0\end{pmatrix}$ Apply $\mathbf{A}^{n-1}$ to both sides of equation $\mathbf{A}^{n}\begin{pmatrix}p\\q\\r\end{pmatrix}=k\mathbf{A}^{n-1}\begin{pmatrix}1\\0\\1\end{pmatrix}+m\mathbf{A}^{n-1}\begin{pmatrix}1\\2\\0\end{pmatrix}$ $\mathbf{A}^{n}\begin{pmatrix}p\\q\\r\end{pmatrix}=\underbrace{k\left(-2\right)^{n-1}}_{\lambda}\begin{pmatrix}1\\0\\1\end{pmatrix}+\underbrace{m\left(-1\right)^{n-1}}_{\mu}\begin{pmatrix}1\\2\\0\end{pmatrix}$ Thus $\lambda=\left(-4p+2q+2r\right)\left(-2\right)^{n-1}$, $\mu=\left(-p+r\right)\left(-1\right)^{n-1}$	

(d)	$\mathbf{B}=4\mathbf{A}\left(\mathbf{A}-\mathbf{I}\right)^{-1}$ $\mathbf{B}\left(\mathbf{A}-\mathbf{I}\right)\mathbf{x}=4\mathbf{A}\mathbf{x}$ $\mathbf{B}\left(\mathbf{A}\mathbf{x}-\mathbf{I}\mathbf{x}\right)=4\mathbf{A}\mathbf{x}$ $\mathbf{B}\left(\alpha\mathbf{x}-\mathbf{x}\right)=4\alpha\mathbf{x}$ $\mathbf{B}\left(\left(\alpha-1\right)\mathbf{x}\right)=4\alpha\mathbf{x}$ $\mathbf{B}\mathbf{x}=\dfrac{4\alpha}{\alpha-1}\mathbf{x}$ Thus the eigenvalues of **B** are 0, 2 and $\dfrac{8}{3}$.	

3	Source: RVHS FM Promo 2025 Qn 8	
(a)	$\mathrm{f}(x,y)=xy-3y\ln(x+1)$ $\mathrm{f}_x=y-\dfrac{3y}{x+1}$ $\mathrm{f}_y=x-3\ln(x+1)$ $\mathrm{f}_{xx}=\dfrac{3y}{(x+1)^2}$ $\mathrm{f}_{yy}=0$ $\mathrm{f}_{xy}=\mathrm{f}_{yx}=1-\dfrac{3}{x+1}$	
(b)	For stationary points, $\mathrm{f}_x=0$ and $\mathrm{f}_y=0$. $\mathrm{f}_x=0 \Rightarrow y-\dfrac{3y}{x+1}=0$ $\Rightarrow y\left(1-\dfrac{3}{x+1}\right)=0$ $\Rightarrow y=0$ or $x=2$ $\mathrm{f}_y=0 \Rightarrow x-\ln(x+1)=0$ $\Rightarrow x=0$ or $x=5.71144$ $\therefore$ Stationary points occur at $(5.71,\ 0)$ and $(0,\ 0)$ (reject as we want $x > 0$). At $(5.71,\ 0)$, $D=\mathrm{f}_{xx}\mathrm{f}_{yy}-\left(\mathrm{f}_{xy}\right)^2=-0.306<0$. Hence, $(5.71,\ 0,\ 0)$ is a saddle point.	
(c)	At (2, 1), $\mathrm{f}(x,y)=2-3\ln 3$, $\mathrm{f}_x=0$ and $\mathrm{f}_y=2-3\ln 3$ Equation of tangent plane: $z=2-3\ln 3+(2-3\ln 3)(y-1)$ Therefore, an approximate the value of $\mathrm{f}(2.05,0.98)$	

	$=2-3\ln 3+(2-3\ln 3)(0.98-1)$ $=-1.27$ (to 3 sf)	

