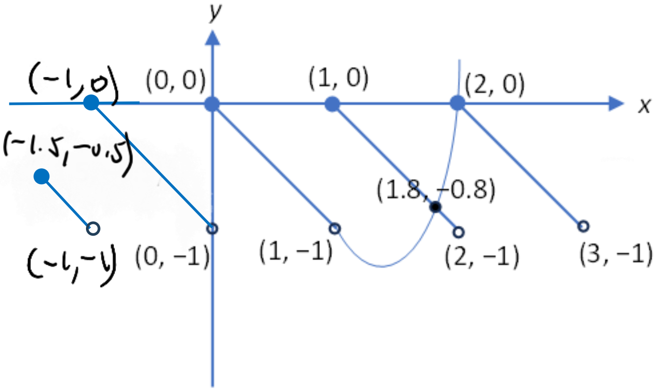
Qn	Suggested Solution	Mark Scheme
1a	Use Cauchy-Schwartz Inequality $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$ Let $a_i = x_i$ and $b_i = 1$ for $i = 1,2,3,\ldots,n$ $\left(\sum_{i=1}^{n} x_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} 1^2\right) \le n\left(\sum_{i=1}^{n} x_i^2\right)$ Divide both sides by n^2, $\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 \le \left(\frac{\sum_{i=1}^{n} x_i^2}{n}\right)$ OR Let $a_i = x_i$ and $b_i = \frac{1}{n}$ for $i = 1, 2, 3, \ldots, n$.	M1 A1: [AG]
1b	Let $a_i = \sqrt{\lambda_i}\ x_i$ and $b_i = \sqrt{\lambda_i}$ for $i = 1, 2, 3, \ldots, n$. $\left(\sum_{i=1}^{n} \lambda_i x_i\right)^2 \le \left(\sum_{i=1}^{n} \lambda_i x_i^2\right)\left(\sum_{i=1}^{n} \lambda_i\right)$ The necessary and sufficient condition for the equality to hold is $\frac{\sqrt{\lambda_1}\ x_1}{\sqrt{\lambda_1}} = \frac{\sqrt{\lambda_2}\ x_2}{\sqrt{\lambda_2}} = \ldots. = \frac{\sqrt{\lambda_n}\ x_n}{\sqrt{\lambda_n}}$ i.e. $x_1 = x_2 = \ldots. = x_n$	M1 A1 B1
1c	In **(ii)**, let $\lambda_i = \frac{1}{t^i}$ and $x_i = y_i$ for $i = 1,2,3,\ldots,n$ $\left(\sum_{i=1}^{n} \frac{1}{t^i} y_i\right)^2 \le \left(\sum_{i=1}^{n} \frac{1}{t^i}\right)\left(\sum_{i=1}^{n} \frac{y_i^2}{t^i}\right)$ $\sum_{i=1}^{n} \frac{1}{t^i} = \frac{\frac{1}{t}\left(1-\left(\frac{1}{t}\right)^n\right)}{1-\frac{1}{t}}$ (sum of geometric series) $= \frac{1}{t-1}\left(1-\left(\frac{1}{t}\right)^n\right)$ < 1 as $t \ge 2$, $\frac{1}{t-1} \le 1$ and $0 < 1-\left(\frac{1}{t}\right)^n < 1$ Thus $\left(\sum_{i=1}^{n} \frac{1}{t^i} y_i\right)^2 < \sum_{i=1}^{n} \frac{y_i^2}{t^i}$. OR $\sum_{i=1}^{n} \frac{1}{t^i} \le \sum_{i=1}^{n} \frac{1}{2^i}$ $\because t \ge 2$ $= \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 1-\left(\frac{1}{2}\right)^n$ < 1 since $0 < \left(\frac{1}{2}\right)^n < 1$ Thus $\left(\sum_{i=1}^{n} \frac{1}{t^i} y_i\right)^2 < \sum_{i=1}^{n} \frac{y_i^2}{t^i}$.	M1: Replacement M1: Sum of GP A1: Show sum is less than 1, leading to answer [AG] [A1]: Show sum is less than 1, leading to answer [AG]

2a i	Let $x = \lfloor x \rfloor + b,$ for some $0 \le b < 1$. Then $x+1 = \lfloor x+1 \rfloor + b \Rightarrow \lfloor x+1 \rfloor = x - b + 1 = \lfloor x \rfloor + 1.$ $\therefore \text{f}(x+1) = \lfloor x+1 \rfloor - (x+1)$ $= \lfloor x \rfloor + 1 - x - 1$ $= \lfloor x \rfloor - x = \text{f}(x)$ Hence, f is periodic with a period of 1.	B1 M1 A1: [AG]
2a ii		B1: Shape of f B1: Open circles indicated B1: Intersections and endpoints indicated
2a iii	$\text{Area} = -\int_{1}^{1.8} \frac{15}{4}x^2 - \frac{41}{4}x + \frac{11}{2}\ \text{d}x - \frac{1}{2}(0.8)(0.8)$ $= 1.04 - 0.32$ $= 0.72$	M1 M1 A1
