



H2 Math Year 6 Preliminary Examination Paper 1: Solutions with comments

- 1 A curve has parametric equations

$$x = t^2 + 1, \quad y = at^3 + bt^2 + ct,$$

where $t > -2$ and a, b and c are constants.

The curve has stationary points at $t = -1$ and $t = 2$. Given that the curve passes through the point $(5, -2)$, find the values of a, b and c . [4]

Solution	Comments
[4] $y = at^3 + bt^2 + ct$ $\frac{dy}{dt} = 3at^2 + 2bt + c$ At stationary points, $\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dt} = 0$ When $t = -1$, $3a - 2b + c = 0$ ----- (1) When $t = 2$, $12a + 4b + c = 0$ ----- (2) Since curve passes through the point $(5, -2)$, we have $5 = t^2 + 1 \Rightarrow t = \pm 2.$ Since $t > -2$, $t = 2$. When $t = 2$, $a(2)^3 + b(2)^2 + c(2) = -2$ $8a + 4b + 2c = -2$ ----- (3) Using GC, $a = \frac{1}{5}, b = -\frac{3}{10}, c = -\frac{6}{5}$	This question was generally well-done. Mistakes were mostly due to carelessness in simplification.

- 2 (a) The sequence $u_1, u_2, u_3, \dots$ is defined by $u_1 = 4$ and $u_{n+1} = -\frac{1}{3}u_n + 2$, for $n \geq 1$.

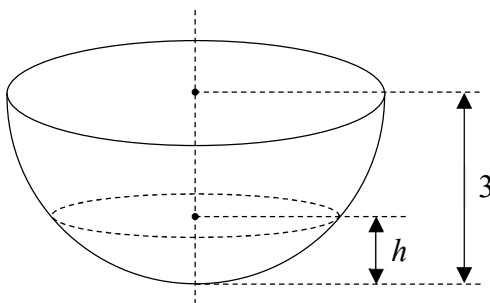
(i) Write down the value of u_6 . [1]

(ii) Given that as $n \rightarrow \infty$, $u_n \rightarrow l$, find the exact value of l using an algebraic method. [2]

- (b) Using the result $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$, find $\sum_{r=0}^n (n+r)(r+1)$, giving your answer in terms of n , in fully factorised form. [3]

Solution	Comments
(a) (i) [1]	A handful of students did not realise that this part can be done using GC.
(a) (ii) [2]	The critical step in this part is to state that “As $n \rightarrow \infty$, $u_n \rightarrow l$, $u_{n+1} \rightarrow l$ ” which many failed to do so.
(b) [3]	The common mistake in this part was not realizing that there are a total of $n+1$ terms in $\sum_{r=0}^n n$. There were students who did not read the question carefully, failing to leave the answer in fully factorized form.

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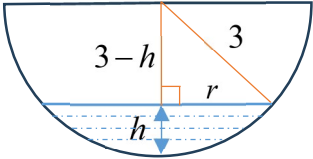


Water is poured at a rate of 0.5 m^3 per minute into an empty container in the form of an open hemisphere with inside radius 3 m (see diagram). At time t minutes after the start, the radius of the water surface is r m and the volume of the water in the hemispherical container is $\frac{1}{3}\pi h^2(9-h)$, where h m is the depth of the water. When the depth of the water is 1 m,

(a) find the rate of change of the depth of water, [3]

(b) find the rate of change of the radius of the water surface. [3]

Solution	Comments
(a) [3] Given $V = \frac{1}{3}\pi h^2(9-h) = \pi\left(3h^2 - \frac{1}{3}h^3\right)$ $\frac{dV}{dh} = \pi(6h - h^2)$ Now, $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $= \frac{1}{\pi(6h - h^2)} \times 0.5$ When $h = 1$, $\frac{dh}{dt} = \frac{0.5}{5\pi} = \frac{1}{10\pi} = 0.0318 \text{ (3 sf)}$ $\therefore$ The depth of water is increasing at a rate of 0.0318 m per minute.	You need to simplify and give the final answer as $\frac{1}{10\pi}$ (exact) or 0.0318 (3 s.f.). Note that the units in this case is metres per minute.

	OR Given $V = \frac{1}{3}\pi h^2(9-h) = 3\pi h^2 - \frac{1}{3}\pi h^3$ Differentiate w.r.t t, $\frac{dV}{dt} = 3\pi \left(2h \frac{dh}{dt} \right) - \pi h^2 \left(\frac{dh}{dt} \right)$ When $h = 1$, $\frac{dV}{dt} = 0.5$. $0.5 = 3\pi \left(2 \frac{dh}{dt} \right) - \pi \left(\frac{dh}{dt} \right)$ $\therefore \frac{dh}{dt} = \frac{0.5}{5\pi} = \frac{1}{10\pi} = 0.0318 \text{ (3 sf)}$	
(b) [3]	 From the diagram, by Pythagoras Theorem, $r^2 = 3^2 - (3-h)^2 = 6h - h^2$ Since $r > 0$, $r = \sqrt{6h - h^2}$ $\frac{dr}{dh} = \frac{1}{2} \frac{1}{\sqrt{6h - h^2}} (6 - 2h) = \frac{3-h}{\sqrt{6h - h^2}}$ $\frac{dr}{dt} = \frac{dr}{dh} \times \frac{dh}{dt}$ $= \frac{3-h}{\sqrt{6h - h^2}} \times \frac{1}{10\pi}$ When $h = 1$, $\frac{dr}{dt} = \frac{2}{10\sqrt{5}\pi} = \frac{1}{5\sqrt{5}\pi} = \frac{\sqrt{5}}{25\pi} = 0.0285 \text{ (3 s.f.)}$ $\therefore$ The radius of water surface is increasing at a rate of 0.0285 m per minute.	Common mistake: Equating the volume of water to that of the volume of a hemisphere. $V = \frac{2}{3}\pi r^3 = \frac{1}{3}\pi h^2(9-h)$ The shape of the lower part of the container is not a hemisphere. Many students tried to apply similar triangles (though incorrectly) to find the relation between r and h. See alternative for the correct version.