Qn	Suggested Solution	
4		
(a)	Equation of the perpendicular bisector $\frac{y-1}{x+3}=-\frac{2}{3}$ $y-1=-\frac{2}{3}x-2$ $y=-\frac{2}{3}x-1$ The foot of perpendicular from $\left(-7,1\right)$ to $y=-\frac{2}{3}x-1$ will give the minimum $\lvert z+7-i\rvert$ Consider D^2, the square of the distance from $\left(-7,1\right)$ to $y=-\frac{2}{3}x-1$, $D^2=\left(x+7\right)^2+\left(y-1\right)^2$ But $y=-\frac{2}{3}x-1$, So $K=D^2=\left(x+7\right)^2+\left(-\frac{2}{3}x-2\right)^2$ $\frac{\mathrm{d}K}{\mathrm{d}x}=2\left(x+7\right)+2\left(\frac{2}{3}x+2\right)\left(\frac{2}{3}\right)=\frac{26}{9}x+\frac{50}{3}$ Set $\frac{\mathrm{d}K}{\mathrm{d}x}=0$, we have $x=-\frac{50\times 9}{26\times 3}=-\frac{75}{13}$ $y=-\frac{2}{3}\left(-\frac{75}{13}\right)-1=\frac{37}{13}$	

$\frac{\mathrm{d}^2K}{\mathrm{d}x^2}=\frac{10}{3}>0$ minimum *K*, hence the complex number is $-\frac{75}{13}+\frac{37}{13}i$

Alternative

Eqn of perp bisector: $\mathbf{r}=\begin{pmatrix}-3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2/3\\0\end{pmatrix}$

To find shortest distance from $(-7,1,0)$ from the line

We have

$$\left(\begin{pmatrix}-3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2/3\\0\end{pmatrix}-\begin{pmatrix}-7\\1\\0\end{pmatrix}\right)\bullet\begin{pmatrix}1\\-2/3\\0\end{pmatrix}=0$$

$$\left(\begin{pmatrix}4\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2/3\\0\end{pmatrix}\right)\bullet\begin{pmatrix}1\\-2/3\\0\end{pmatrix}=0$$

$$4+\lambda\left(1+\frac{4}{9}\right)=0 \quad \Rightarrow \lambda=-\frac{36}{13}$$

Thus the position vector on the line is

$$\begin{pmatrix}-3\\1\\0\end{pmatrix}+\left(-{}^{36}\!/_{13}\right)\begin{pmatrix}1\\-2/3\\0\end{pmatrix}=\begin{pmatrix}-{}^{75}\!/_{13}\\{}^{37}\!/_{13}\\0\end{pmatrix}$$

Alternative:

Equation of $\perp$ bisector:

