1 It is given that $\mathrm{f}(x)=\ln(a+x)$, $x\in\mathbb{R}$, $x>-a$, where a is a constant.

(a) Using the standard series from the List of Formulae (MF27), find the series expansion for $\mathrm{f}(x)$, up to and including the term in x^3. [2]

It is given that $a=1$.

(b) Hence, or otherwise, show that the series expansion of $\sin\left[\mathrm{f}(x)\right]$, up to and including the term in x^3 is given by $x-\frac{1}{2}x^2+\frac{1}{6}x^3$. [2]

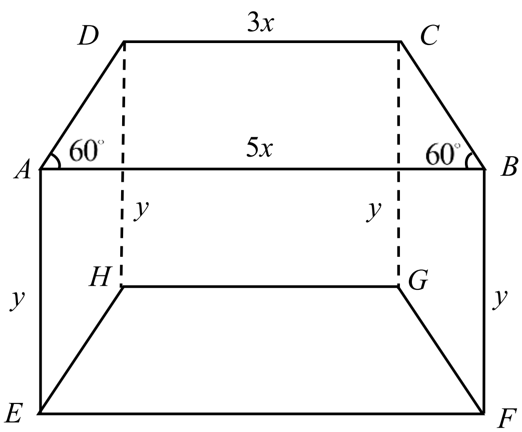
(c) Deduce the Maclaurin series for $\cos\left[\mathrm{f}(x)\right]$ up to and including the term in x^2. [2]

(d) Find $\int_1^3\left(x-\frac{1}{2}x^2+\frac{1}{6}x^3\right)\mathrm{d}x$. Without the use of a calculator or any further calculation, explain, whether this value is a good approximation to the value of $\int_1^3\sin\left[\mathrm{f}(x)\right]\mathrm{d}x$. [2]

1a	$\ln(a+x)=\ln\left[a\left(1+\frac{x}{a}\right)\right]$ $=\ln a+\ln\left(1+\frac{x}{a}\right)$ $=\ln a+\left(\frac{x}{a}-\frac{1}{2}\left(\frac{x}{a}\right)^2+\frac{1}{3}\left(\frac{x}{a}\right)^3+\ldots\right)$ $=\ln a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}+\ldots$	
1b	$\sin\left[\ln(1+x)\right]=\sin\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$ $=\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)-\frac{1}{3!}\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)^3+\ldots$ $=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{1}{6}\left(x^3+\ldots\right)$ $=x-\frac{x^2}{2}+\frac{x^3}{6}+\ldots$	
1c	$\sin\left[\ln(1+x)\right]=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots..$ Differentiate w.r.t. x: $\frac{1}{1+x}\cos\left[\ln(1+x)\right]=1-x+\frac{1}{2}x^2+\ldots$ $\cos\left[\ln(1+x)\right]=(1+x)\left(1-x+\frac{1}{2}x^2+\ldots\right)=1-\frac{1}{2}x^2+\ldots$	

1d	$\int_1^3 x-\frac{x^2}{2}+\frac{x^3}{6}\ \mathrm{d}x=3$ For $x\in(1,3)\not\subseteq(-1,1]$, the expansion $\sin\left[\ln(1+x)\right]=\sin\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$ is not valid and hence, the approximation is not a good approximation.	

2 The following diagram shows the dimensions of a trapezoidal prism with fixed volume $4k\sqrt{3}$ units^3, with variables x and y.

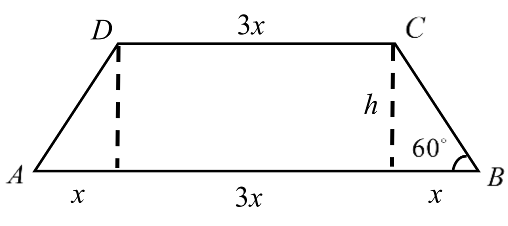


The top surface of the prism, *ABCD*, is an isosceles trapezoid with *AB* of length $5x$ units, *DC* of length $3x$ units, $AD=BC$ and $\angle ABC=\angle BAD=60^\circ$. The rectangular sides *ABFE* and *BCGF* are perpendicular to both the top surface *ABCD* and the bottom surface *EFGH*, with $AE=BF=DH=CG=y$ units.

(a) Show that the total external surface area A of the trapezoidal prism is given by $A=8x^2\sqrt{3}+\frac{12k}{x}$. [4]

(b) Using differentiation, find the value of x in terms of k at which A is a minimum. [4]

(c) It is given instead that the volume of the prism is 1000 units^3 and its external surface area is 800 units^2. Find the two possible values of x. [2]

No.	Solution	Marks
2a	 Let the height of the isosceles trapezoid be h. $h=x\tan 60^\circ=\sqrt{3}x$ and $BC=\frac{x}{\cos 60^\circ}=2x$	

$$V=\frac{1}{2}(5x+3x)\times\sqrt{3}x\times y$$

$$4\sqrt{3}k=4\sqrt{3}x^2y$$

$$y=\frac{k}{x^2}$$

$$A=\frac{1}{2}(5x+3x)\left(\sqrt{3}x\right)\times 2+3xy+5xy+2(2xy)$$

$$=8\sqrt{3}x^2+12xy$$

$$=8\sqrt{3}x^2+12x\left(\frac{k}{x^2}\right)$$

$$=8\sqrt{3}x^2+\frac{12k}{x}\quad\text{(shown)}$$

2b

For minimum A, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$.

$$16\sqrt{3}x-\frac{12k}{x^2}=0\quad\Rightarrow\quad x^3=\frac{12k}{16\sqrt{3}}\quad\Rightarrow\quad x=\left(\frac{3k}{4\sqrt{3}}\right)^{\frac{1}{3}}=\left(\frac{\sqrt{3}k}{4}\right)^{\frac{1}{3}}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=16\sqrt{3}+\frac{24k}{x^3}>0\qquad\left(\because k,x>0\Rightarrow\frac{24k}{x^3}>0\right)$$

Hence, A is minimum when $x=\left(\frac{\sqrt{3}k}{4}\right)^{\frac{1}{3}}$.