OR

From the diagram, by Pythagoras Theorem,

$$r^2 = 3^2 - (3 - h)^2$$

Differentiate w.r.t t ,

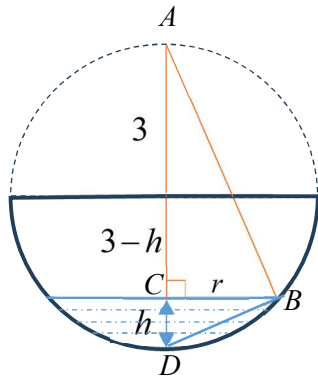
$$2r \frac{dr}{dt} = -2(3 - h)(-1) \frac{dh}{dt}$$

$$r \frac{dr}{dt} = (3 - h) \frac{dh}{dt}$$

When $h = 1$, $r^2 = 5 \Rightarrow r = \sqrt{5}$ (since $r > 0$).

$$\sqrt{5} \frac{dr}{dt} = (2) \frac{1}{10\pi}$$

$$\frac{dr}{dt} = \frac{1}{5\sqrt{5}\pi} = \frac{\sqrt{5}}{25\pi} = 0.0285 \text{ (3 s.f.)}$$

Alternative method to obtain relation between r and h
(via similar triangles): $\triangle ABC$ is similar to $\triangle BDC$. By similar triangles,

$$\frac{6 - h}{r} = \frac{r}{h}$$

$$r^2 = h(6 - h) = 6h - h^2$$

- 4 With reference to the origin O , three distinct points A , B and C are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. The mid-point of AC is D .
- (a) Express $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. [1]
- (b) Given that $|\mathbf{a}| = |\mathbf{c}|$, $|\mathbf{b}| \neq 0$ and BD is perpendicular to AC , show that OB is also perpendicular to AC by using a suitable scalar product. [3]
- (c) Given further that point D lies on line segment OB , $|\mathbf{a}| = 2$, $|\mathbf{b}| = 6$ and $\angle AOC = 60^\circ$, find the exact area of quadrilateral $OABC$. [2]

Solution		Comments
(a) [1]	$\overrightarrow{OD} = \frac{\mathbf{a} + \mathbf{c}}{2}$	Note: $\overrightarrow{OD} \neq \frac{1}{2} \overrightarrow{AC}$
(b) [3]	$\overrightarrow{BD} = \frac{\mathbf{a} + \mathbf{c}}{2} - \mathbf{b}$ Since BD is perpendicular to AC, $\overrightarrow{BD} \cdot \overrightarrow{AC} = 0$ $\Rightarrow \left(\frac{\mathbf{a} + \mathbf{c}}{2} - \mathbf{b} \right) \cdot (\mathbf{c} - \mathbf{a}) = 0$ $\Rightarrow \frac{1}{2} \mathbf{a} \cdot \mathbf{c} - \frac{1}{2} \mathbf{a} \cdot \mathbf{a} + \frac{1}{2} \mathbf{c} \cdot \mathbf{c} - \frac{1}{2} \mathbf{c} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a} = 0$ $\Rightarrow \frac{1}{2} \mathbf{a} \cdot \mathbf{c} - \frac{1}{2} \mathbf{a} ^2 + \frac{1}{2} \mathbf{c} ^2 - \frac{1}{2} \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a} = 0$ $\Rightarrow -\mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{a} = 0$ $\Rightarrow \mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0$ Since $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a} \neq \mathbf{c}$, OB is perpendicular to AC. (shown) Alternative: $\overrightarrow{OB} \cdot \overrightarrow{AC} = (\overrightarrow{OD} - \overrightarrow{BD}) \cdot \overrightarrow{AC}$ $= \overrightarrow{OD} \cdot \overrightarrow{AC} - \overrightarrow{BD} \cdot \overrightarrow{AC}$ As $\mathbf{a} = \mathbf{c}$, $\triangle AOC$ is isosceles and since the mid-point of AC is D, OD is perpendicular to AC. Thus $\overrightarrow{OD} \cdot \overrightarrow{AC} = 0$.	Note: $\mathbf{a} \cdot \mathbf{a} = \mathbf{a} ^2$, $\mathbf{c} \cdot \mathbf{c} = \mathbf{c} ^2$ $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \mathbf{c} \cos \angle AOC$ $\neq \mathbf{a} \mathbf{c} $

It is incorrect to assume $\overrightarrow{OB} = k \overrightarrow{BD}$ (or O , B and D are collinear) even though OD and BD are both perpendicular to AC . This information is

	Also, as BD is perpendicular to AC, $\overrightarrow{BD} \cdot \overrightarrow{AC} = 0$. Thus, $\overrightarrow{OB} \cdot \overrightarrow{AC} = \overrightarrow{OD} \cdot \overrightarrow{AC} - \overrightarrow{BD} \cdot \overrightarrow{AC}$ $= 0$ That is, OB is perpendicular to AC. (shown)	only given in part (c). In (b), AC is perpendicular to the plane that contains points O, B and D.
(c) [2]	Given that point D lies on line segment OB, then OB (or DB) is the perpendicular bisector of AC. Area of $\triangle OAB$ = Area of $\triangle OCB$ Therefore, Area of $OABC$ $= 2 \times \text{Area of } \triangle OAB$ $= 2 \left(\frac{1}{2} \mathbf{a} \times \mathbf{b} \right)$ $= 2 \times \frac{1}{2} \mathbf{a} \mathbf{b} \sin \left(\frac{60}{2} \right)^\circ$ $= 2 \times 6 \times \frac{1}{2} = 6 \text{ units}^2$ Alternative: Since $\triangle OAC$ is isosceles (as $\mathbf{a} = \mathbf{c}$) and $\angle AOC = 60^\circ$, $\triangle OAC$ is an equilateral triangle. Hence $AC = 2$. Or $AD = 2 \sin 30^\circ = 1$. Hence $AC = 2$. Quadrilateral $OABC$ forms a kite. Therefore, Area of $OABC$ $= \frac{1}{2} (AC)(OD) + \frac{1}{2} (AC)(DB)$ $= \frac{1}{2} (AC)(OD + DB)$ $= \frac{1}{2} (AC)(OB)$ $= \frac{1}{2} \times 2 \times 6 = 6 \text{ units}^2$	Note: Area of $\triangle OAC$ is not the same as area of $\triangle BAC$. D is given to be the midpoint of AC but it is not the midpoint of OB. The diagonals do not bisect each other. In fact, $OD = 2 \cos 30^\circ = \sqrt{3}$ and $BD = 6 - \sqrt{3}$. $OABC$ is not a parallelogram (or rhombus). Thus area of $OABC \neq \mathbf{a} \times \mathbf{c}$.