2b i	$\lfloor \log_3 k \rfloor = 2 \Rightarrow 2 \le \log_3 k < 3$ Hence $3^2 \le k < 3^3 \Rightarrow k = 9,\ 10,\ \ldots,\ 26.$	M1 A1
2b ii	$x_1 = 2 + 1 = 3;\ x_2 = 3;$ $x_3 = x_4 = \ldots = x_7 = x_8 = 4;\ x_9 = 5.$ $x_n = x_{\lfloor \frac{n}{3} \rfloor} + 1 = \begin{cases} x_0 + 1 & \text{for } n = 1,2 \\ x_1 + 1 & \text{for } n = 3,4,5 \\ x_2 + 1 & \text{for } n = 6,7,8 \\ x_3 + 1 & \text{for } n = 9,10,11 \\ \vdots & \end{cases} = \begin{cases} 3 & \text{for } n = 1,2 \\ 4 & \text{for } n = 3,4,5 \\ 4 & \text{for } n = 6,7,8 \\ 5 & \text{for } n = 9,10,11 \\ \vdots & \end{cases}$ Observe that $\lfloor \log_3 n \rfloor = \begin{cases} 0 & \text{for } n = 1,2 \\ 1 & \text{for } n = 3,4,\ldots,8 \\ 2 & \text{for } n = 9,10,\ldots,26 \\ \vdots & \end{cases} \Rightarrow 3 + \lfloor \log_3 n \rfloor = \begin{cases} 3 & \text{for } n = 1,2 \\ 4 & \text{for } n = 3,4,\ldots,8 \\ 5 & \text{for } n = 9,10,\ldots,26 \\ \vdots & \end{cases}$ Hence, $x_n = 3 + \lfloor \log_3 n \rfloor$.	B1 B1 B1

3a	$\binom{2n}{n}^{-1} = \left[\frac{(2n)!}{n!n!}\right]^{-1} = \frac{n!n!}{(2n)!} = \frac{(n!)^2}{(2n)!}$.	B1: Apply definition
3b	$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, \mathrm{d}x = \frac{4^n}{2n+1}\binom{2n}{n}^{-1}$ for all $n \in \mathbb{Z}_0^+$. For n = 0, LHS = $\int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}x = 1$. RHS = $\frac{4^0}{2(0)+1}\binom{0}{0}^{-1} = 1$ whereby definition $\binom{0}{0} = \frac{0!}{0!(0-0)!} = 1$. So statement holds for n = 0.	B1: Base case
	Assume $I_k = \int_0^{\frac{\pi}{2}} \sin^{2k+1} x \, \mathrm{d}x = \frac{4^k}{2k+1}\binom{2k}{k}^{-1}$ for some $k \in \mathbb{Z}_0^+$. $I_{k+1} = \int_0^{\frac{\pi}{2}} \sin^{2k+3} x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin x \cdot \sin^{2k+2} x \, \mathrm{d}x$.	M1: Split integrand
	Let $u = \sin^{2k+2} x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = (2k+2)\sin^{2k+1} x \cos x$ $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$ $I_{k+1} = \int_0^{\frac{\pi}{2}} \sin^{2k+3} x \, \mathrm{d}x$ $= -\left[\sin^{2k+2} x \cos x\right]_0^{\frac{\pi}{2}} + (2k+2)\int_0^{\frac{\pi}{2}} \sin^{2k+1} x \cos^2 x \, \mathrm{d}x$	M1: Integrate by parts
	$= (2k+2)\int_0^{\frac{\pi}{2}} \sin^{2k+1} x \left(1 - \sin^2 x\right)\mathrm{d}x$ $= (2k+2)I_k - (2k+2)I_{k+1}$ $\Rightarrow (2k+3)I_{k+1} = (2k+2)I_k$ $\Rightarrow I_{k+1} = \frac{2k+2}{2k+3}I_k$	M1: I_{k+1} in terms of I_k
	$= \left(\frac{2k+2}{2k+3}\right)\frac{4^k}{2k+1}\binom{2k}{k}^{-1}$ by (IH) $= \left(\frac{4^{k+1}}{2k+3}\right) \cdot \frac{k+1}{2(2k+1)}\binom{2k}{k}^{-1}$	M1: Use IH