<u>Alternative</u> (for 2nd derivative test)

At $x=\left(\frac{\sqrt{3}}{4}k\right)^{\frac{1}{3}}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=16\sqrt{3}+\frac{24k}{\frac{\sqrt{3}k}{4}}=48\sqrt{3}>0$.

<u>Alternative</u> (for 1st derivative test)

x	$0.75k^{\frac{1}{3}}$	$\left(\frac{\sqrt{3}}{4}k\right)^{\frac{1}{3}}$	$0.76k^{\frac{1}{3}}$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	$-0.549k^{\frac{1}{3}}$	0	$0.286k^{\frac{1}{3}}$
sign (since $k>0$)	$-$	0	$+$
slope	\	—	/

2c	When $V=1000$, $k=\dfrac{1000}{4\sqrt{3}}=\dfrac{250}{\sqrt{3}}$. $800=8\sqrt{3}x^2+\dfrac{12}{x}\left(\dfrac{250}{\sqrt{3}}\right)$ From GC, $x=-8.5102$ (N.A. since $x>0$) or $x=6.10$ or $x=2.41$	

3 With reference to the point O as the origin and the x-y plane as a horizontal plane, the pyramid $OPQRV$ has a parallelogram base $OPQR$ and height OV. The position vectors of the points P and R are $-3\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ and $5\mathbf{i}-2\mathbf{j}-\mathbf{k}$ respectively.

(a) Find the coordinates of the point S that lies on the line PR such that the distance from O to S is a minimum. [3]

(b) Find the cartesian equations of the planes such that the perpendicular distance from each plane to the base $OPQR$ is $2\sqrt{86}$ units. [3]

(c) Find the acute angle between OV and the vertical. [2]

(d) Given that QV is parallel to the vector $-5\mathbf{j}+8\mathbf{k}$, find the position vector of the point V. Hence find the exact volume of the pyramid $OPQRV$. [5]

[Volume of a pyramid = $\dfrac{1}{3}\times\text{base area}\times\text{height}$]

No.	Solution	Marks
3a	$\overrightarrow{PR}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}-\begin{pmatrix}-3\\4\\3\end{pmatrix}=\begin{pmatrix}8\\-6\\-4\end{pmatrix}=-2\begin{pmatrix}-4\\3\\2\end{pmatrix}$ Line PR: $\underset{\sim}{r}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-4\\3\\2\end{pmatrix}$, $\lambda\in\mathbb{R}$ $OS\perp PR$ for OS to be minimum: $\overrightarrow{OS}\bullet\overrightarrow{PR}=0$ Since S lies on line PR, $\left[\begin{pmatrix}5\\-2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-4\\3\\2\end{pmatrix}\right]\bullet\begin{pmatrix}-4\\3\\2\end{pmatrix}=0$. $(-20-6-2)+\lambda(16+9+4)=0 \Rightarrow \lambda=\dfrac{28}{29}$ $\overrightarrow{OS}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}+\dfrac{28}{29}\begin{pmatrix}-4\\3\\2\end{pmatrix}=\dfrac{1}{29}\begin{pmatrix}33\\26\\27\end{pmatrix}$ Coordinates of S are $\left(\dfrac{33}{29},\dfrac{26}{29},\dfrac{27}{29}\right)$.	

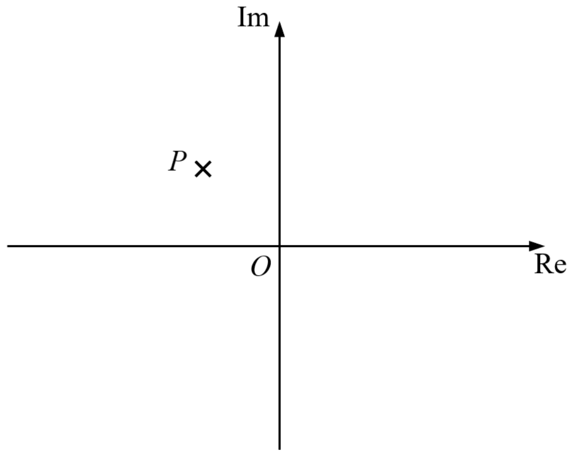
	Alternative $\overrightarrow{PS}=\left(\dfrac{\overrightarrow{PO}\bullet\overrightarrow{PR}}{\left\lvert\overrightarrow{PR}\right\rvert}\right)\dfrac{\overrightarrow{PR}}{\left\lvert\overrightarrow{PR}\right\rvert}$ $\overrightarrow{PS}=\dfrac{\begin{pmatrix}3\\-4\\-3\end{pmatrix}\bullet\begin{pmatrix}-4\\3\\2\end{pmatrix}}{\sqrt{29}}\;\dfrac{\begin{pmatrix}-4\\3\\2\end{pmatrix}}{\sqrt{29}}=\dfrac{-30}{29}\begin{pmatrix}-4\\3\\2\end{pmatrix}$ $\overrightarrow{OS}=\overrightarrow{OP}+\overrightarrow{PS}=\begin{pmatrix}3\\-4\\-3\end{pmatrix}+\dfrac{-30}{29}\begin{pmatrix}-4\\3\\2\end{pmatrix}=\dfrac{1}{29}\begin{pmatrix}33\\26\\27\end{pmatrix}$ Coordinates of S are $\left(\dfrac{33}{29},\ \dfrac{26}{29},\dfrac{27}{29}\right)$.	
3b	$\overrightarrow{OP}\times\overrightarrow{OR}=\begin{pmatrix}-3\\4\\3\end{pmatrix}\times\begin{pmatrix}5\\-2\\-1\end{pmatrix}=\begin{pmatrix}2\\12\\-14\end{pmatrix}=2\begin{pmatrix}1\\6\\-7\end{pmatrix}$ Let the equation of a plane be $\underset{\sim}{r}\cdot\begin{pmatrix}1\\6\\-7\end{pmatrix}=t$, and note that $(t,0,0)$ is a point on the plane. $\dfrac{\left\lvert\left(\begin{pmatrix}t\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\6\\-7\end{pmatrix}\right\rvert}{\sqrt{1+36+49}}=2\sqrt{86}$ $\lvert t\rvert=2(86)$ $t=\pm172$ **OR** $\lvert\underset{\sim}{r}\cdot\underset{\sim}{\hat{n}}\rvert=2\sqrt{86}$ $\dfrac{\underset{\sim}{r}\cdot\begin{pmatrix}1\\6\\-7\end{pmatrix}}{\sqrt{1+36+49}}=\pm2\sqrt{86}$ $\underset{\sim}{r}\cdot\begin{pmatrix}1\\6\\-7\end{pmatrix}=\pm2(86)=\pm172$ The planes have equations $x+6y-7z=172$ and $x+6y-7z=-172$. **Alternative** $\overrightarrow{OP}\times\overrightarrow{OR}=\begin{pmatrix}-3\\4\\3\end{pmatrix}\times\begin{pmatrix}5\\-2\\-1\end{pmatrix}=\begin{pmatrix}2\\12\\-14\end{pmatrix}=2\begin{pmatrix}1\\6\\-7\end{pmatrix}$ A normal to the plane $OPQR$ = $\begin{pmatrix}1\\6\\-7\end{pmatrix}$ Let T and U be different points on the two planes such that $OT=2\sqrt{86}=OU$.	