5 Do not use a calculator in answering this question.

It is given that $1 + 2i$ is a complex root of the quartic equation

$$z^4 + pz^3 + qz^2 + rz + 45 = 0,$$

where p , q and r are real numbers.

- (a) State, with justification, another complex root of the equation in the form $a + ib$, where a and b are real numbers and $b \neq 0$. Hence find a quadratic factor corresponding to these two roots. [2]

The quartic equation has a repeated negative real root.

- (b) Find the value of the repeated real root and hence find the values of p , q and r . [4]
- (c) By means of a suitable substitution, solve $45w^4 + rw^3 + qw^2 + pw + 1 = 0$. [2]

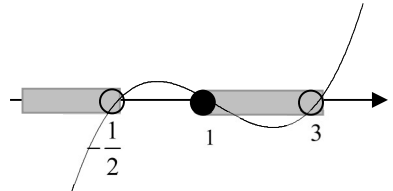
Solution	Comments
(a) [2] Since all the coefficients of the polynomial are real, non-real roots occur in conjugate pairs. Hence $1 - 2i$ is also a root. $\begin{aligned} & [z - (1 + 2i)][z - (1 - 2i)] \\ &= [(z - 1) - 2i][(z - 1) + 2i] \\ &= (z - 1)^2 - (2i)^2 \\ &= z^2 - 2z + 1 + 4 \\ &= z^2 - 2z + 5 \end{aligned}$	Some spent way too much time on the justification. Do refer to the solution and write down your justification succinctly. When finding the quadratic factor, rearrange the terms as shown in the 2nd step and apply the identity $(x - y)(x + y) = x^2 - y^2$ to find the factor effortlessly.
(b) [4] Let the repeated real root be k, where $k < 0$. Then we have $\begin{aligned} z^4 + pz^3 + qz^2 + rz + 45 &= (z^2 - 2z + 5)(z - k)^2 \\ &= (z^2 - 2z + 5)(z^2 - 2kz + k^2) \end{aligned}$ Comparing constant terms $45 = 5k^2 \Rightarrow k = \pm 3$	Make use of the information given, that k is a repeated negative real root, so $k < 0$ and then write out the factorized form (the RHS of the equation) accordingly.

	Since k is negative, $k = -3$. We now have $z^4 + pz^3 + qz^2 + rz + 45 = (z^2 - 2z + 5)(z^2 + 6z + 9)$ Comparing coefficients of $z^3 : p = 6 - 2 = 4$ $z^2 : q = 9 - 12 + 5 = 2$ $z : r = -18 + 30 = 12$	Failure to consider the fact the k is a negative root often led to errors/long and tedious methods.
(c) [2]	$45w^4 + rw^3 + qw^2 + pw + 1 = 0$ Divide throughout by w^4, $45 + r\frac{1}{w} + q\frac{1}{w^2} + p\frac{1}{w^3} + \frac{1}{w^4} = 0$ Replace w with $\frac{1}{z}$, Hence the roots are $w = \frac{1}{1+2i} \times \frac{1-2i}{1-2i} = \frac{1}{5} - \frac{2}{5}i$ or $w = \frac{1}{1-2i} \times \frac{1+2i}{1+2i} = \frac{1}{5} + \frac{2}{5}i$ or $w = -\frac{1}{3}$.	As the calculator is not allowed for this question, students are reminded to show clear working on how they arrive at the final answer.

- 6 (a) Show that $2x^2 - 4x + 3$ is always positive for all real values of x .

Without using a calculator, solve the inequality $\frac{x^2 - 10x + 3}{(2x+1)(x-3)} \geq x$. [6]

- (b) Hence solve the inequality $\frac{x^2 - 10|x| + 3}{(2|x|+1)(|x|-3)} > |x|$. [2]

Solution	Comments
(a) [6] $2x^2 - 4x + 3 = 2(x^2 - 2x) + 3$ $= 2[(x-1)^2 - 1] + 3$ $= 2(x-1)^2 + 1$ Since $(x-1)^2 \geq 0 \quad \forall x \in \mathbb{R} \Rightarrow 2(x-1)^2 + 1 > 0$ Thus $2x^2 - 4x + 3$ is always positive for all real values of x. (shown) $\frac{x^2 - 10x + 3}{(2x+1)(x-3)} \geq x$ $\frac{x^2 - 10x + 3}{(2x+1)(x-3)} - x \geq 0$ $\frac{x^2 - 10x + 3 - x(2x+1)(x-3)}{(2x+1)(x-3)} \geq 0$ $\frac{x^2 - 10x + 3 - x(2x^2 - 5x - 3)}{(2x+1)(x-3)} \geq 0$ $\frac{-2x^3 + 6x^2 - 7x + 3}{(2x+1)(x-3)} \geq 0$ $\frac{2x^3 - 6x^2 + 7x - 3}{(2x+1)(x-3)} \leq 0$ By observation, $2(1)^3 - 6(1)^2 + 7(1) - 3 = 0$. $\frac{(x-1)(2x^2 - 4x + 3)}{(2x+1)(x-3)} \leq 0$ Since $2x^2 - 4x + 3 > 0 \quad \forall x \in \mathbb{R}$, the inequality reduces to $\frac{(x-1)}{(2x+1)(x-3)} \leq 0, \quad x \neq -\frac{1}{2}, 3$ $(x-1)(2x+1)(x-3) \leq 0$ $\therefore x < -\frac{1}{2} \quad \text{or} \quad 1 \leq x < 3.$ 	There are a couple of ways to show that $2x^2 - 4x + 3$ is always positive, but the method shown here is the most efficient. Students ought to read the question carefully and note that the calculator is not allowed in the solving of this part. As such, there must be working showing evidence that $x-1$ is a factor of $2x^3 - 6x^2 + 7x - 3$. You can then write down the quadratic factor $2x^2 - 4x + 3$ by observation if you are able to since the calculator is not able to help you with that. Clear workings also should be shown for the subsequent part to show how you arrived at the final answer. Otherwise, no credit will be given.

(b) [2]	To solve $\frac{x^2 - 10 x + 3}{(2 x + 1)(x - 3)} > x$, using (a), replace x by x, $ x < -\frac{1}{2} \quad \text{or} \quad 1 < x < 3$ No solution since $-3 < x < -1$ or $1 < x < 3$ $ x \geq 0 \quad \forall x \in \mathbb{R}$	Students should be more careful when reading the question - notice that the inequality sign in part (b) has changed to a strict inequality. Students should also exercise care with what they write. Do not write "Let $x = x$" as the solutions for this equation is $x \geq 0$ which is not true.
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- 7 Conservationists are investigating the effect of habitat loss on the number of deer on an island. They observe that the population of the deer on the island decreases at a rate of 2% every year. The number of deer on the island is P at a time t years after conservationists begin observations.