	$\frac{k+1}{2(2k+1)}\binom{2k}{k}^{-1} = \frac{k+1}{2(2k+1)} \cdot \frac{(k!)(k!)}{(2k)!}$ $= \frac{(2k+2)}{2(k+1)} \cdot \frac{(k+1)!(k+1)!}{(2k+2)!}$ $= \binom{2k+2}{k+1}^{-1}$	M1: Apply ${}^nC_r$ formula and simplify
	Hence $I_{k+1} = \left(\frac{4^{k+1}}{2k+3}\right)\binom{2k+2}{k+1}^{-1}$ proving the statement by induction.	A1: Final form

3c	For $\lvert x\rvert<1$, $\displaystyle\sum_{n=0}^{\infty}\frac{4^n}{2n+1}\binom{2n}{n}^{-1}x^{2n+1}=\sum_{n=0}^{\infty}\left(\int_0^{\frac{\pi}{2}}\sin^{2n+1}t\ \mathrm{d}t\right)x^{2n+1}$ $\displaystyle=\sum_{n=0}^{\infty}\left(\int_0^{\frac{\pi}{2}}\left(x\sin t\right)^{2n+1}\mathrm{d}t\right)$ $\displaystyle\overset{(*)}{=}\int_0^{\frac{\pi}{2}}\left[\sum_{n=0}^{\infty}\left(x\sin t\right)^{2n+1}\right]\mathrm{d}t$ (*) we have swapped the summation and integral symbols. $\displaystyle=\int_0^{\frac{\pi}{2}}\left[\frac{x\sin t}{1-\left(x\sin t\right)^2}\right]\mathrm{d}t$ since $\lvert x\sin t\rvert<1$ $\displaystyle=\int_0^{\frac{\pi}{2}}\left[\frac{x\sin t}{1-x^2\left(1-\cos^2 t\right)}\right]\mathrm{d}t$	M1: Use part (b) M1: Sum of GP A1: [AG]
3d	$\displaystyle\int_0^{\frac{\pi}{2}}\left[\frac{x\sin t}{1-x^2\left(1-\cos^2 t\right)}\right]\mathrm{d}t\overset{u=\cos t}{=}-x\int_1^0\left[\frac{1}{1-x^2\left(1-u^2\right)}\right]\mathrm{d}u$ $\displaystyle=\frac{1}{x}\int_0^1\left(\frac{1}{\frac{1-x^2}{x^2}+u^2}\right)\mathrm{d}u$ $\displaystyle=\frac{1}{x}\cdot\frac{x}{\sqrt{1-x^2}}\left[\tan^{-1}\left(\frac{xu}{\sqrt{1-x^2}}\right)\right]_0^1$ $\displaystyle=\frac{1}{\sqrt{1-x^2}}\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ Let $t=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\Rightarrow\tan t=\frac{x}{\sqrt{1-x^2}}$ $\Rightarrow\sin t=x$ $\Rightarrow t=\sin^{-1}x$ $\displaystyle\sum_{n=0}^{\infty}\frac{4^n}{2n+1}\binom{2n}{n}^{-1}x^{2n+1}=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$.	M1: Substitution M1: Integrate A1: Justify $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ $=\sin^{-1}x$ to get answer [AG]
3e	$\displaystyle\sum_{n=0}^{\infty}\frac{4^n}{2n+1}\binom{2n}{n}^{-1}x^{2n+1}=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$. Differentiate w.r.t. x the equation in **(c)** and LHS term by term for $\lvert x\rvert<1$ $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}\left[\frac{4^n}{2n+1}\binom{2n}{n}^{-1}x^{2n+1}\right]=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin^{-1}x}{\sqrt{1-x^2}}\right)$	M2: Differentiate both sides (M1 for each side)

