EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2025

General Certificate of Education Advanced Level

Higher 3

CANDIDATE NAME	

CIVICS GROUP		INDEX NO.	

MATHEMATICS

9820/01

Paper 1 [80 marks]

24 September 2025

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

1 Let a, b, c be positive real numbers such that $abc = 36$.

(a) Show that $2a + b + 3c \geq 18$. [2]

(b) Deduce the minimum value of $\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b}$ and find the values of a, b and c where this minimum is attained. [5]

(c) Hence, or otherwise, prove that if p, q and r are positive real numbers such that $pqr = \dfrac{1}{36}$, then

$$\frac{1}{9p^3(r+3q)} + \frac{1}{36q^3(3p+2r)} + \frac{1}{4r^3(2q+p)} \geq 9.$$ [4]

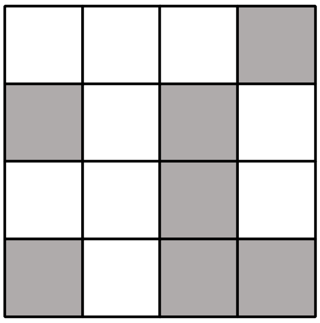
2 **(a)** The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 - 2y^2}, \quad x \neq \sqrt{2}y \text{ for all } x, y \in \mathbb{R}.$$

Using the substitution $y = ux$, find the general solution of the differential equation. [7]

(b) Describe geometrically the set of points that satisfy the general solution obtained in part **(a)**. [4]

3 Consider 4-by-4 unit-square grids where each unit-square is either shaded or white.
For example, the following diagram illustrates an example of such a grid.



(a) **(i)** How many possible 4-by-4 unit-square grids are there? [1]

(ii) How many 4-by-4 unit-square grids are there which contain at least one entirely shaded 3-by-3 square? [4]

(b) Determine the minimum number of shaded unit-squares in a 4-by-4 unit-square grid to guarantee that it contains at least one entirely shaded 2-by-2 square. Justify your answer. [3]

(c) Define c_i to be the number of shaded unit squares in the ith column from the left of a given 4-by-4 unit-square grid.
For example, for the 4-by-4 unit-square above, $c_1 = 2, c_2 = 0, c_3 = 3, c_4 = 2$.
How many distinct possible tuples (c_1, c_2, c_3, c_4) are there if the given 4-by-4 unit-square grid has

(i) exactly 4 shaded unit-squares, [1]

(ii) at least 12 shaded unit-squares? [3]

4 Let S be the set of all integers from 1 to n and T be the set of all subsets of S, including the empty set $\varnothing$ and S itself.

(a) Explain why $|T| = 2^n$. [1]

The sets A_1 and A_2, which are not necessarily distinct, are chosen randomly and independently from the 2^n subsets in T.

(b) In this part of the question, we will calculate $\mathrm{P}(A_1 \cap A_2 = \varnothing)$ in 2 ways.

(i) Let i be an integer such that $0 \le i \le n$. Show that $\mathrm{P}(i \notin A_1 \cap A_2) = \frac{3}{4}$. [2]

(ii) Hence, find $P(A_1 \cap A_2 = \varnothing)$. [1]

(iii) Let k be an integer such that $0 \le k \le n$. Find $\mathrm{P}(A_1 \cap A_2 = \varnothing \,|\, |A_1| = k)$. [1]

(iv) Deduce that $\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n+k}} = \left(\frac{3}{4}\right)^n$. [3]

(c) **(i)** Let k be an integer such that $0 \le k \le n$. Find $\mathrm{P}(|A_1 \cap A_2| = k)$. [2]

(ii) Find $\mathrm{E}(|A_1 \cap A_2|)$. [2]

5 **(a)** Let $\mathrm{f}(x)$ be a polynomial with integer coefficients.

(i) For integers α and β show that $\mathrm{f}(x) - \mathrm{f}(\alpha)$ has a factor of $x - \alpha$ and hence, or otherwise, show that

$$\mathrm{f}(\beta) - \mathrm{f}(\alpha) \equiv 0 \text{ (modulo } \beta - \alpha\text{).}$$ [2]

(ii) Deduce that there does not exist a polynomial f with integer coefficients such that $\mathrm{f}(1) = 2,\ \mathrm{f}(3) = 5$. [1]

(iii) Explain why there does not exist a polynomial f with integer coefficients such that $\mathrm{f}(1) = 0,\ 0 < \mathrm{f}(2025) < 2024$. [2]

(b) A monic degree n polynomial is a polynomial where the coefficient of x^n is 1.
Let p be an odd prime. Let $\mathrm{h}(x)$ be a monic degree n polynomial with integer coefficients.
We call an integer γ a *root modulo p* of $\mathrm{h}(x)$ if $\mathrm{h}(\gamma) \equiv 0$ (modulo p).
Moreover, if $\alpha \not\equiv \beta$ (modulo p), we call integers α, β *incongruent*.