	$\overrightarrow{OT} = 2\sqrt{86} \times \dfrac{\begin{pmatrix}1\\6\\-7\end{pmatrix}}{\sqrt{1+36+49}} = \begin{pmatrix}2\\12\\-14\end{pmatrix}$ $\overrightarrow{OU} = 2\sqrt{86} \times \dfrac{-\begin{pmatrix}1\\6\\-7\end{pmatrix}}{\sqrt{1+36+49}} = \begin{pmatrix}-2\\-12\\14\end{pmatrix}$ The equations of the planes are $\underset{\sim}{r} \bullet \begin{pmatrix}1\\6\\-7\end{pmatrix} = \begin{pmatrix}2\\12\\-14\end{pmatrix} \bullet \begin{pmatrix}1\\6\\-7\end{pmatrix}$ and $\underset{\sim}{r} \bullet \begin{pmatrix}1\\6\\-7\end{pmatrix} = \begin{pmatrix}-2\\-12\\14\end{pmatrix} \bullet \begin{pmatrix}1\\6\\-7\end{pmatrix}$, i.e. $x+6y-7z=172$ and $x+6y-7z=-172$.	
3c	Let the acute angle between OV and the vertical be θ. $\cos\theta = \dfrac{\left\lvert \begin{pmatrix}1\\6\\-7\end{pmatrix} \bullet \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rvert}{\sqrt{86}} = \dfrac{7}{\sqrt{86}}$ $\theta = 41.0^\circ$	
3d	$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OR} = \begin{pmatrix}-3\\4\\3\end{pmatrix} + \begin{pmatrix}5\\-2\\-1\end{pmatrix} = \begin{pmatrix}2\\2\\2\end{pmatrix}$ $\begin{pmatrix}2\\2\\2\end{pmatrix} = \overrightarrow{OQ} = \overrightarrow{OV} + \overrightarrow{VQ} = a\begin{pmatrix}1\\6\\-7\end{pmatrix} + b\begin{pmatrix}0\\-5\\8\end{pmatrix}$, for some $a, b \in \mathbb{R}$ By comparing LHS and RHS, $a=2$ (and $b=2$). $\therefore \overrightarrow{OV} = 2\begin{pmatrix}1\\6\\-7\end{pmatrix} = \begin{pmatrix}2\\12\\-14\end{pmatrix}$	

	Volume of pyramid $= \frac{1}{3} \times \left\| \overrightarrow{OP} \times \overrightarrow{OR} \right\| \times \left\| \overrightarrow{OV} \right\|$ $= \frac{1}{3} \times \left\| \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} \right\| \times \left\| \begin{pmatrix} 2 \\ 12 \\ -14 \end{pmatrix} \right\|$ $= \frac{1}{3} \times \left\| \begin{pmatrix} 2 \\ 12 \\ -14 \end{pmatrix} \right\| \times \left\| \begin{pmatrix} 2 \\ 12 \\ -14 \end{pmatrix} \right\|$ $= \frac{1}{3} \times (4 + 144 + 196)$ $= \frac{344}{3}$ units^3 **<u>Alternative</u> (to find position vector of *V*)** $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OR} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ Line *QV*: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ -8 \end{pmatrix}, \quad \alpha \in \mathbb{R}$ Line *OV*: $\underset{\sim}{r} = \beta \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix}, \quad \beta \in \mathbb{R}$ *V* is the intersection of lines *QV* and *OV*: $\beta \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 5 \\ -8 \end{pmatrix}$, for some $\alpha, \beta \in \mathbb{R}$ Solving gives $\beta = 2,\ \alpha = 2$. $\overrightarrow{OV} = 2 \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ -14 \end{pmatrix}$	

4 Do not use a calculator in answering this question.

The complex number z has modulus 1 and argument θ, where $\frac{\pi}{2} < \theta < \pi$, and the complex number w is given by $w = \mathrm{i}\sqrt{3}\, z$. The point P on the Argand diagram represents z.