	$\sum_{n=0}^{\infty}\left[\frac{4^n}{2n+1}\binom{2n}{n}^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2n+1}\right)\right]=\frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x)\sin^{-1}x$ $\sum_{n=0}^{\infty}4^n\binom{2n}{n}^{-1}x^{2n}=\frac{1}{1-x^2}+\frac{x\sin^{-1}x}{\left(1-x^2\right)^{\frac{3}{2}}}$ for $\lvert x\rvert<1$ Putting $x=\frac{1}{2}$ into above, $\sum_{n=0}^{\infty}4^n\binom{2n}{n}^{-1}\frac{1}{2^{2n}}=\frac{1}{1-\frac{1}{4}}+\frac{\frac{1}{2}\sin^{-1}\frac{1}{2}}{\left(1-\frac{1}{4}\right)^{\frac{3}{2}}}$ $\Rightarrow\sum_{n=0}^{\infty}\binom{2n}{n}^{-1}=\frac{4}{3}+\frac{2\pi}{9\sqrt{3}}$ Hence by **(a)**, $\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}=\sum_{n=0}^{\infty}\binom{2n}{n}^{-1}=\frac{4}{3}+\frac{2\pi}{9\sqrt{3}}$.	M1: Substitute $x=\frac{1}{2}$ to get $\sum_{n=0}^{\infty}\binom{2n}{n}^{-1}$ A1

4a	$x^3 \equiv y^3 \equiv z^3 \pmod{p}$ $p \mid \left(x^3 - y^3\right)$ $p \mid (x-y)\left(x^2+y^2+xy\right)$ Since $-p < x-y < 0$, p does not divide $x-y$ and so $p \mid \left(x^2+y^2+xy\right)$.	B1: p divides difference M1: Factorisation A1: Explain to show result [AG]
4b	WLOG, by the result in **(a)**, $p \mid \left(y^2+z^2+yz\right)$ too. Hence, p divides $\left(x^2+y^2+xy\right)-\left(y^2+z^2+yz\right)=x^2-z^2+xy-yz=(x-z)(x+y+z)$. Again, $-p < x-z < 0$, p does not divide $x-z$ and so $p \mid (x+y+z)$.	B1: p divides y^2+z^2+yz B1: p divides difference to show result [AG]
4c	By part **(b)**, p divides $x+y+z$. Furthermore, $0 < x+y+z < 3p$ and so, $x+y+z=p$ or $x+y+z=2p$. <u>Case 1</u>: $x+y+z=p$ Here, p divides $(x+y+z)^2 = x^2+y^2+z^2+2(xy+xz+yz)$. By the result in **(a)** again, WLOG, we see that p divides x^2+y^2+xy, x^2+z^2+xz and y^2+z^2+yz. Hence, p divides the sum given by $2\left(x^2+y^2+z^2\right)+2(xy+xz+yz)$. Therefore, $x+y+z=p$ divides $\left[2\left(x^2+y^2+z^2\right)+2(xy+xz+yz)\right]-\left[x^2+y^2+z^2+2(xy+xz+yz)\right]$ $=x^2+y^2+z^2$. <u>Case 2</u>: $x+y+z=2p$ Similarly, $p \mid (x+y+z) \Rightarrow p \mid (x+y+z)^2 \Rightarrow p \mid \left(x^2+y^2+z^2\right)$ by the same argument given in Case 1. Since $x+y+z=2p$ is even, $x^2+y^2+z^2=(x+y+z)^2-2(xy+xz+yz)$ is even too. Now since $2 \mid \left(x^2+y^2+z^2\right)$, $p \mid \left(x^2+y^2+z^2\right)$, and $\gcd(2,p)=1$, $x+y+z=2p$ must divide $x^2+y^2+z^2$.	B1 M1: Attempt to show p divides a sum/diff involving $x^2+y^2+z^2$ A1: Show p divides $2\left(x^2+y^2+z^2\right)$ $+2(xy+xz+yz)$ B1: Show p divides $x^2+y^2+z^2$ B1: Explain that $2 \mid \left(x^2+y^2+z^2\right)$ B1: Show $2p$ divides $x^2+y^2+z^2$