Line with grad= $-\frac{2}{3}$ passing through $(-3,1)$

$$y-1=-\frac{2}{3}(x+3)\cdots(1)$$

A line passing through $(-7,1)$. grad $=\frac{3}{2}$

$$y-1=\frac{3}{2}(x+7)\cdots(2)$$

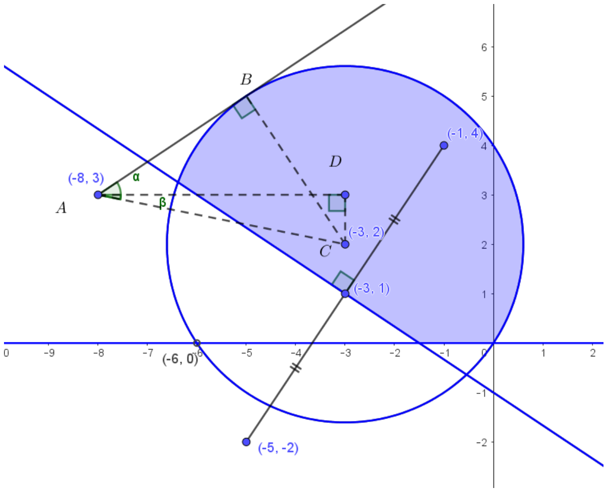
Solving (1) & (2), we have

$$-\frac{2}{3}(x+3)=\frac{3}{2}(x+7)$$

$$-4x-12=9x+63$$

$$x=-\frac{75}{13},\ \text{thus } y=\frac{37}{13}$$

Therefore the complex number is $-\frac{75}{13}+\frac{37}{13}\mathrm{i}$

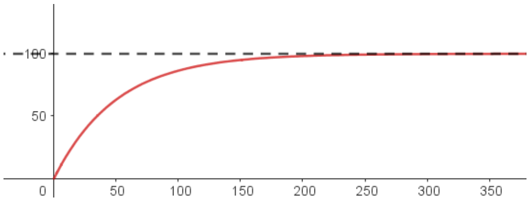
(b)	 $AC=\sqrt{\left(-8+3\right)^2+\left(3-2\right)^2}=\sqrt{26}$ Hence angle $CAB=\alpha=\sin^{-1}\left(\frac{\sqrt{13}}{\sqrt{26}}\right)=\frac{\pi}{4}$ Angle $CAD=\beta=\tan^{-1}\left(\frac{1}{5}\right)$ $\tan\left(\alpha-\beta\right)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ $=\frac{1-\frac{1}{5}}{1+\left(1\right)\left(\frac{1}{5}\right)}=\frac{4}{6}$ Thus max $\arg\left(z+8-3\text{i}\right)=\alpha-\beta=\tan^{-1}\left(\frac{2}{3}\right)$	

Qn	Suggested Solution	
5(i)	Given that x kg be the amount of salt in the tank t minutes. In 1 min, 2 kg of salt is going into the tank. $\frac{\mathrm{d}x}{\mathrm{d}t}_{\mid\text{in}}=2$ Since 2 gallons of mixture is flowing out per min, hence amount of salt flowing out per min is $\frac{2}{100}x$, Thus $\frac{\mathrm{d}x}{\mathrm{d}t}_{\mid\text{out}}=-\frac{1}{50}x$ Thus $\frac{\mathrm{d}x}{\mathrm{d}t}=2-\frac{1}{50}x$ $\int\frac{50}{100-x}\,\mathrm{d}x=\int\mathrm{d}t$ $-50\ln\left\|100-x\right\|=t+C$	

$100-x=Ae^{-\frac{1}{50}t}$ where $A=\pm e^{-\frac{1}{50}C}$

When $t=0, x=0$, $A=100$

$$x=100-100e^{-\frac{1}{50}t}$$



(ii) Given that *x* kg be the amount of salt in the tank *t* minutes.

In 1 min, 2 kg of salt is going into the tank. $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{\text{in}}=2$

At time *t*, amount of mixture in the tank is $100+2t-t=100+t$ gallons.

Since 1 gallon of the mixture flows out of the tank per minute,, amount of salt flowing out of the tank is $\frac{1}{100+t}x$ kg.

Hence we have the DE: $\frac{\mathrm{d}x}{\mathrm{d}t}=2-\frac{1}{100+t}x$

$$\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{1}{100+t}x=2$$

IF = $e^{\int\frac{1}{100+t}\,\mathrm{d}t}=e^{\ln(100+t)}=100+t$. Multiplying the DE by IF gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(100+t)\right]=2(100+t)$$

$$x(100+t)=2\int(100+t)\ \mathrm{d}t$$

$$=(100+t)^2+C$$

Since when *t* = 0, *x* = 0, $C=-100^2$.

Thus the particular solution is

$$x(100+t)=(100+t)^2-100^2\Rightarrow x=\frac{t(200+t)}{100+t}.$$

When amount of mixture is 150, $100+t=150\quad\Rightarrow t=50$.

Amount of salt in the tank at this instant = $\frac{50(200+50)}{100+50}\approx 83.3\text{ kg}$.