(a) Write down a differential equation relating P and t . [1]

To maintain population stability, the conservationists relocate deer to the island at a constant uniform rate of h deer per year.

(b) Write down a differential equation to model the new situation. [1]

(c) Without solving the differential equation, determine the value of h if the conservationists would like the number of deer to settle down to 600 after many years. [2]

(d) Solve the differential equation to find an expression for P in terms of t and h . Hence, state the population of the deer, in terms of h , in the long term. [5]

Solution		Comments
(a) [1]	$\frac{dP}{dt} = -0.02P$	If you write in fractions, simplify $\frac{2}{100}$ to $\frac{1}{50}$.

(b) [1]	$\frac{dP}{dt} = -0.02P + h$	Rate of change = rate of increase – rate of decrease
(c) [2]	When the number of deer settles down to 600 after many years, we have $P = 600$ when $\frac{dP}{dt} = 0$. $\therefore -0.02(600) + h = 0 \Rightarrow h = 12$	This is a continuous model, not a discrete model, so you should not be using sequences or series.
(d) [5]	$\frac{dP}{dt} = -0.02P + h$ $\frac{1}{-0.02P + h} \frac{dP}{dt} = 1$ Integrate w.r.t. t, $\frac{1}{-0.02} \int \frac{-0.02}{-0.02P + h} dP = \int 1 dt$ $\frac{1}{-0.02} \ln -0.02P + h = t + c, \text{ where } c \in \mathbb{R}$ $\ln -0.02P + h = -0.02(t + c)$ $ -0.02P + h = e^{-0.02(t+c)}$ $-0.02P + h = \pm e^{-0.02t} \cdot e^{-0.02c}$ $-0.02P + h = Ae^{-0.02t}, \text{ where } A = \pm e^{-0.02c}$ $0.02P = h - Ae^{-0.02t}$ $P = 50(h - Ae^{-0.02t})$ When $t \rightarrow \infty$, $e^{-0.02t} \rightarrow 0$. Hence $P \rightarrow 50h$. $\therefore$ The population of the deer approaches $50h$ in the long term.	Need to have modulus as $-0.02P + h$ may be positive or negative. Remember to have $\pm$ when you remove the modulus and simplify $\pm e^{-0.02c}$ to another constant, e.g. A. The question is asking for the general solution as there is no condition given to find a particular solution. Express your answer as “$P = \dots$” and not “$0.02P = \dots$” etc. Do simplify $\frac{h}{0.02}$ to $50h$.

	Alternative solution: $\frac{dP}{dt} = -\frac{1}{50}P + h = -\frac{1}{50}(P - 50h)$ $\int \frac{1}{P - 50h} dP = \int -\frac{1}{50} dt$ $\ln P - 50h = -\frac{1}{50}t + c, \text{ where } c \in \mathbb{R}$ $ P - 50h = e^{-\frac{1}{50}t + c}$ $P - 50h = \pm e^{-\frac{1}{50}t} \cdot e^c = Ae^{-\frac{1}{50}t}, \text{ where } A = \pm e^c$ $P = 50h + Ae^{-\frac{1}{50}t}$	
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- 8 The curve C has parametric equations

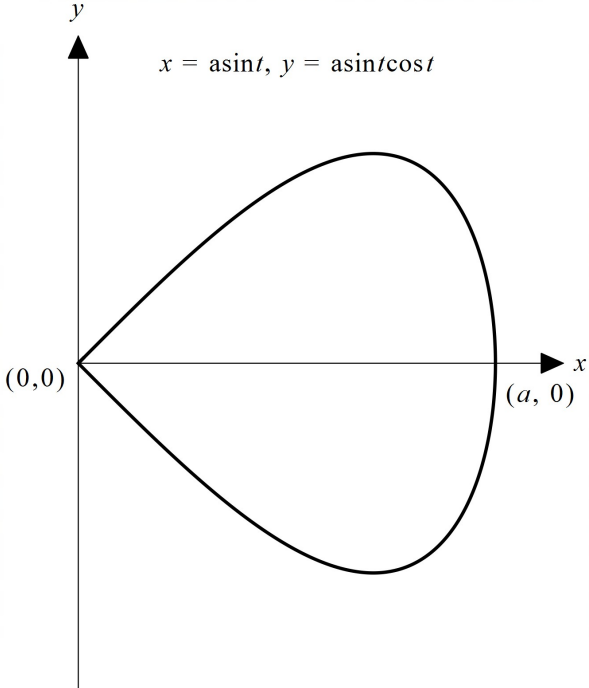
$$x = a \sin t, \quad y = a \sin t \cos t,$$

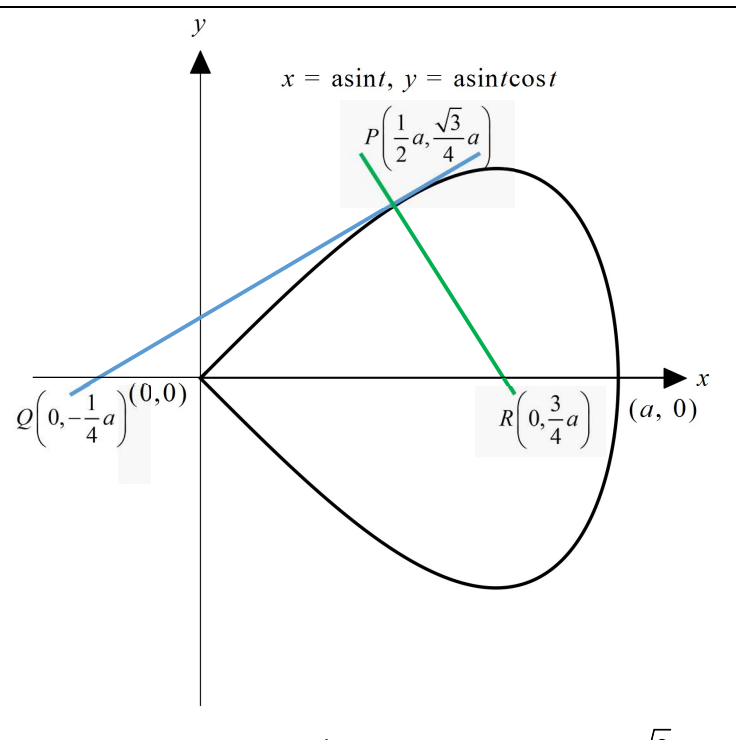
for $0 \leq t < \pi$ and a is a positive constant.