5a	Suppose that $\mathrm{f}(x_0)>1$ for some $x_0\in\mathbb{R}^+$. Let $y_0=\dfrac{x_0}{\mathrm{f}(x_0)-1}$, then $\mathrm{f}(x_0)\mathrm{f}(y_0\,\mathrm{f}(x_0))=\mathrm{f}(x_0+y_0)$ $\Rightarrow \mathrm{f}(x_0)\mathrm{f}\left(\dfrac{x_0}{f(x_0)-1}\times \mathrm{f}(x_0)\right)=\mathrm{f}\left(x_0+\dfrac{x_0}{f(x_0)-1}\right)=\mathrm{f}\left(\dfrac{x_0 f(x_0)}{f(x_0)-1}\right).$ Hence, we have $\mathrm{f}(x_0)=1$, a contradiction. Hence, $\mathrm{f}(x)\le 1$ for all $x\in\mathbb{R}^+$.	M1: Attempt to express *y* in terms of *x* and substitute M1: Find equation in $\mathrm{f}(x_0)$ only A1: Derive a contradiction correctly
5b	By part **(a)**, $\mathrm{f}(x)\le 1$ for all $x\in\mathbb{R}^+$ and so $\mathrm{f}(x+y)=\mathrm{f}(x)\mathrm{f}(y\,\mathrm{f}(x))\le \mathrm{f}(x)$ for all $x,y\in\mathbb{R}^+$. This shows that f is non-increasing on $\mathbb{R}^+$. Now suppose $\mathrm{f}(x_0)=1$ for some $x_0\in\mathbb{R}^+$. Then we have $\mathrm{f}((k+1)x_0)=\mathrm{f}(x_0+kx_0)=\mathrm{f}(x_0)\mathrm{f}(kx_0\,\mathrm{f}(x_0))=\mathrm{f}(kx_0)$ for all $k\in\mathbb{R}^+$. In particular, $1=\mathrm{f}(x_0)=\mathrm{f}(2x_0)=\mathrm{f}(3x_0)=\ldots$, and by the non-increasing property of f on $\mathbb{R}^+$, this shows that $\mathrm{f}(x)=1$ for all $x\in\mathbb{R}^+$.	B1: Show f is non-increasing M1: Attempt to show more than 2 *x*'s such that $\mathrm{f}(x)=1$ A1: Show infinitely many *x*'s with $\mathrm{f}(x)=1$ and conclude correctly
5c	Replace *x* with 1 and *y* with $x+\dfrac{1}{\mathrm{f}(x)}-1$ in $\mathrm{f}(x)\mathrm{f}(y\,\mathrm{f}(x))=\mathrm{f}(x+y)$: $\mathrm{f}(1)\mathrm{f}\left(\left(x+\dfrac{1}{\mathrm{f}(x)}-1\right)\mathrm{f}(1)\right)=\mathrm{f}\left(1+x+\dfrac{1}{\mathrm{f}(x)}-1\right)$ $=\mathrm{f}\left(x+\dfrac{1}{\mathrm{f}(x)}\right)$ $=\mathrm{f}(x)\mathrm{f}\left(\dfrac{1}{\mathrm{f}(x)}\mathrm{f}(x)\right)$ (by the original eqn.) $=\mathrm{f}(x)\mathrm{f}(1)$ $\mathrm{f}\left(\left(x+\dfrac{1}{\mathrm{f}(x)}-1\right)\mathrm{f}(1)\right)=\mathrm{f}(x)\quad(\because \mathrm{f}(1)\neq 0)$	B1: Use given replacement and simplify B1: Apply formula given to get answer [AG]
5d	Given $\mathrm{f}(1)=\dfrac{1}{2}$, so by parts **(a)** and **(b)**, $\mathrm{f}(x)<1$ for all $x\in\mathbb{R}^+$. Hence, $\mathrm{f}(x+y)=\mathrm{f}(x)\mathrm{f}(y\,\mathrm{f}(x))<\mathrm{f}(x)$ for all $x,y\in\mathbb{R}^+$, which shows that f is decreasing. Thus, f is injective, and by the result in part **(c)**: $\mathrm{f}\left(\left(x+\dfrac{1}{\mathrm{f}(x)}-1\right)\mathrm{f}(1)\right)=\mathrm{f}(x)$ $\Rightarrow\left(x+\dfrac{1}{\mathrm{f}(x)}-1\right)\mathrm{f}(1)=x\quad(\because \text{f is injective})$ $\Rightarrow\dfrac{1}{\mathrm{f}(x)}=2x+1-x\quad\left(\because \mathrm{f}(1)=\dfrac{1}{2}\right)$ $\Rightarrow \mathrm{f}(x)=\dfrac{1}{x+1}$	B1: Use previous results to explain $\mathrm{f}(x)<1$ for all $x\in\mathbb{R}^+$ B1: Use eqn. to show strictly decreasing [AG] M1: State and apply that f is injective A1: $\mathrm{f}(x)=\dfrac{1}{x+1}$