(a) On the copy of the Argand diagram with origin O in the Printed Answer Booklet, plot the points Q and R to represent w and $z - w$ respectively. Show clearly the geometrical relationship between the points P, Q and R. [3]

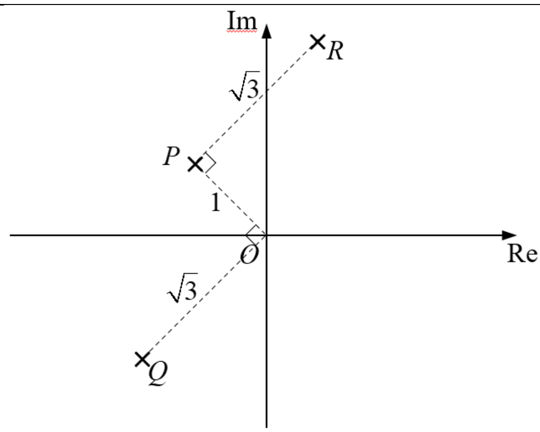
(b) Find the area of the quadrilateral $ORPQ$. [1]

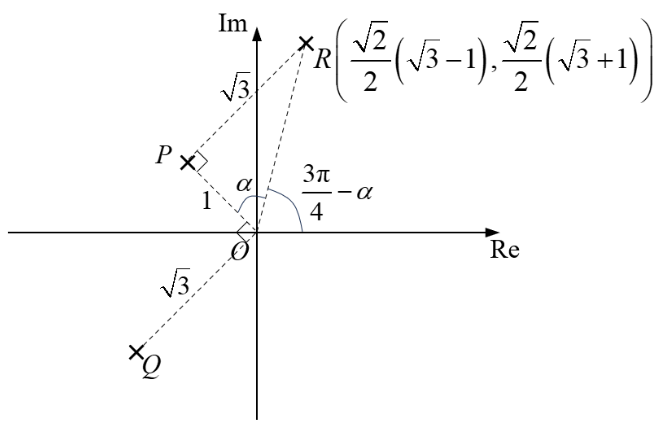
It is given that $\theta = \frac{3\pi}{4}$.

(c) Find z in the form $x + y\mathrm{i}$, where x and y are real numbers. [2]

(d) Show that $z - w = k\left[\left(\sqrt{3}-1\right) + \left(\sqrt{3}+1\right)\mathrm{i}\right]$, where k is a constant to be determined. [2]

(e) Hence show that $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$. [1]

No.	Solution	Marks
4a		
4b	Area $ORPQ = \frac{1}{2}(1)\left(\sqrt{3}+\sqrt{3}\right) = \sqrt{3}$	

4c	Length $x = 1\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ Length $y = 1\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\therefore z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}$	
4d	$z - w = \left(z - \sqrt{3}\mathrm{i}z\right)$ $= \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right)\left(1 - \sqrt{3}\mathrm{i}\right)$ $= \frac{1}{\sqrt{2}}\left(-1 + \mathrm{i}\right)\left(1 - \sqrt{3}\mathrm{i}\right)$ $= \frac{1}{\sqrt{2}}\left(-1 + \sqrt{3} + \mathrm{i} + \sqrt{3}\mathrm{i}\right)$ $= \frac{1}{\sqrt{2}}\left(\left(\sqrt{3} - 1\right) + \left(\sqrt{3} + 1\right)\mathrm{i}\right)$ $k = \frac{1}{\sqrt{2}}$	
4e	 $\alpha = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$ $\tan\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) = \frac{\frac{\sqrt{2}}{2}\left(\sqrt{3}+1\right)}{\frac{\sqrt{2}}{2}\left(\sqrt{3}-1\right)}$ $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$	

5 A code consists of 10 characters.

The first 5 characters of the code is formed using 5 letters chosen from the set {A, B, C, D, E, F, G, H} and the last 5 characters of the code is formed using 5 digits chosen from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 0}. The code allows for repetitions of letters and/or digits.

(a) Find the number of different codes that can be formed. [1]

(b) Find the probability that a code chosen at random

(i) contains the letter E exactly thrice and the digit 5 exactly once, [2]

(ii) has E as the first character or 4 as its tenth character, but not both. [2]

5a	Number of codes $=8^5\times10^5=3276800000$	
5bi	P(E exactly thrice and 5 once) $=\dfrac{{}^5C_3\times7^2\times{}^5C_1\times9^4}{8^5 10^5}\approx0.00491$ **Alternative** P(E exactly thrice and 5 once) $={}^5C_3\left(\dfrac{1}{8}\right)^3\left(\dfrac{7}{8}\right)^2\times{}^5C_1\left(\dfrac{1}{10}\right)\left(\dfrac{9}{10}\right)^4$ ≈0.00491	
5bii	Required probability $=\dfrac{1}{8}+\dfrac{1}{10}-2\left(\dfrac{1}{8}\times\dfrac{1}{10}\right)$ $=0.2$ **Alternative** Required probability $=\dfrac{1}{8}\times\dfrac{9}{10}+\dfrac{7}{8}\times\dfrac{1}{10}$ $=0.2$ **Alternative** Required probability $=\dfrac{8^4\times10^5+8^5\times10^4-2\left(8^4\times10^4\right)}{8^5 10^5}$ $=0.2$ **Alternative** P(E first, 4 not tenth character) $=\dfrac{1\times8^4\times10^4\times9}{8^5 10^5}$ or $\dfrac{1}{8}\times\dfrac{9}{10}=0.1125$ P(4 tenth, E not first character) $=\dfrac{10^4\times1\times7\times8^4}{8^5 10^5}$ or $\dfrac{7}{8}\times\dfrac{1}{10}=0.0875$ Required probability $=0.1125+0.0875=0.2$	