(i) Find two incongruent roots modulo 7 for $x^2 + x + 1$. [1]

(ii) For $n \ge 1$, show that if γ is a root modulo p of $\mathrm{h}(x)$, then there exists a monic degree $n-1$ polynomial $\mathrm{h}_0(x)$ with integer coefficients, such that

$$\mathrm{h}(x) \equiv (x - \gamma)\mathrm{h}_0(x) \text{ (modulo } p\text{).}$$ [3]

(iii) Prove, by induction, that any monic degree n polynomial $\mathrm{h}(x)$ has at most n incongruent roots modulo p for $n \ge 0$. [5]

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

On the irrationality of π

We denote $\mathrm{f}^{(k)}(x)$ to be the kth derivative of $\mathrm{f}(x)$ with respective to x and $\mathrm{f}^{(0)}(x)=\mathrm{f}(x)$.

A number is called *rational* if it can be simplified to the form $\frac{a}{b}$ where a is an integer and b is a positive integer where a and b do not share any prime factors. A number is called irrational if it is not a rational number. For example, the numbers $0.1, \frac{4}{3}, -\frac{39}{13}=-3$ are rational numbers but $\sqrt{2}$ and π are not.

However, showing that π is irrational, is not trivial.
In 1947, the Canadian-American mathematician Ivan Morton Niven produced an elementary yet elegant proof that π is irrational. His methods could be traced back to French mathematician Charles Hermite's 1873 paper on proving irrationality of e^{r}, where r is a rational number. We will investigate a modified version of Niven's proof of the irrationality of π.

We suppose that $\pi=\frac{a}{b}$ for positive integers a, b, for contradiction.
In his proof, for each positive integer n, Niven considered the polynomial

$$\mathrm{f}(x)=\frac{x^n(a-bx)^n}{n!}=\frac{b^n\left[x(\pi-x)\right]^n}{n!}.$$

Using binomial expansion, we can write this polynomial in the form

$$\mathrm{f}(x)=\frac{1}{n!}\sum_{r=n}^{2n}c_r x^r,$$

where c_r are integers for all r.
By considering the expansion, we can show that $\mathrm{f}^{(k)}(0)$ is an integer, for $0\le k\le 2n$. This would then imply that $\mathrm{f}^{(k)}(\pi)$ is an integer, for $0\le k\le 2n$, as well.

Niven also define another function $\mathrm{F}(x)$ as

$$\mathrm{F}(x)=\mathrm{f}(x)-\mathrm{f}^{(2)}(x)+\mathrm{f}^{(4)}(x)-\ldots+(-1)^n\mathrm{f}^{(2n)}(x).$$

This is carefully chosen so that the identity $\int_0^{\pi}\mathrm{f}(x)\sin(x)\,\mathrm{d}x=\mathrm{F}(0)+\mathrm{F}(\pi)$ holds for all n.

For $0<x<\pi$, we have $0<\mathrm{f}(x)<\frac{\pi^n m^n}{n!}$ for some positive real constant m.
We can then develop the bound

$$0<\int_0^{\pi}\mathrm{f}(x)\sin(x)\,\mathrm{d}x<\frac{2\pi^n m^n}{n!}\quad\text{for all } n\ge 1.$$

It is a known fact that if $\lim_{n\to\infty}u_n=0$, this implies that for sufficiently large n, we have $u_n<1$.

By selecting a sufficiently large n, we will attain the required contradiction to prove that π is irrational.

6 In this question you are asked to prove that π is irrational.

For the entirety of the question, we will assume that $\pi = \frac{a}{b}$, where a, b are positive integers, for contradiction.

(a) **(i)** Consider any c_r in the sum $\frac{1}{n!}\sum_{r=n}^{2n} c_r x^r$.

Briefly explain why c_r is an integer. [2]

(ii) Hence, explain why $\mathrm{f}^{(k)}(0)$ is an integer for $0 \le k \le 2n$. [2]

(b) By showing that $\mathrm{f}(x) = \mathrm{f}(\pi - x)$, prove that $\mathrm{f}^{(k)}(\pi)$ is an integer for $0 \le k \le 2n$. [3]

(c) **(i)** Prove that $\mathrm{F}''(x) = \mathrm{f}(x) - \mathrm{F}(x)$. [2]

(ii) Deduce that $\int_0^{\pi} \mathrm{f}(x)\sin(x)\,\mathrm{d}x = \mathrm{F}(0) + \mathrm{F}(\pi)$. [3]

(d) **(i)** Prove that for some positive real constant m, $0 < \mathrm{f}(x) < \frac{\pi^n m^n}{n!}$ for $0 < x < \pi$. [2]

(ii) Hence, show that $0 < \int_0^{\pi} \mathrm{f}(x)\sin(x)\,\mathrm{d}x < \frac{2\pi^n m^n}{n!}$. [1]

(iii) By considering the series expansion of e^x, or otherwise, explain why $\lim_{n\to\infty} \frac{2\pi^n m^n}{n!} = 0$. [2]

(e) Prove that π is irrational. [3]

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