(iii)	$\dfrac{\frac{t(200+t)}{100+t}}{100+t}=0.9$ Using G.C. $t=216.23$ min	

Qn	Suggested Solution	
6 (a)	α is the **proportion** of adults of breeding age in the previous month that become ageing adults. β is the **proportion** of ageing adults that died in the previous month. γ is the **proportion** of children in the previous month that become adults of breeding age. δ is the **proportion** of adults of breeding age in the previous month that gave birth to **one** offspring each. The model assumes that the proportions ($\alpha,\beta,\gamma,\delta$) interpreted in (a)(i) are the same every month, which may not be true as time progresses given external factors like the environment and seasons. OR Since each adult of breeding age only give birth to **one** offspring at most per month under this model, there may be insufficient adults of breeding age eventually to breed and replace those that aged and died.	
(b)	$x_n=4x_{n-1}-5x_{n-2}$ Auxiliary Equation: $m^2-4m+5=0$ $m=\dfrac{4\pm\sqrt{16-20}}{2}=2\pm i$ General solution: $x_n=A(2+i)^n+B(2-i)^n$ $x_0=A+B=2$ $x_1=A(2+i)+B(2-i)=2$ $\Rightarrow 2\underbrace{(A+B)}_{=2}+i(A-(2-A))=2$ $\Rightarrow A=1+i,\quad B=1-i$ $x_n=(1+i)(2+i)^n+(1-i)(2-i)^n$ $x_n=(1+i)(2+i)^n+(1-i)(2-i)^n$ $=\sqrt{2}e^{\frac{\pi}{4}i}\left(\sqrt{5}e^{\theta i}\right)^n+\sqrt{2}e^{-\frac{\pi}{4}i}\left(\sqrt{5}e^{-\theta i}\right)^n$ where $\theta=\tan^{-1}\left(\dfrac{1}{2}\right)$ $=2^{\frac{1}{2}}5^{\frac{n}{2}}e^{\left(\frac{\pi}{4}+n\theta\right)i}+2^{\frac{1}{2}}5^{\frac{n}{2}}e^{-\left(\frac{\pi}{4}+n\theta\right)i}$ $=2^{\frac{1}{2}}5^{\frac{n}{2}}(e^{\left(\frac{\pi}{4}+n\theta\right)i}+e^{-\left(\frac{\pi}{4}+n\theta\right)i})$ $=2^{\frac{1}{2}}5^{\frac{n}{2}}\left(2\cos\left(\dfrac{\pi}{4}+n\theta\right)\right)$	

	$= 2^{\frac{3}{2}}5^{\frac{n}{2}}\cos\left(\frac{\pi}{4}+n\tan^{-1}\frac{1}{2}\right)$	
	Alternative $1\pm i=\sqrt{2}\left(\cos\frac{\pi}{4}\pm\mathrm{i}\sin\frac{\pi}{4}\right)$ $2\pm i=\sqrt{5}\left(\cos\theta\pm\mathrm{i}\sin\theta\right)$ where $\theta=\tan^{-1}\frac{1}{2}$ $x_n=\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\left(\sqrt{5}\left(\cos\theta+\mathrm{i}\sin\theta\right)\right)^n$ $\quad+\sqrt{2}\left(\cos\frac{\pi}{4}-\mathrm{i}\sin\frac{\pi}{4}\right)\left(\sqrt{5}\left(\cos\theta-\mathrm{i}\sin\theta\right)\right)^n$ $x_n=2^{\frac{1}{2}}5^{\frac{n}{2}}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\left(\cos n\theta+\mathrm{i}\sin n\theta\right)$ $\quad+2^{\frac{1}{2}}5^{\frac{n}{2}}\left(\cos\frac{\pi}{4}-\mathrm{i}\sin\frac{\pi}{4}\right)\left(\cos n\theta-\mathrm{i}\sin n\theta\right)$ $x_n=2^{\frac{1}{2}}5^{\frac{n}{2}}\left(\cos\left(\frac{\pi}{4}+n\theta\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+n\theta\right)\right)$ $\quad+2^{\frac{1}{2}}5^{\frac{n}{2}}\left(\cos\left(\frac{\pi}{4}+n\theta\right)-\mathrm{i}\sin\left(\frac{\pi}{4}+n\theta\right)\right)$ $x_n=2\left[2^{\frac{1}{2}}5^{\frac{n}{2}}\cos\left(\frac{\pi}{4}+n\theta\right)\right]=2^{\frac{3}{2}}5^{\frac{n}{2}}\cos\left(\frac{\pi}{4}+n\tan^{-1}\frac{1}{2}\right)$	
(d)	As $n\to\infty,\lvert x_n\rvert\to\infty.$ Given a finite map area, this is not realistic.	