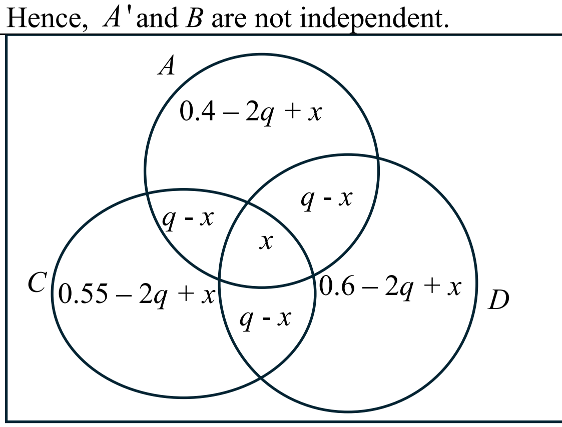
6 Two events A and B are such that $\mathrm{P}(A) = p$ and $\mathrm{P}(B) = \frac{5}{4}p$. It is given that $\mathrm{P}(A \cup B) = \mathrm{P}(A \mid B) = 0.6$.

(a) Find the value of p. [3]

(b) Explain whether the events A' and B are independent. [1]

The events C and D are such that $\mathrm{P}(C) = 0.55$, $\mathrm{P}(D) = 0.6$ and $\mathrm{P}(A \cap C) = \mathrm{P}(A \cap D) = \mathrm{P}(C \cap D) = q$.

(c) Find the value of q that gives the minimum value of $\mathrm{P}(A \cup C \cup D)$ and state the minimum value of $\mathrm{P}(A \cup C \cup D)$. [2]

6a	$\mathrm{P}(A \mid B) = 0.6 \Rightarrow \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = 0.6$ $\dfrac{\mathrm{P}(A \cap B)}{\frac{5}{4}p} = 0.6 \Rightarrow \mathrm{P}(A \cap B) = 0.75p$ $\mathrm{P}(A \cup B) = 0.6$ $\Rightarrow \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.6$ $\Rightarrow p + \frac{5}{4}p - 0.75p = 0.6$ $\Rightarrow p = 0.4$	
6b	$\mathrm{P}(A \mid B) = 0.6 \neq 0.4 = \mathrm{P}(A)$ Since A and B are not independent, then A' and B are also not independent. **<u>Alternative</u>** $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.5 - 0.3 = 0.2 \neq 0.6 \times 0.5 = \mathrm{P}(A')\mathrm{P}(B)$ Hence, A' and B are not independent.	
6c	 Let x be $\mathrm{P}(A \cap C \cap D)$. For minimum $\mathrm{P}(A \cup C \cup D)$, $0.4 - 2q + x = 0$ and $q - x = 0$, i.e. $q = x$. $\therefore 0.4 - 2q + q = 0 \Rightarrow q = 0.4$ $\mathrm{P}(A \cup C \cup D) = \mathrm{P}(A) + \mathrm{P}(C \cap A' \cap D') + \mathrm{P}(C \cap A' \cap D) + \mathrm{P}(C' \cap A' \cap D)$ $= 0.4 + (0.55 - 2q + x) + (q - x) + (0.6 - 2q + x)$ $= 1.55 - 3q + x$ $\mathrm{P}(A \cup C \cup D) = 1.55 - 3(0.4) + (0.4) = 0.75$	

7 A shop sells apples in bags of 12. The average number of unripe apples in a bag is 2.16.

(a) State two assumptions needed for the number of unripe apples in a bag to be well modelled by a binomial distribution. [2]

A bag of apples is sold at a reduced price if more than 3 apples are unripe.

(b) Find the probability that a randomly selected bag of apples is sold at a reduced price. [2]

(c) Twenty bags of apples are selected at random. Find the probability that fewer than 4 of these bags will be sold at a reduced price. [1]

(d) The shop also sells oranges in bags of n oranges. On average, the proportion of oranges that is unripe is 0.16. It is known that the modal number of unripe oranges in a bag is 2. Find the set of possible values of n. [2]

<table>
<tr><td>7a</td><td>1. The probability of each apple being unripe is constant.
2. The ripeness of an apple is independent of other apples.
OR Unripe apples occur independently of other apples.</td><td></td></tr>
<tr><td>7b</td><td>$p = \frac{2.16}{12} = 0.18$
Let X be the number of unripe apples in a bag of 12.
$X \sim \mathrm{B}(12, 0.18)$
$\mathrm{P}(X > 3) = 1 - \mathrm{P}(X \le 3) \approx 0.15515 = 0.155$ (3 s.f.)</td><td></td></tr>
<tr><td>7c</td><td>Let Y be the number of bags that are sold at a reduced price out of 20 bags. $Y \sim \mathrm{B}(20, 0.15515)$.
$\mathrm{P}(Y < 4) = \mathrm{P}(Y \le 3) = 0.623$ (3 s.f.)</td><td></td></tr>
<tr><td>7d</td><td>Let W be the number of unripe oranges in a bag out of n.
$W \sim \mathrm{B}(n, 0.16)$

Since modal number of unripe oranges in a bag is 2,
$\mathrm{P}(W=1) < \mathrm{P}(W=2)$ and $\mathrm{P}(W=3) < \mathrm{P}(W=2)$.
<table>
<tr><td>n</td><td>$\mathrm{P}(W=1)$</td><td>$\mathrm{P}(W=2)$</td><td>$\mathrm{P}(W=3)$</td></tr>
<tr><td>11</td><td>0.308</td><td>0.293</td><td>0.168</td></tr>
<tr><td>12</td><td>0.282</td><td>0.296</td><td>0.188</td></tr>
<tr><td>⋮</td><td>⋮</td><td>⋮</td><td>⋮</td></tr>
<tr><td>17</td><td>0.167</td><td>0.255</td><td>0.243</td></tr>
<tr><td>18</td><td>0.149</td><td>0.241</td><td>0.244</td></tr>
</table>
$\{n \in \mathbb{Z} : 12 \le n \le 17\}$