- (a) Sketch the graph of C , stating clearly the coordinates of the points where the curve C meets the axes. [2]
- (b) State the equation of the line of symmetry. [1]
- (c) Find the equation(s) of the tangent to C which is parallel to the x -axis. [3]

The tangent and normal to C at the point P where $t = \frac{\pi}{6}$ cut the x -axis at points Q and R respectively.

- (d) Find the exact area of triangle PQR . [4]

Solution		Comments
(a) [2]	 $x = a \sin t, y = a \sin t \cos t$	Students are reminded to take note of the range of values that the parameter can take when sketching graphs given in parametric equations. Many students did not change the values of t to $0 \leq t < \pi$ and used the default range of values of t instead (after resetting GC) which were $0 \leq t < 2\pi$ and thus resulting in another “loop” on the negative side of the x axis. For students who replaced the value of a with a number to key into the GC to help them to sketch the graph, do remember to indicate the x-intercept as $(a, 0)$ and not the number.
(b) [1]	Line of symmetry: $y = 0$	Not as well done as expected for this part with many students putting their answers down as $x = 0$ even when they have sketched the correct graph in part (a). Students are reminded that the x axis has equation $y = 0$ and not $x = 0$.
(c) [3]	$x = a \sin t \quad y = a \sin t \cos t$ $\frac{dx}{dt} = a \cos t \quad \frac{dy}{dt} = a(\cos^2 t - \sin^2 t)$ $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{a(\cos^2 t - \sin^2 t)}{a \cos t} = \frac{\cos^2 t - \sin^2 t}{\cos t} \quad \text{OR} \quad \frac{\cos 2t}{\cos t}$ For tangent to be parallel to the x-axis, gradient of tangent is 0. $\cos^2 t - \sin^2 t = 0$ $\tan t = 1 \quad \text{or} \quad \tan t = -1$ $t = \frac{\pi}{4} \quad t = \frac{3\pi}{4}$ OR $\cos 2t = 0$ Since $0 \leq t < \pi$, $0 \leq 2t < 2\pi$. $2t = \frac{\pi}{2} \quad 2t = \frac{3\pi}{2}$ $t = \frac{\pi}{4} \quad t = \frac{3\pi}{4}$	Many students were able to perform parametric differentiation well. However, when it comes to solving for the values of t that will result in $\frac{dy}{dx} = 0$, many students missed out on the value of $t = \frac{3\pi}{4}$.

	Equations of tangent which are parallel to the x-axis are $y = a \sin \frac{\pi}{4} \cos \frac{\pi}{4} = \frac{a}{2}$ or $y = a \sin \left(\frac{3\pi}{4} \right) \cos \left(\frac{3\pi}{4} \right) = -\frac{a}{2}$	
(d) [4]	 At $t = \frac{\pi}{6}$, $x = a \sin \frac{\pi}{6} = \frac{1}{2}a$, $y = a \sin \frac{\pi}{6} \cos \frac{\pi}{6} = \frac{\sqrt{3}}{4}a$ $\frac{dy}{dx} = \frac{\cos \frac{\pi}{3}}{\cos \frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$ Equation of tangent at P is $y - \frac{\sqrt{3}}{4}a = \frac{1}{\sqrt{3}}\left(x - \frac{1}{2}a\right)$ When $y = 0$, $-\frac{\sqrt{3}}{4}a = \frac{1}{\sqrt{3}}\left(x - \frac{1}{2}a\right) \Rightarrow x = \frac{1}{2}a - \frac{3}{4}a = -\frac{1}{4}a$ $\therefore Q\left(-\frac{1}{4}a, 0\right)$ Gradient of normal $= -\sqrt{3}$. Equation of normal at P is $y - \frac{\sqrt{3}}{4}a = -\sqrt{3}\left(x - \frac{1}{2}a\right)$	Two common mistakes seen: 1. Area of triangle $= \frac{\sqrt{3}}{8}a$ as a result of solving for the x-intercepts wrongly for Q and R. 2. Solving Q and R wrongly, often taking the y-intercepts as the x-intercepts.

When $y = 0$, $-\frac{\sqrt{3}}{4}a = -\sqrt{3}\left(x - \frac{1}{2}a\right) \Rightarrow x = \frac{1}{2}a + \frac{1}{4}a = \frac{3}{4}a$ $\therefore R\left(\frac{3}{4}a, 0\right)$ $\therefore \text{Area of triangle } PQR = \frac{1}{2}(a)\left(\frac{\sqrt{3}}{4}a\right) = \frac{\sqrt{3}}{8}a^2$.	
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9 A curve C has equation $y = xe^{-x}$, $x \geq 0$.

(a) Show that C has exactly one stationary point. [2]

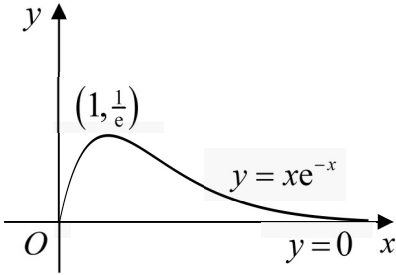
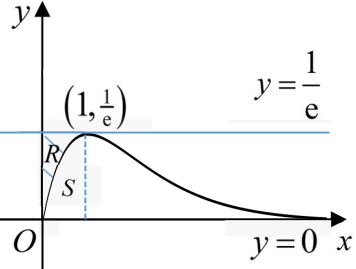
(b) Sketch C , giving the exact coordinates of the stationary point. [1]

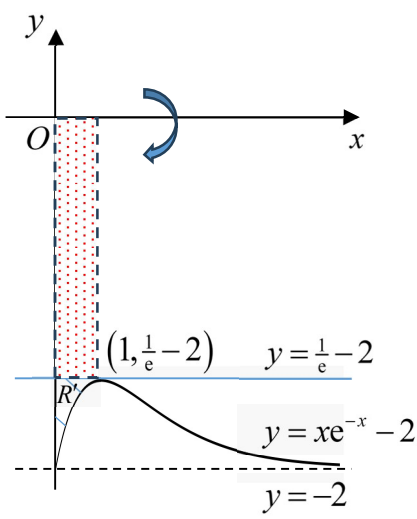
The region R is bounded by C , the y -axis and the line $y = \frac{1}{e}$.