6a i	Since the primes are all odd, the common difference k must be even, i.e. k is either 2, 4, 6, 8 or 0 (mod 10). If $k \equiv 2 \pmod{10}$, the last digits of a_1, a_2, a_3, a_4, a_5 are consecutive terms of the sequence $1, 3, 5, 7, 9, 1, 3, 5, 7, 9, \ldots$. If $k \equiv 4 \pmod{10}$, the last digits of a_1, a_2, a_3, a_4, a_5 are consecutive terms of the sequence $1, 5, 9, 3, 7, 1, 5, 9, 3, 7, \ldots$. If $k \equiv 6 \pmod{10}$, the last digits of a_1, a_2, a_3, a_4, a_5 are consecutive terms of the sequence $1, 7, 3, 9, 5, 1, 7, 3, 9, 5, \ldots$. If $k \equiv 8 \pmod{10}$, the last digits of a_1, a_2, a_3, a_4, a_5 are consecutive terms of the sequence $1, 9, 7, 5, 3, 1, 9, 7, 5, 3, \ldots$. In all the cases above, there exists $a_i \equiv 0 \pmod 5$ which implies that a_i is divisible by 5, a contradiction. Hence, by exhaustion and listing, $k \equiv 0 \pmod{10}$.	B1: Identify k as odd B1: Explain why k cannot be just any even number – see at least 2 cases B1: Correct conclusion that 10 divides k
6a ii	11, 41, 71, 101, 131	B1: Any correct sequence
6b i	Since each b_i is prime and is greater than p, p does not divide b_i for any i. By pigeonhole principle, there exist b_m and b_n that are equal modulo p, i.e. $b_m \equiv b_n \pmod p$. Hence, p divides $b_n - b_m = d(n-m)$. Since $n - m < p$, p cannot divide $n - m$ and hence must divide d.	B1: p not dividing any b_i B1: Apply pigeonhole principle B1: Property of AP, $d(n-m)$ B1: Explain why p divides d
6b ii	Consider the sequence 5, 11, 17, 23, 29. Then $b_1 = p = 5$ and $d = 6$, but clearly, p does not divide d.	B1: Any valid sequence B1: Show p not dividing d
6b iii	By the result in part **(b)(i)**, we see consider $b_1, b_2, b_3, \ldots, b_k$ for $k = 2, 3, 5, \ldots, 17$ and see that each k divides d. Since 2, 3, 5, …, 17 divide d, and they are all coprime, the product $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 = 510510$ must divide d too.	B1: Consider sub-sequences B1: Show and evaluate product [AG]

7a i	Let $a, b, c, d \in S$ such that $a < b < c < d$. Since $a-1 \notin S$, it follows that $a+1 \in S$. Hence $b = a+1$. Similarly, since $d+1 \notin S$, it follows that $d-1 \in S$. Hence $c = d-1$. Therefore, a connected set has the form $\{a, a+1, d-1, d\}$ with $d-a > 2$. For *n* = 7, the 10 connected subsets of the set $\{1, 2, 3, \ldots, 7\}$ are $\{1,2,3,4\}$, $\{1,2,4,5\}$, $\{1,2,5,6\}$, $\{1,2,6,7\}$ $\{2,3,4,5\}$, $\{2,3,5,6\}$, $\{2,3,6,7\}$ $\{3,4,5,6\}$, $\{3,4,6,7\}$ $\{4,5,6,7\}$ $\therefore C_7 = 10$.	M1 M1 A1