<u>Alternative</u>
$\mathrm{P}(W=1) - \mathrm{P}(W=2) < 0$ and $\mathrm{P}(W=3) - \mathrm{P}(W=2) < 0$
<table>
<tr><td colspan="2">For $\mathrm{P}(W=1) - \mathrm{P}(W=2)$:</td></tr>
<tr><td>n</td><td>$\mathrm{P}(W=1) - \mathrm{P}(W=2)$</td></tr>
<tr><td>11</td><td>0.0147</td></tr>
<tr><td>12</td><td>–0.0134</td></tr>
<tr><td>13</td><td>–0.0367</td></tr>
</table>
<table>
<tr><td colspan="2">For $\mathrm{P}(W=3) - \mathrm{P}(W=2)$:</td></tr>
<tr><td>n</td><td>$\mathrm{P}(W=3) - \mathrm{P}(W=2)$</td></tr>
<tr><td>16</td><td>–0.0297</td></tr>
<tr><td>17</td><td>–0.0121</td></tr>
<tr><td>18</td><td>0.00382</td></tr>
</table>
$\{n \in \mathbb{Z} : 12 \le n \le 17\}$</td><td></td></tr>
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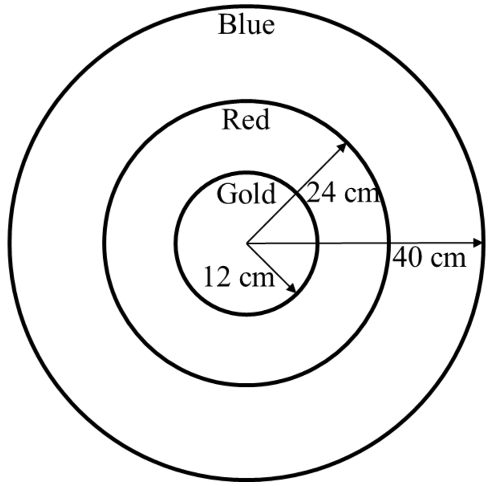
8 The diagram below shows a circular target board of radius 40 cm, divided by two concentric circles of radii 12 cm and 24 cm, into three regions, Gold, Red and Blue.



Arrows are shot at the target board and the scores obtained for hitting the Gold, Red and Blue regions are 10 points, 6 points and 3 points respectively. Arrows that land outside the target board score 0 points.

Alex shoots one arrow at the target board. The probability that his shot lands on the target board is p, where $p > 0.5$. If his shot lands on the target board, the arrow is equally likely to hit any position on the board. Let X represent the score obtained from a single shot.

You may assume that the arrow **does not land on the boundary** of any region.

(a) Show that the $\mathrm{P}(X = 6) = 0.27p$. [1]

(b) Given that $\mathrm{Var}(X) = 5.464476$, find the value of p. [4]

8a	$\mathrm{P}(X=6)=(p)\left(\dfrac{\pi(24^2)-\pi(12^2)}{\pi(40^2)}\right)=0.27p$	
8b	$\mathrm{P}(X=10)=\dfrac{12^2\pi}{40^2\pi}p=0.09p$ <table><tr><td>x</td><td>0</td><td>3</td><td>6</td><td>10</td></tr><tr><td>$\mathrm{P}(X=x)$</td><td>$1-p$</td><td>$0.64p$</td><td>$0.27p$</td><td>$0.09p$</td></tr></table> $\mathrm{E}(X)=(3)(0.64)p+(6)(0.27)p+(10)(0.09)p=4.44p$ $\mathrm{Var}(X)=\left(0+3^2(0.64p)+6^2(0.27p)+10^2(0.09p)\right)-(4.44p)^2$ $5.464476=24.48p-19.7136p^2$ $19.7136p^2-24.48p+5.464476=0$ $p=0.292$ $(\text{rej.} \because p>0.5)$ or $p=0.95$	

9 The random variable X has distribution $\mathrm{N}\left(\mu,\ \sigma^2\right)$.

(a) Given that $\mathrm{P}(X>k)=0.45$, find the value of $\mathrm{P}(X>2\mu-k)$. [2]

It is given that $\mu=4$ and $\mathrm{P}(1<X<4)=0.2475$.

(b) Find the value of σ. [2]

(c) Draw a sketch to show the distribution of X for x between -11 and 19. [2]

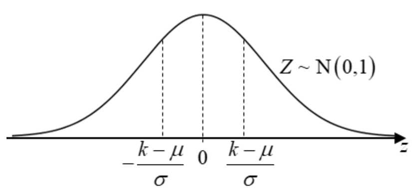
<table>
<tr><td>9a</td><td>

$\mathrm{P}(X>k)=0.45 \Rightarrow \mathrm{P}\left(Z>\dfrac{k-\mu}{\sigma}\right)=0.45$

$\mathrm{P}(X>2\mu-k)=\mathrm{P}\left(Z>\dfrac{2\mu-k-\mu}{\sigma}\right)$

$=\mathrm{P}\left(Z>\dfrac{\mu-k}{\sigma}\right)$

$=\mathrm{P}\left(Z<\dfrac{k-\mu}{\sigma}\right)$



$=1-0.45$

$=0.55$

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<tr><td>9b</td><td>

$\mathrm{P}(1<X<4)=0.2475$

$\mathrm{P}(X<1)=0.5-0.2475=0.2525$

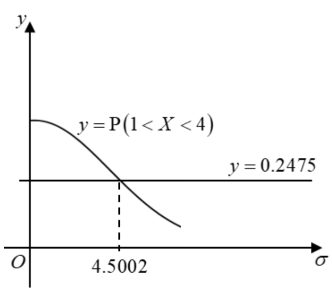
$\mathrm{P}\left(Z<\dfrac{1-4}{\sigma}\right)=0.2525$

From the GC, $\mathrm{P}(Z<-0.66664)=0.2525$

$\dfrac{1-4}{\sigma}=-0.66664$

$\sigma=4.50$ (to 3 s.f.)

