NANYANG JUNIOR COLLEGE

JC2 COMMON TEST 2

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **1st July 2021**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 Find the general solution of the differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+3y=\cos x$. [5]

2 The linear transformation T: $\mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A}=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}.$$

(i) Find the image of $\begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$ where $r \geq 0$ and $0 \le \alpha < 2\pi$ under T and give a geometrical description of the transformation T. [2]

(ii) Find all values of the angle θ for which the matrix $\mathbf{A}$ has real eigenvalues and state the corresponding eigenvalue. [5]

3 Let e_1, e_2, e_3 be the standard basis for $\mathbb{R}^3$ and let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with the properties

$$T(e_1)=e_2, \quad T(e_2)=2e_1+e_2, \quad T(e_1+e_2+e_3)=e_3.$$

(i) Find a vector v such that $T(v)=kv$ for some $k\in\mathbb{R}$. Explain the significance of this result. [2]

(ii) Find A, the matrix representation of T with respect to the standard basis. [2]

(iii) State the kernel of the linear transformation T_1 with matrix representation $A-2I$ and obtain a basis for the range space, R_1, of T_1. [3]

4 Let $\mathbf{A}$ and $\mathbf{B}$ be $n\times n$ matrices.

(i) Prove that $\ker(\mathbf{A})\subseteq\ker(\mathbf{BA})$. [2]

(ii) If **BA** is an invertible matrix, prove that both **A** and **B** are invertible. [2]

(iii) If $\mathbf{A}+\mathbf{B}$ is invertible, show that $\mathbf{A}\left(\mathbf{A}+\mathbf{B}\right)^{-1}\mathbf{B}=\mathbf{B}\left(\mathbf{A}+\mathbf{B}\right)^{-1}\mathbf{A}$. [3]

5 A research laboratory is studying a population of hamsters. The hamsters can be grouped into three age classes of equal duration defined as follows: Hamsters in the first age class are for those with $0 \le \text{age} < 1$, second age class are for those with $1 \le \text{age} < 2$ and the third age class are for those with $2 \le \text{age} \le 3$. The maximum life span of the hamsters is assumed to be 3 years.

The population of hamsters in the laboratory has the characteristics listed below:

- Half of the population survives its first year. Of those that survive the first year, half survives the second year.
- There is no offspring produced by the hamsters in the first age group. The average number of offspring for each hamster of the population is 6 for the second age class and 8 for the third age class.

The number of hamsters in the first age class, second age class and third age class after r years are denoted by a_r, b_r and c_r respectively.
The information above can be represented by the matrix equation

$$\mathbf{M}\mathbf{x}_n = \mathbf{x}_{n+1}$$

where $\mathbf{M}$ is a 3×3 matrix, $\mathbf{x}_r = \begin{pmatrix} a_r \\ b_r \\ c_r \end{pmatrix}$ and r is a positive integer.

(i) Write down the matrix $\mathbf{M}$.

Given that there are 120 hamsters in each of the three age class initially, show that $\mathbf{x}_1 = \begin{pmatrix} 1680 \\ 60 \\ 60 \end{pmatrix}$.

Find the number of hamsters in each age class after 2 years. [3]

(ii) The laboratory would like to achieve a stable growth pattern, one in which the proportion in each age class remains the same each year. For this stable growth pattern to be achieved, $\mathbf{M}\mathbf{x}_n = \mathbf{x}_{n+1} = \lambda \mathbf{x}_n$. By finding a suitable eigenvalue and eigenvector, determine the smallest initial population of the hamsters in each age class so that the proportion of hamsters in each class remains the same each year. [4]

(iii) Using the initial population values found in **(ii)**, determine the number of years taken for the total population of the hamsters to exceed 5000. [3]

6 The 'Chinese rings puzzle' is a disentanglement puzzle featuring a loop which must be disentangled from a sequence of rings on interlinked pillars. The number x_n of steps required to solve the 'Chinese rings puzzle' with n rings satisfies $x_1 = 1$ and

$$x_{n+1} = \begin{cases} 2x_n, & n \text{ is odd} \\ 2x_n + 1, & n \text{ is even} \end{cases}$$

(i) Prove that $x_{n+2} = x_{n+1} + 2x_n + 1$. [2]

(ii) Find the maximum number of rings such that the number of steps required to solve the puzzle is less than 2021. [2]

(iii) Given that the solution of the recurrence system is of the form $x_n = A(2^n) + B(-1)^n + C$, find the values of *A, B* and *C.* [5]