Qn	**Suggested Solution**	
7 (a)	$\frac{\mathrm{d}x}{\mathrm{d}t}=-2\sin t+2\sin 2t$ $\frac{\mathrm{d}y}{\mathrm{d}t}=2\cos t-2\cos 2t$ $\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$ $=\left(2\cos t-2\cos 2t\right)^2+\left(-2\sin t+2\sin 2t\right)^2$ $=4\cos^2 t-8\cos 2t\cos t+4\cos^2 2t+4\sin^2 t-8\sin 2t\sin t+4\sin^2 2t$ $=4\left(\cos^2 t+\sin^2 t\right)+4\left(\cos^2 2t+\sin^2 2t\right)-8\left(\cos 2t\cos t+\sin 2t\sin t\right)$ $=8-8\cos\left(2t-t\right)$ $=8\left(1-\cos t\right)=8\left(2\sin^2\frac{t}{2}\right)$	

$$\text{Length } = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$$

$$= \int_0^{2\pi} \sqrt{16\sin^2\frac{t}{2}}\ \mathrm{d}t$$

$$= 4\int_0^{2\pi} \left|\sin\frac{t}{2}\right| \mathrm{d}t$$

$$= 4\int_0^{2\pi} \sin\frac{t}{2}\ \mathrm{d}t \qquad \text{since } \sin\frac{t}{2} \geq 0 \text{ for } 0 \leq t \leq 2\pi$$

$$= -8\left[\cos\frac{t}{2}\right]_0^{2\pi} = -8\left(-1-1\right) = 16$$

(b) Surface Area about x-axis

$$= \int_0^{\pi} 2\pi y\sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$$

$$= \int_0^{\pi} 2\pi\left(2\sin t - \sin 2t\right)4\sin\frac{t}{2}\ \mathrm{d}t$$

$$= 8\pi\int_0^{\pi} \left(2\sin t - \sin 2t\right)\sin\frac{t}{2}\ \mathrm{d}t$$

$$= 8\pi\int_0^{\pi} \left(2\sin t\sin\frac{t}{2} - \sin 2t\sin\frac{t}{2}\right)\mathrm{d}t$$

Using $\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$$= 8\pi\int_0^{\pi} \left(-\cos\frac{3}{2}t + \cos\frac{t}{2} + \frac{1}{2}\left(\cos\frac{5}{2}t - \cos\frac{3}{2}t\right)\right)\mathrm{d}t$$

$$= 8\pi\int_0^{\pi} \left(-\frac{3}{2}\cos\frac{3}{2}t + \cos\frac{t}{2} + \frac{1}{2}\cos\frac{5}{2}t\right)\mathrm{d}t$$

$$= 8\pi\left[-\sin\frac{3}{2}t + 2\sin\frac{t}{2} + \frac{1}{5}\sin\frac{5}{2}t\right]_0^{\pi} = 8\pi\left[1 + 2 + \frac{1}{5}\right] = \frac{128}{5}\pi$$