<u>Alternative</u>



By GC, $\sigma=4.50$ (to 3 s.f.).

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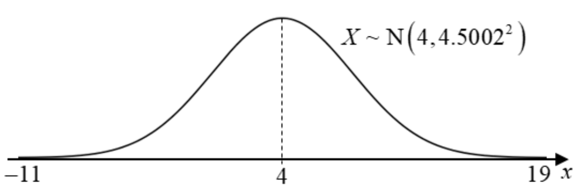
<tr><td>9c</td><td>



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10 In this question you should state the parameters of any distributions you use.

An orchard sells kiwis of two varieties, Type A and Type B. The masses, in grams, of Type A and Type B kiwis follow the distributions $\mathrm{N}\left(110, 6^2\right)$ and $\mathrm{N}\left(85, 8^2\right)$ respectively. It is assumed that these two distributions are independent.

Kiwis are selected randomly and packed into bags. Type A kiwis are packed in bags of 30, while Type B kiwis are packed in bags of 60.

(a) In a randomly chosen bag of Type A kiwis, find the probability that the mean mass of kiwis is between 109 grams and 115 grams. [2]

The selling prices of Type A and Type B kiwis are \$25 per kilogram and \$15 per kilogram respectively.

(b) Find the probability that the average selling price of a kiwi from a bag of Type A kiwis is more than the average selling price of a kiwi from a bag of Type B kiwis by at most \$1.50. [3]

The orchard also produces genetically modified Type G kiwis, which are packed in bags of 50. The masses of Type G kiwis are identically distributed with mean and standard deviation 90 grams and 5 grams respectively.

Each Assortment Box contains one bag of Type A kiwis and one bag of Type G kiwis. A box is fit for sale if the total mass of the kiwis in the box is at least 7.85 kilograms.

(c) Ten Assortment Boxes are selected at random. Find the probability that none of the boxes are fit for sale. [3]

(d) State, in context, the approximation(s) and assumption(s) needed for the distribution(s) used in part **(c)**. [2]

10a	Let A g be the mass of one Type A kiwi. $\bar{A} = \dfrac{A_1 + A_2 + \ldots + A_{30}}{30} \sim \mathrm{N}(110, 1.2)$ $\mathrm{P}(109 < \bar{A} < 115) = 0.819$ (to 3 s.f.)	
10b	Let B g be the mass of one Type B kiwi. $\bar{B} = \dfrac{B_1 + B_2 + \ldots + B_{60}}{60} \sim \mathrm{N}\left(85, \dfrac{16}{15}\right)$ $0.025\bar{A} \sim \mathrm{N}(2.75, 0.00075)$ $0.015\bar{B} \sim \mathrm{N}(1.275, 0.00024)$ $0.025\bar{A} - 0.015\bar{B} \sim \mathrm{N}(1.475, 0.00099)$ $\mathrm{P}\left(0 < 0.025\bar{A} - 0.015\bar{B} \le 1.50\right) \approx 0.787$	
10c	Let G g be the mass of one Type G kiwi. Since $n = 50$ is sufficiently large, $T = G_1 + G_2 + \ldots + G_{50} \sim \mathrm{N}(4500, 1250)$ approximately. Together with $S = A_1 + A_2 + \ldots + A_{30} \sim \mathrm{N}(3300, 1080)$, $S + T \sim \mathrm{N}(7800, 2330)$ approximately. Required probability $= \left[\mathrm{P}(A + G < 7850)\right]^{10}$ ≈ 0.197	

	Alternative $30\overline{A} \sim \mathrm{N}(3300, 1080)$ Since $n = 50$ is sufficiently large, $\overline{G} \sim \mathrm{N}(90, 0.5)$ approximately. $30\overline{A} + 50\overline{G} \sim \mathrm{N}(7800, 2330)$ approx. Required probability $= \mathrm{P}(A + G < 7850)^{10}$ ≈ 0.197	
10d	Approximation: By the **Central Limit Theorem**, the **sample sum** of masses of Type G kiwis is **approximately normally distributed**, since the **number of kiwis in each bag, 50, is sufficiently large**. Assumption: The **masses** of kiwis are **independent**.	

11 An energy company is studying the relationship between the average daily temperature, x (in degrees Celsius), and the amount of electricity consumed, y (in megawatt-hours MWh). For a random sample of ten days, the records are shown in the table below.

Average daily temperature ($x^{\circ}\mathrm{C}$)	20	22	24	26	28	30	32	34	36	38
Electricity consumption (y MWh)	12.5	12.8	13.2	14.0	14.7	16.2	16.2	19.9	22.3	29.5

(a) Draw a scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the electricity consumption and average daily temperature can be modelled by one of the formulae

$$y = a + bx^2, \qquad y = c + dx,$$

where a, b, c and d are constants.

(b) Find the value of the product moment correlation coefficient between

(i) x and y,

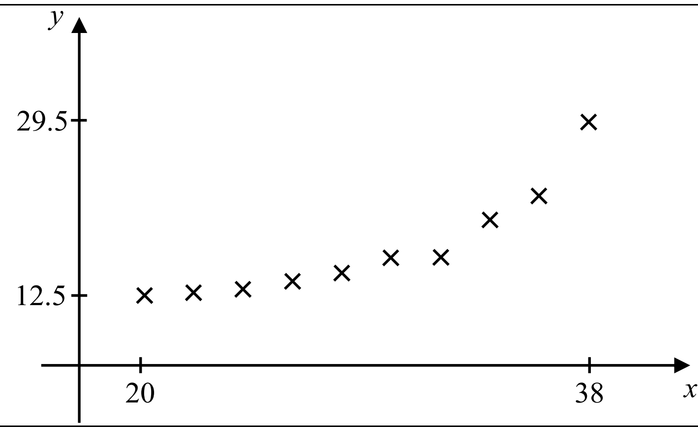
(ii) x^2 and y. [2]

(c) Using your answers in parts **(a)** and **(b)**, explain which of $y = a + bx^2$ and $y = c + dx$ is the better model and find the equation of a suitable regression line for this model. [3]

(d) Use the equation of your regression line to estimate the electricity consumption on a day with an average temperature of $37^{\circ}\mathrm{C}$. Comment on the reliability of your estimate. [2]

In some regions, temperature is measured in degrees Fahrenheit instead of degrees Celsius. A temperature of C degrees Celsius is equivalent to F degrees Fahrenheit, where $F = \frac{9}{5}C + 32$.