(c) Find the exact area of R . [4]

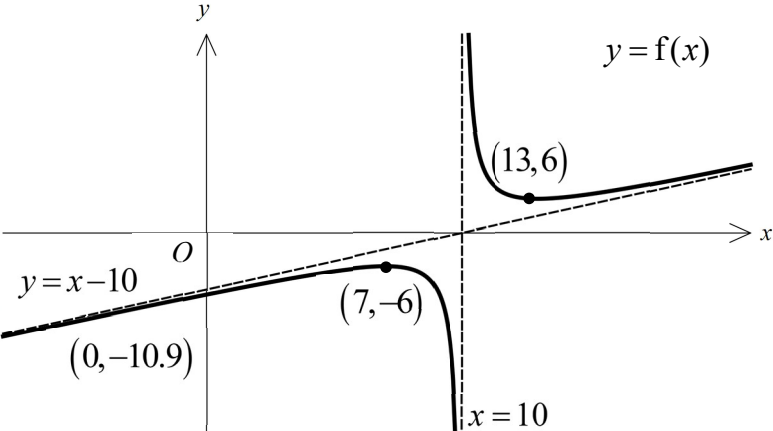
(d) An ornament is formed by rotating R through 2π radians about the line $y = 2$.
By considering a suitable translation of C , find the volume of the ornament. [3]

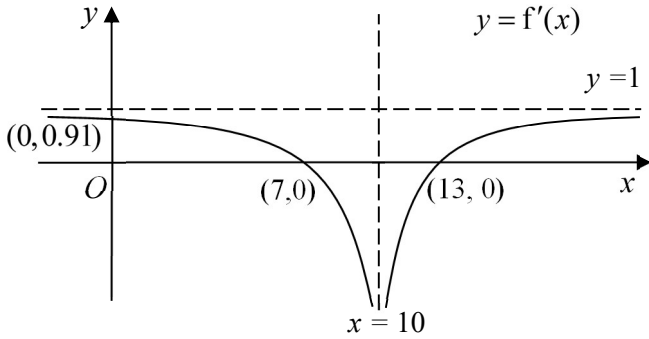
Solution	Comments
(a) [2] $y = xe^{-x}$ $\frac{dy}{dx} = e^{-x} + x(-e^{-x}) = e^{-x}(1 - x)$ When $\frac{dy}{dx} = 0$, $e^{-x}(1 - x) = 0$ Since $e^{-x} > 0 \quad \forall x \in \mathbb{R}$, $1 - x = 0 \Rightarrow x = 1$ Hence, C has exactly one stationary point. (shown)	As this is a show question, a proper mathematical reason must be provided for rejecting one of the possibilities. This eventually is equivalent to one of the key properties of the exponential function, that it is always positive (and thus nonzero).

(b) [1]	 When $x = 1$, $y = \frac{1}{e}$.	Care should be taken to exhibit the asymptotic behaviour of the curve as x approaches infinity. Again, this asymptote is a result of the exponential function being nonzero and decreasing faster than the term in x.
(c) [4]	 Area of S $= \int_0^1 x e^{-x} \, dx$ $= \left[-x e^{-x} \right]_0^1 - \int_0^1 -e^{-x} \, dx$ $= -e^{-1} + \left[-e^{-x} \right]_0^1$ $= -\frac{1}{e} + \left(-\frac{1}{e} + 1 \right) = 1 - \frac{2}{e}$ $\therefore$ Area of R $= \text{Area of rectangle} - \text{Area of } S$ $= (1) \left(\frac{1}{e} \right) - \left(1 - \frac{2}{e} \right)$ $= \frac{3}{e} - 1$ $u = x \quad \frac{dv}{dx} = e^{-x}$ $\frac{du}{dx} = 1 \quad v = -e^{-x}$	Students who did not do well in this question often 1) mistook the required region to be S instead of R, what we are trying to find. When a sketch of the region is not provided by the question, it is important for you to do so to identify the correct region yourself. Correctly evaluating incorrect regions does not give you much credit. 2) When performing integration by parts, when the terms involve multiple negative signs like this one, you should use brackets judiciously or split up the work as shown into 2 parts to reduce unnecessary errors which are often costly as the question requires an exact answer.

(d) [3]	Translate C 2 units in the negative y-direction: $y = xe^{-x} - 2$  Volume of ornament = Volume generated when R is rotated through 2π radians about the line $y = 2$ = Volume generated when R' is rotated through 2π radians about the line $y = 0$ $= \pi \int_0^1 \left[(xe^{-x} - 2)^2 - \left(\frac{1}{e} - 2 \right)^2 \right] dx$ $= 1.13 \text{ units}^3 \text{ (3 s.f.)}$ OR $= \pi \int_0^1 y^2 dx - \text{Volume of cylinder}$ $= \pi \int_0^1 (xe^{-x} - 2)^2 dx - \pi \left(2 - \frac{1}{e} \right)^2 (1)$ $= 1.13 \text{ units}^3 \text{ (3 s.f.)}$	This part of the question was poorly done. Many first translated the curve 2 units in the positive y direction. This should not be the case. When treating a problem where the desired axis of revolution is not the x or y axis, the correct method to go about solving it is to perform a transformation so that the axis of revolution is the x or y axis. Once again, a brief sketch of the new region obtained and thus seeing that the required volume is the difference between the volume of revolution and a cylinder is strongly recommended. Incorrect transformations do not lead to any credit given, as the resultant region obtained will not make sense.
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- 10 (a) It is given that $f(x) = \frac{x^2 - 20x + 109}{x - 10}$. Describe the transformation which transforms the graph of $y = x + \frac{9}{x}$ onto the graph of $y = f(x)$. [2]
- (b) Sketch the graph of $y = f(x)$, labelling clearly the equations of any asymptotes and the coordinates of any axial intercepts and turning points. [3]
- (c) On a separate diagram, sketch the graph of $y = f'(x)$, labelling clearly the equations of any asymptotes and the coordinates of any axial intercepts and turning points. [3]
- (d) By considering the graph of $(x - 13)^2 + \frac{y^2}{m} = 1$, where $m > 0$, determine the value of m for which the equation $m(x - 13)^2 + \left(\frac{x^2 - 20x + 109}{x - 10}\right)^2 = m$ has exactly one real root. [2]

Solution	Comments
(a) [2] $y = \frac{x^2 - 20x + 109}{x - 10} = \frac{(x - 10)^2 + 9}{x - 10} = x - 10 + \frac{9}{x - 10}$ The transformation is a translation of 10 units in the positive x-direction.	Most used the correct terms such as “translate”. However, some did not read the question carefully and transformed from $y = f(x)$ instead.
(b) [3] 	Most managed to have both parts of the graph. Many lost mark(s) for missing labels and/or for unreasonable scale and symmetry.