(c) $x = 2\cos t - \cos 2t - 1$

$y = 2\sin t - \sin 2t$

$$\frac{y}{x} = \frac{2\sin t - \sin 2t}{2\cos t - \cos 2t - 1}$$

$$= \frac{2\sin t\left(1-\cos t\right)}{2\cos t - 2\cos^2 t}$$

$$= \frac{2\sin t\left(1-\cos t\right)}{2\cos t\left(1-\cos t\right)} = \tan t$$

	But $\dfrac{y}{x} = \tan\theta$, hence $\theta = t$ $r^2 = x^2 + y^2$ $= \left(2\cos t - 2\cos^2 t\right)^2 + \left(2\sin t - 2\sin t\cos t\right)^2$ $= 4\cos^2 t\left(1-\cos t\right)^2 + 4\sin^2 t\left(1-\cos t\right)^2$ $= 4(1-\cos t)^2$ $r = 2\left(1-\cos t\right) = 2\left(1-\cos\theta\right)$ since $r \geq 0$ and $t = \theta$	
(d)	Area $= 2\times\dfrac{1}{2}\int_0^{\pi}\left(2\left(1-\cos\theta\right)\right)^2 \,\mathrm{d}\theta$ $= 4\int_0^{\pi} 1 - 2\cos\theta + \cos^2\theta \,\mathrm{d}\theta$ $= 4\int_0^{\pi} 1 - 2\cos\theta + \dfrac{1+\cos 2\theta}{2} \,\mathrm{d}\theta$ $= 4\left[\dfrac{3}{2}\theta - 2\sin\theta + \dfrac{1}{4}\sin 2\theta\right]_0^{\pi} = 6\pi$	

<table>
<tr><th>Qn</th><th>Suggested Solution</th><th></th></tr>
<tr><td>8</td><td>
Using GC, $\bar{x} = 1.5$, $s^2 = 1.5657$
Since $\bar{x}$ and s^2 is approximately the same, a Poisson distribution is a possible model for x.

To test $H_0 : X \sim \text{Po}(1.5)$ models the data
$H_1 : X \sim \text{Po}(1.5)$ does not model the data
Significance Level: 5%
Combining adjacent cells with expected frequencies < 5
<table>
<tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>≥ 4</td></tr>
<tr><td>o_x</td><td>22</td><td>37</td><td>20</td><td>13</td><td>8</td></tr>
<tr><td>e_x</td><td>22.313</td><td>33.470</td><td>25.102</td><td>12.551</td><td>4.707+1.412+0.445
= 6.564</td></tr>
</table>
Degree of freedom = $5-1-1=3$.

Under H_0, $X^2 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(3)$.

Reject H_0 if p-value $\leq$ 0.10

Using GC, $\chi^2_{calc} = 1.74$, p-value = 0.627

Since p-value > 0.10, we do not reject H_0. There is insufficient evidence at 10% significance level, to conclude that the number of thunderstorms does not have a Poisson distribution with mean 1.5.
</td><td></td></tr>
</table>

<table>
<tr><th>Qn</th><th>Suggested Solution</th><th></th></tr>
<tr><td>9</td><td>
(a) From GC, the unbiased estimate of population mean and variances are
$\bar{x} = 55$ and $s^2 = 17.9355^2 = 321.68 = 322$ (3 s.f.)

(b) Let X be the survival times (in days) of a randomly chosen guinea pig in the farm, with population mean μ.
$H_0 : \mu = 60$
$H_1 : \mu < 60$
To perform a t-test at 10% significance level.
Under H_0 $T = \frac{\bar{X} - 60}{S / \sqrt{20}} \sim t(19)$.
Reject H_0 if $t_{\text{cal}} \leq -1.3277$
From sample, $\bar{x} = 55$, $s = 17.9355$ and $n = 20$
$t_{\text{cal}} = \frac{55-60}{17.9355/\sqrt{20}} = -1.2467$
</td><td>In hypothesis testing, it is required for you to “work out the test statistics” and “show the values that you use in the calculation”.
Note:
p-value $= 0.11382$</td></tr>
</table>

	Since $t_{cal} > -1.3277,$ we do not reject H_0 and conclude that there is insufficient evidence, at 10% significance level, that the **average/mean** survival times of guinea pigs from the farm is not at least 60 days. (c) The data seems to be skewed to the right (i.e. longer tail at the right) and hence does not seem to come from a normal distribution. The data seems to be not symmetrical around the mean which suggest that a normal distribution may not be appropriate.	