(e) Rewrite the equation of your regression line found in part **(c)** in terms of y and F, where F is the average daily temperature in degrees Fahrenheit. [1]

11a		
11bi	From GC, $r = 0.89056 \approx 0.891$.	
11bii	From GC, $r = 0.92229 \approx 0.922$.	
11c	From the scatter diagram, it is observed that as x increases, y increases by increasing amounts, and the product moment correlation coefficient between x^2 and y is 0.922, which is closer to 1 than that between x and y, which is 0.891 . Hence $y = a + bx^2$ is the better model. $a = 4.8291 \approx 4.83$ $b = 0.014074 \approx 0.0141$ $y = 4.83 + 0.0141x^2$	
11d	$y = 4.8291 + 0.014074(37)^2$ ≈ 24.1 Since $x = 37$ lies within the given range of x values, $20 \le x \le 38$, and the value of $r \approx 0.922$ is close to 1, the estimated electricity consumption is reliable.	
11e	$F = \frac{9}{5}x + 32 \Rightarrow x = \frac{5}{9}(F - 32)$ Replace x by $\frac{5}{9}(F-32)$: $y = 4.8291 + 0.014074\left[\frac{5}{9}(F-32)\right]^2$ $= 4.8291 + 0.0043438(F-32)^2$ $\approx 4.83 + 0.00434(F-32)^2$	

12 A hospital introduces a new medication regimen to help patients recover from a viral infection more quickly. Previously, the average recovery time was 7.2 days.

After adopting the new regimen, a random sample of 40 patients is observed, and their recovery times, x, in days, are recorded. The results are summarised below.

$$\sum(x-7.2)=-42 \quad \text{and} \quad \sum(x-7.2)^2=648$$

(a) Explain whether the hospital should carry out a one-tailed test or a two-tailed test. [1]

(b) Calculate unbiased estimates of the population mean and variance of the recovery times under the new regimen. [2]

(c) Carry out an appropriate test at the 5% significance level. You should state clearly the hypotheses for your test and define any parameters that you use. [4]

(d) Explain, in the context of the question, the meaning of 'at the 5% significance level'. [1]

(e) Explain whether it would have been sufficient for the hospital to take a random sample of 10 patients and record their recovery times under the new regimen in order to carry out the hypothesis test. [1]

It was discovered that an intern at the hospital had made an error in recording the recovery times. A random sample of another 30 patients was taken and their recovery times under the new regimen was taken and recorded by a senior researcher. The sample standard deviation was found to be 1.7 days and the sample mean time was denoted by k.

(f) A test at the 1% significance level concludes that the average recovery time did not improve from 7.2 days. Find the set of values of k. [3]

12a	The researchers should carry out a one-tailed test as they want to test whether the new regimen has **reduced mean recovery time.**	
12b	Let $y=x-7.2$ Unbiased estimate of population mean, $\bar{x}=\bar{y}+7.2=\frac{-42}{40}+7.2=6.15$ Unbiased estimate of population variance, $s_x^2=s_y^2=\frac{1}{39}\left(648-\frac{(-42)^2}{40}\right)=\frac{603.9}{39}=15.4846\approx 15.5$	
12c	Let X days be the recovery time of a patient and μ days be the population mean of X. $H_0: \mu=7.2$ $H_1: \mu<7.2$ Level of Significance: 5% Test Statistic: Since n = 40 is large, by Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z=\frac{\bar{X}-7.2}{S/\sqrt{n}} \sim N(0,1)$ approximately. Computation: $\bar{x}=6.15$, $s_x^2\approx 15.4846$, p-value = 0.045744 (or $z=-1.6876$)	

	Conclusion: Since p-value = 0.0457 < 0.05, H_0 is rejected at 5% level of significance. Hence there is sufficient evidence to conclude that the mean recovery time has reduced under the new regimen.	
12d	A 5% significance level means that there is a probability of 0.05 that the test concludes that the mean recovery time has reduced when it is actually 7.2 days.	
12e	No. Since the distribution of recovery time under the new regimen (X) is unknown, the sample size would have to be sufficiently large for Central Limit Theorem to be applied so that the **sample mean recovery time** is approximately normally distributed. A sample of 10 is not large enough.	
12f	$H_0 : \mu = 7.2$ $H_1 : \mu < 7.2$ Level of Significance: 1% Test Statistic: Since n = 30 is large, by Central Limit Theorem, $\overline{X}$ is approximately normal. When H_0 is true, $Z = \dfrac{\overline{X} - 7.2}{S/\sqrt{n}} \sim \mathrm{N}(0,1)$ approximately. Computation: $s^2 = \dfrac{n}{n-1}(\text{sample variance}) = \dfrac{30}{29}(1.7)^2 = 2.9897$ Rejection Region: $z \le -2.3263$ Since H_0 is not rejected, $\dfrac{k - 7.2}{\sqrt{2.9897}/\sqrt{30}} > -2.3263$ $k > 6.4656$ $\therefore \{k \in \mathbb{R} : k > 6.47\}$	