7 Find the coordinates of the stationary points of the surface

$$z = \mathrm{e}^{-(y-1)^2} \cos\left(x - \frac{\pi}{3}\right)$$

in the region $\{(x, y) \mid -\pi \le x \le \pi, y \in \mathbb{R}\}$ and determine the nature of these stationary points. [8]

8 The cartesian equation of a curve Γ_1 is given by $x^4 - 6xy + 3 = 0$.
Show that

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{\left(x^4 + 1\right)^2}{4x^4}.$$ [1]

(i) The points A and B on Γ_1 correspond to $x = 1$ and $x = 2$ respectively. Show that the length of the arc AB is $\dfrac{17}{12}$. [2]

(ii) The arc AB is rotated through one complete revolution about the x-axis. Find, in exact form, the area of the curved surface generated. [3]

Another curve Γ_2 has cartesian equation $x^4 - 6xy + 24x + 3 = 0$.
State the relationship between the curves Γ_1 and Γ_2. [1]

The region bounded by the curves Γ_1 and Γ_2 and the lines $x = 1$ and $x = 2$, is denoted by R.

(iii) Find the perimeter of R, showing your working clearly. [2]

(iv) Obtain the volume of the solid formed when R is rotated completely about the y-axis, giving your answer in an exact form. [2]

9 In many practical situations, we are interested to fit a polynomial curve to a set of given points obtained via experiments. Suppose we have n data points given by (x_1, y_1), (x_2, y_2), …, (x_n, y_n). We attempt to find a degree $n-1$ polynomial that will pass through all these points. In another words, we have to find a polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ such that $p(x_i) = y_i$ for each $i = 1, 2, \ldots, n$.

(i) Show that the values of a_0, a_1, …, a_{n-1} can be found by solving the system of equations

$$\mathbf{Q}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}, \text{ where } \mathbf{Q} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$ [1]

(ii) For the case where $n = 3$, use Gaussian row elimination to show that the determinant of $\mathbf{Q}$ is

$$(x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$ [4]

It can be proven that the determinant of $\mathbf{Q}$ is $\prod_{1 \le i < j \le n} (x_j - x_i)$.

(iii) If the n data points are such that $x_i \neq x_j$ if $i \neq j$, show that the polynomial that passes through all the n data points is unique. [2]

(iv) For the case where $n = 4$, $p(0) = -1$, $p(1) = 0$, $p(3) = 2$ and $p(4) = 5$, find the equation of $p(x)$. [3]

10 The Richardson's Arms Race Model is a model for how countries may react to each other based on the measure of a particular interest. For example, countries may acquire and accumulate weapons due to mutual fear of conflict. This model can also be used to look at gang wars or advertising wars between rivals.

Let $\$x$ and $\$y$ represent the wealth being spent on arms by Nation X and Nation Y respectively. Consider the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3y - 2x$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 3y$$

(i) Show that the general solution for x is $x = A\mathrm{e}^{t} + B\mathrm{e}^{-6t}$. Hence, find the general solution for y, leaving your answer in terms of $t, A,$ and B. [4]

(ii) Given that $x = y = 1$ when $t = 0$, find the particular solutions for x and y. [2]

(iii) Using your answer in **(ii)**, explain what will happen in the long run. [1]

Nations tend to avoid war, and may have a sense of goodwill to other nations. Consider the following system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3y - 2x - 10$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4x - 3y - 10$$

(iv) Find the general solutions for x and y. [2]

(v) With the same initial conditions, show that it is possible for both nations to disarm completely in the long run. [3]

11 The complex number z has real part x and imaginary part y.

(i) Show that

$$\tan\left[\arg\left(z-1\right)\right] = \frac{y}{x-1},$$

distinguishing the cases between which the point representing z lies in each of the four quadrants. [4]

(ii) Show also that

$$\tan\left[\arg\left(1-\frac{4}{z}\right)\right] = \frac{4y}{x^2+y^2-4x}.$$ [1]

The points P and Q in an Argand diagram represent z and $\frac{4}{z}$ respectively. Given that the line PQ passes through the point representing 1, show that

(iii) $y = 0$ or $\left(x-4\right)^2 + y^2 = 12$. [3]

(iv) Sketch the locus of P. [2]

(v) Show that, for any position of P on this locus for which $y \neq 0$,

$$\left|1-\frac{1}{z}\right| = \frac{1}{2}\sqrt{3}.$$ [4]

———END OF PAPER———