Qn 10	**Suggested Solution**	
(a)	$\text{P}(T \le t) = \text{P}\left(\dfrac{-\ln(1-X)}{\lambda} \le t\right)$ $= \text{P}(1-X \ge \text{e}^{-\lambda t})$ $= \text{P}(X \le 1-\text{e}^{-\lambda t})$ $= 1-\text{e}^{-\lambda t}, \quad t > 0$ There $T \sim \exp(\lambda)$	
(b)	$T \sim \text{Exp}(2)$, $\text{E}(T) = 0.5$, $\text{Var}(T) = 0.25$ Since n = 60 is large, by Central Limit Theorem, $T_1 + T_2 + \ldots + T_{60} \sim \text{N}(30,\ 15)$ approximately $\text{P}(T_1 + T_2 + \ldots + T_{60} \le 40) = 0.99509 = 0.995$	
(c)	$\text{P}(Y \le y) = \text{P}(T \le y) + \text{P}(T \ge 1)\text{P}(T \le y) + \left[\text{P}(T \ge 1)\right]^2 \text{P}(T \le y) + \ldots$ $= \text{P}(T \le y)\left[1 + \text{P}(T \ge 1) + \left[\text{P}(T \ge 1\right]^2 + \ldots\right]$ $= \left(1-\text{e}^{-2y}\right)\left(1+\text{e}^{-2}+\text{e}^{-4}+\ldots\right)$ $= \dfrac{1-\text{e}^{-2y}}{1-\text{e}^{-2}}$	

(d)

$$\begin{aligned}
\mathrm{E}(W) &= \sum_{r=1}^{k} \frac{2r-1}{2}\,\mathrm{P}\left(W=\frac{2r-1}{2}\right) \\
&= \sum_{r=1}^{k} \frac{2r-1}{2}\,\mathrm{P}(r-1 \le V \le r) \\
&= \sum_{r=1}^{k} \frac{2r-1}{2}\left(\mathrm{F}(r)-\mathrm{F}(r-1)\right) \\
&= \frac{1}{2}\left(\mathrm{F}(1)-\mathrm{F}(0)\right) \\
&\quad +\frac{3}{2}\left(\mathrm{F}(2)-\mathrm{F}(1)\right) \\
&\quad +\frac{5}{2}\left(\mathrm{F}(3)-\mathrm{F}(2)\right) \\
&\quad +\ldots \\
&\quad +\frac{2k-1}{2}\left(\mathrm{F}(k)-\mathrm{F}(k-1)\right) \\
&= -\mathrm{F}(1)-\mathrm{F}(2)-\mathrm{F}(3)-\ldots-\mathrm{F}(k-1)+\frac{2k-1}{2}\mathrm{F}(k) \\
&= \frac{2k-1}{2}(1)-\left[\mathrm{F}(1)+\ldots+\mathrm{F}(k-1)\right] \\
&= \frac{2k-1}{2}-\left[1-\frac{(k-1)^2}{k^2}+1-\frac{(k-2)^2}{k^2}+\ldots+1-\frac{1}{k^2}\right] \\
&= \frac{2k-1}{2}-(k-1)+\frac{1}{k^2}\left[(k-1)^2+(k-2)^2+\ldots+1\right] \\
&= \frac{1}{2}+\frac{1}{k^2}\sum_{r=1}^{k-1} r^2
\end{aligned}$$

For students reference: If we further evaluate

$$\begin{aligned}
&= \frac{1}{2}+\frac{1}{k^2}\left[\frac{k-1}{6}(k)\left(2(k-1)+1\right)\right] \\
&= \frac{1}{2}+\frac{1}{6k}\left[(k-1)(2k-1)\right] \\
&= \frac{3k}{6k}+\frac{2k^2-3k+1}{6k} \\
&= \frac{k}{3}+\frac{1}{6k}
\end{aligned}$$