7a ii	Let $D = d - a$. Clearly $D > 2$ and $D \le n-1$. For $D = n-1$, there is only 1 connected set, namely $\{1, 2, n-1, n\}$ For $D = n-2$, there are 2 connected sets, namely $\{2, 3, n-1, n\}, \{1, 2, n-2, n-1\}$ and so on. For $D = 3$, there are $n-3$ connected sets. Adding up, we have $C_n = 1+2+3+\cdots+(n-3)$ $= \frac{n-3}{2}(1+n-3)$ $= \frac{(n-3)(n-2)}{2}$.	M1 M1 A1
7b	Let $\|S\|$ be the set of all possible arrangements of the set $\{1, 2, 3, \ldots, n\}$. We have $\|S\| = n!$. For $i = 1, 2, \cdots, n$, let A_i be the set of all permutations where the number *i* is at the *i*th position. Clearly, we are interested in $\left\|A_1' \cap A_2' \cap \ldots \cap A_n'\right\|$ $= \|S\| - \|A_1 \cup A_2 \cup \ldots \cup A_n\|$ $= \|S\| - \sum_{i=1}^{n} \|A_i\| + \sum_{i<j} \|A_i \cap A_j\| - \sum_{i<j<k} \|A_i \cap A_j \cap A_k\|$ $+ \cdots + (-1)^n \|A_1 \cap A_2 \cap \ldots \cap A_n\|$ Let *r* be an integer such that $1 \le r \le n$. Suppose that that we are required to find the number of permutations with $i_1, i_2, \cdots, i_r$ in their original position. Then the number of permutations to arrange the remaining $(n-r)$ numbers is $(n-r)!$. Hence, we have $\left\|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}\right\| = (n-r)!$. This means that	M1 M1 A1

	$\left\|A_1^{'}\cap A_2^{'}\cap\ldots\cap A_n^{'}\right\| = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^n\binom{n}{n}1!$ $= n!\left[1 - \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n(n-1)} - \binom{n}{3}\frac{1}{n(n-1)(n-2)} + \ldots + (-1)^n\binom{n}{n}\frac{1}{n!}\right]$ $= n!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}\right]$ As $n \to \infty$, $1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!} \to \mathrm{e}^{-1}$. Hence $\left\|A_1^{'}\cap A_2^{'}\cap\ldots\cap A_n^{'}\right\| = n!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}\right] \to \frac{n!}{\mathrm{e}}$.	A1 B1: [AG]

8a	**Method 1:** Let $a = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$ be the prime factorisation of *a* where $p_1, p_2, \dots, p_m$ are distinct primes and $k_1, k_2, \dots, k_m$ are some positive integers. Thus $a^n = p_1^{nk_1} p_2^{nk_2} \dots p_m^{nk_m}$. Since $p \mid a^n$ and if *p* is one of $p_1, p_2, \dots, p_m$, we are done as clearly $p \mid a$ also. If *p* is not one of $p_1, p_2, \dots, p_m$, since *p* does not divide $p_1, p_2, \dots, p_m$ and $p \mid a^n$, this implies that $p \mid 1 \Rightarrow p = 1$ which is a contraction. Thus *p* must one of $p_1, p_2, \dots, p_m$, $p \mid a^n \;\Rightarrow\; p \mid a$ **Method 2:** We prove by induction that if *p* is a prime *a* is an integer, then (*) $p \mid a^n \Rightarrow p \mid a$ where *n* is a positive integer. For *n* = 1, $p \mid a^1 \Rightarrow