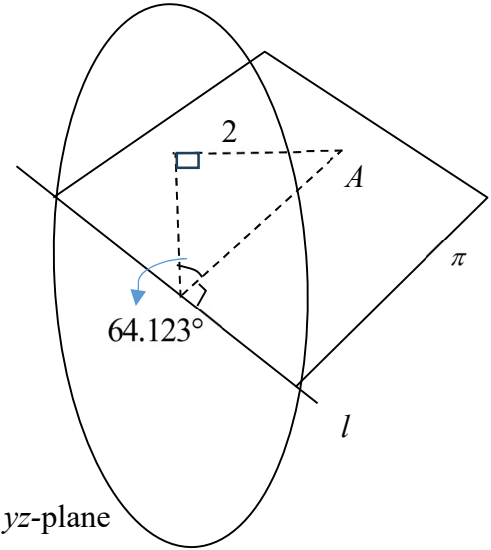
(c) [3]		Most drew the correct shape. Many lost mark(s) for missing labels and/or for unreasonable scale and symmetry. Note that the coordinate (0,0.91) can be obtained via the use of the GC.
(d) [2]	$(x-13)^2 + \frac{y^2}{m} = 1$ is an ellipse centered at (13, 0) and with radius 1 along x-axis and radius $\sqrt{m}$ along y-axis. Therefore when $m = 36$, the two graphs have exactly 1 intersection and the given equation has exactly 1 solution.	A reasonable number of students obtained the correct answer. A mention of the word “ellipse” will not gain any credit.

- 11 With reference to the origin O , the coordinates of A , B and C are $(2, 8, -a)$, $(1, -10, 3)$ and $(5, 6, a)$ respectively. The plane π contains points O , A , B and C .

- (a) Show that $a = 1$. [4]
- (b) Find the acute angle between π and the yz -plane. [2]
- (c) π and the yz -plane meet in the line l . By evaluating an appropriate cross product, find a vector equation for l . [2]
- (d) Write down the distance from A to the yz -plane. Hence or otherwise, find the distance from A to l . [3]

Solution	Comments
(a) [4] $\mathbf{n} = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 2 \\ 8 \\ -a \end{pmatrix} \times \begin{pmatrix} 1 \\ -10 \\ 3 \end{pmatrix} = \begin{pmatrix} 24 - 10a \\ -a - 6 \\ -28 \end{pmatrix}$	Again, as this is a “show” question, students need to be rigorous in their workings to show that $a = 1$.

	$\begin{pmatrix} 24-10a \\ -a-6 \\ -28 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ a \end{pmatrix} = 0$ $\Rightarrow 120 - 50a - 6a - 36 - 28a = 0$ $\Rightarrow 84 - 84a = 0$ $\Rightarrow a = 1 \text{ (shown)}$ <u>Alternative 1:</u> $\mathbf{n} = \overrightarrow{OC} \times \overrightarrow{OB} = \begin{pmatrix} 5 \\ 6 \\ a \end{pmatrix} \times \begin{pmatrix} 1 \\ -10 \\ 3 \end{pmatrix} = \begin{pmatrix} 18+10a \\ a-15 \\ -56 \end{pmatrix}$ $\begin{pmatrix} 18+10a \\ a-15 \\ -56 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 8 \\ -a \end{pmatrix} = 0$ $\Rightarrow 36 + 20a + 8a - 120 + 56a = 0$ $\Rightarrow 84a - 84 = 0$ $\Rightarrow a = 1 \text{ (shown)}$ <u>Alternative 2:</u> $\mathbf{n} = \overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} 2 \\ 8 \\ -a \end{pmatrix} \times \begin{pmatrix} 5 \\ 6 \\ a \end{pmatrix} = \begin{pmatrix} 14a \\ -7a \\ -28 \end{pmatrix} = -7 \begin{pmatrix} -2a \\ a \\ 4 \end{pmatrix}$ $\begin{pmatrix} -2a \\ a \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -10 \\ 3 \end{pmatrix} = 0$ $\Rightarrow -2a - 10a + 12 = 0 \Rightarrow a = 1 \text{ (shown)}$	Recognizing that O is in the plane is useful in finding the normal of the plane and the RHS of the vector equation of the plane.
(b) [2]	Now, $\mathbf{n} // \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ Let θ be the acute angle between the planes, then $\cos \theta = \frac{\left \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right }{\sqrt{4+1+16}} = \frac{2}{\sqrt{21}}$	Students need to remember the correct formula and ensure that the angle is acute (either by having modulus on the RHS or finding the answer using $\theta = 180^\circ - 115.877^\circ$)

	$\theta = 64.123^\circ$ (3 d.p) $= 64.1^\circ$ (1 d.p)	
(c) [2]	$\pi : \mathbf{r} \cdot \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = 0 \quad \text{and} \quad yz\text{-plane} : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$ $l \text{ is parallel to } \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$ Since the origin lies on both planes, the origin lies on l. Therefore, a vector equation for l is: $\mathbf{r} = \lambda \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$	Question required students to evaluate an appropriate cross product in finding the vector equation of l, that is, using GC to find the required will not receive any credit.
(d) [3]	The distance from A to the yz-plane is 2.  <u>Hence</u> The distance from A to $l = \frac{2}{\sin 64.123^\circ} = 2.22$ (3 s.f.)	Students should be able to observe and state the distance from A to the yz-plane, without any working. Visualizing the diagram is useful for this question. Students are reminded to use at least 3 d.p. (or 5 s.f.) for intermediate workings.

	<u>Otherwise</u> The distance from A to l $= \frac{\left \begin{pmatrix} 2 \\ 8 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} \right }{\sqrt{16+1}} = \frac{\left \begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix} \right }{\sqrt{17}} = \frac{2 \left \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \right }{\sqrt{17}}$ $= \frac{2\sqrt{21}}{\sqrt{17}} = \frac{2\sqrt{357}}{17} = 2.22 \text{ (3 s.f.)}$	
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- 12** A seating specialist was tasked to install seats for a performance arena. In the curved section of the arena, he installed 21 seats in the front row facing the arena and for each subsequent row, there are 2 more seats than the row before it. The seating capacity of this curved section is 1664.

- (a)** Find the total number of rows in this section of the arena, and the number of seats in the tenth row from the back of the section. [3]

At a performance held at this arena, the cost of a ticket for a front row seat at this curved section is \$151. The cost of a ticket for a seat in every subsequent row decreases by 2% compared to the row before. The cost of tickets for seats in the same row are equal.

- (b)** **(i)** Sam can only afford a ticket below \$100 and he would like to sit as near to the front of the section as possible. Find the row nearest to the front for which he can purchase a ticket. [2]

- (ii)** If $F(r)$ is the total cost of tickets for the seats in the r th row from the front of the section, write down $F(r)$ in terms of r . [1]

- (iii)** Find the total cost of tickets for all the seats in the first n rows from the front, in the form of

$$143450(1 - A^n) + B \sum_{r=1}^n (rA^{r-1}),$$

where A and B are constants to be determined. [3]

- (iv) Suppose a company can purchase tickets for this performance before public sales are open and they would like to purchase 600 tickets in this curved section with a budget of \$52 000. Determine whether this budget is sufficient for this purchase. [3]

Solution	Comments						
(a) [3] AP : 21, 23, 25, ... $\frac{n}{2}[2(21) + (n-1)(2)] = 1664$ $21n + n^2 - n - 1664 = 0$ $n^2 + 20n - 1664 = 0$ From GC, $n = 32 \quad \text{or} \quad n = -52 \text{ (n.a.)}$ $\therefore$ There are 32 rows in this section. Counting from the back, the tenth row is $(32 - 10 + 1) = 23^{\text{rd}}$ row. Number of seats in the 23^{rd} row $= u_{23} = 21 + (23 - 1)(2) = 65$ $\therefore$ There are 65 seats.	This part was generally well done, except for students who did not see that we asked for the number of seats in the 10^{th} row from the back, which means it is the $(32 - 10 + 1) = 23^{\text{rd}}$ row.						
(b)(i) [2] A ticket at the n^{th} row cost $151(0.98)^{n-1}$. $151(0.98)^{n-1} < 100$ $n - 1 > \frac{\ln \frac{100}{151}}{\ln 0.98} = 20.4$ $n > 21.4$ OR From GC, <table border="1" data-bbox="358 1402 716 1533"> <tr> <td>n</td><td>$151(0.98)^{n-1}$</td></tr> <tr> <td>21</td><td>$100.81 > 100$</td></tr> <tr> <td>22</td><td>$98.793 < 100$</td></tr> </table> $\therefore$ The row nearest to the front that Sam can purchase is the 22^{nd} row.	n	$151(0.98)^{n-1}$	21	$100.81 > 100$	22	$98.793 < 100$	Students are reminded of the need for clear working, so either they show the working when taking $\ln$, together with the sign change because $\ln 0.98$ is negative, or they should show two rows of the table if they are using the GC to determine when the cost will be below 100.
n	$151(0.98)^{n-1}$						
21	$100.81 > 100$						
22	$98.793 < 100$						

(b) (ii) [1]	The next row has two more seats than the row before it and cost 2% less, which is 0.98 times. $F(1) = 151(21)$ $F(2) = 151(0.98)(23)$ $F(3) = 151(0.98)^2(25)$ $\vdots$ $F(r) = 151(0.98)^{r-1}(21 + 2(r-1)) = 151(0.98)^{r-1}(19 + 2r)$ OR Number of seats in the r^{th} row $= u_r = 21 + (r-1)(2) = 19 + 2r$ $F(r)$ $= \text{cost of ticket in the } r^{\text{th}} \text{ row} \times \text{no. of seats in the } r^{\text{th}} \text{ row}$ $= 151(0.98)^{r-1}(19 + 2r)$	Most did well on this, but many did not simplify $21 + 2(r-1)$
(iii) [3]	Total cost of tickets for all the seats in the first n rows from the front is $\sum_{r=1}^n F(r)$ $= \sum_{r=1}^n [151(0.98^{r-1})(19 + 2r)]$ $= \sum_{r=1}^n [2869(0.98^{r-1})] + \sum_{r=1}^n [302r(0.98^{r-1})]$ $= \frac{2869(1 - 0.98^n)}{1 - 0.98} + 302 \sum_{r=1}^n r(0.98^{r-1})$ $= 143450(1 - 0.98^n) + 302 \sum_{r=1}^n r(0.98^{r-1})$ where $A = 0.98$, $B = 302$ (shown)	Students should write out the sum, either as $F(1) + \dots + F(n)$ or in sigma notation. Starting with the sum of n terms of the GP does not show your understanding from the third to the fourth line in the working.

(iv) [3]	The company should purchase seats from the back rows as the seats have a lower cost. Number of seats in the last row ie 32nd row $= 21 + (32 - 1)(2) = 83$ seats. To get 600 seats, count the last seven rows ($83 + 81 + 79 + 77 + 75 + 73 + 71 = 539$) at full capacity, and 61 seats from the eighth row from the back ie 25th row. Total cost of the cheapest 600 tickets <math>= F(32) - F(25) + 61(\text{cost of a ticket in the 25th row})</math> $= 143450(1 - 0.98^{32}) + 302 \sum_{r=1}^{32} r(0.98^{r-1})$ $- \left[143450(1 - 0.98^{25}) + 302 \sum_{r=1}^{25} r(0.98^{r-1}) \right] + 61 \times 151(0.98^{25-1})$ $= 51\,840$ (nearest dollar) OR calculate directly $151(0.98^{31})(83) + 151(0.98^{30})(81)$ $+ 151(0.98^{29})(79) + \dots + 151(0.98^{25})(71) + 151(0.98^{24})(61)$ $= 51\,840$ (nearest dollar) Yes, the company has sufficient budget for this purchase.	If the purchase of 600 seats is from the front, we will see that the first 16 rows and 24 seats are needed in the 17th row. However, the cost for all seats in the 16th row will exceed budget because $F(16) = 73284$ (5.s.f.) For those counting from the back, there were counting mistakes, for example, some said 61 seats from the 24th row were needed when it was the 25th row. There are other possible methods, such as computing the complement cost of the first 16 rows and 24 seats in 17th.
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