p \mid a$ so (*) holds trivially. Suppose $p \mid a^k \Rightarrow p \mid a$ for some positive integer *k*. (IH) $p \mid a^{k+1} \Rightarrow p \mid a\left(a^k\right) \Rightarrow p \mid a$ or $p \mid a^k$ since *p* is prime by Euclid's Lemma [Euclid's Lemma: If *p* is prime, then $p \mid ab \Rightarrow p \mid a$ or $p \mid b$] If $p \mid a$, then we have our desired conclusion. If $p \mid a^k$, then by (IH), $p \mid a$. Thus in either case, $p \mid a^{k+1} \Rightarrow p \mid a$. Hence (*) holds .	M1: Prime factorisation B1 B1 M1: Euclid's Lemma B1 B1: Use IH
8b	$x^n + c_{n-1}x^{n-1} + \dots + c_1 x + c_0 = 0$ for some integers $c_0, c_1, \dots, c_{n-1}$. Let $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}, b \neq 0$ and assume that $\gcd(a,b) = 1$ without loss of generality. Substitute into above gives $\left(\frac{a}{b}\right)^n + c_{n-1}\left(\frac{a}{b}\right)^{n-1} + \dots + c_1\left(\frac{a}{b}\right) + c_0 = 0$ $a^n + c_{n-1}a^{n-1}b + \dots + c_1 ab^{n-1} + c_0 b^n = 0$ $b\underbrace{\left(c_{n-1}a^{n-1} + \dots + c_1 ab^{n-2} + c_0 b^{n-1}\right)}_{\in \mathbb{Z}} = -a^n$ Thus $b \mid a^n$. <u>Case 1</u>: $\lvert b \rvert = 1 \Rightarrow b = \pm 1$ In this case, $x = \frac{a}{b} = \pm a \in \mathbb{Z}$. <u>Case 2</u>: $\lvert b \rvert > 1$. In this case, there exists *p* prime such that $p \mid \lvert b \rvert$ or simply $p \mid b$. But since $b \mid a^n$, it follows that $p \mid a^n$. By (a), we conclude that $p \mid a$. Hence $p \mid a$ and $p \mid b$ implies *a* and *b* have a common divisor $p > 1$, contradicting the assumption that $\gcd(a,b) = 1$. Thus for case 2, $x \neq \frac{a}{b}$ where $a, b \in \mathbb{Z}, \lvert b \rvert > 1$. So *x* is irrational.	B1: Assumption M1 B1: Conclude $b \mid a^n$ M1: Case 1 B1: Argue $p \mid b$ M1: Conclude $p \mid a$ A1: Get contradiction

8c	Substitute $x = \sqrt{6}-\sqrt{3}-\sqrt{2}$ into $\left(x^2-11\right)^2$: $\left(x^2-11\right)^2 = \left[\left(\sqrt{6}-\sqrt{3}-\sqrt{2}\right)^2-11\right]^2$ $= \left[6+3+2+2\sqrt{6}-2\sqrt{6}\left(\sqrt{3}+\sqrt{2}\right)-11\right]^2$ $= \left[2\sqrt{6}\left(1-\left(\sqrt{3}+\sqrt{2}\right)\right)\right]^2$ $= 24\left[1-2\left(\sqrt{3}+\sqrt{2}\right)+\left(\sqrt{3}+\sqrt{2}\right)^2\right]$ $= 24\left(1-2\sqrt{3}-2\sqrt{2}+5+2\sqrt{6}\right)$ $= 24\left[6+2\left(\sqrt{6}-\sqrt{3}-\sqrt{2}\right)\right]$ $= 24(6+2x)$ Hence $\left(x^2-11\right)^2 = 24(6+2x) \Rightarrow x^4-22x^2-48x-23=0$. Since the coefficients are all integers, by (b), x is either an integer or an irrational number. Since $x \notin \mathbb{Z}$ ($-1 < x < 0$), x must be irrational.	M1: Simplify M1: RHS M1: Obtain polynomial eqn. B1: Argue by **(b)**