2025 EJC H3 Mathematics Prelim Suggested Solutions

1 Let a, b, c be positive real numbers such that $abc = 36$.

(a) Show that $2a + b + 3c \geq 18$. [2]

(b) Deduce the minimum value of $\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b}$ and find the values of a, b and c where this minimum is attained. [5]

(c) Hence, or otherwise, prove that if p, q and r are positive real numbers such that $pqr = \dfrac{1}{36}$, then

$$\frac{1}{9p^3(r+3q)} + \frac{1}{36q^3(3p+2r)} + \frac{1}{4r^3(2q+p)} \geq 9.$$ [4]

1	Solution [11]
(a)	By AM-GM inequality, $2a+b+3c \geq 3\sqrt[3]{2a \cdot b \cdot 3c}$ $= 3\sqrt[3]{6abc}$ $= 3(6)$ $= 18$
(b)	By Cauchy Schwarz inequality $\left[\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b}\right]\left[(b+3c)+(3c+2a)+(2a+b)\right] \geq (2a+b+3c)^2$ $\left[\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b}\right]\left[4a+2b+6c\right] \geq (2a+b+3c)^2$ $\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b} \geq \dfrac{2a+b+3c}{2}$ since $2a+b+3c > 0$ Hence $\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b} \geq 9$ The minimum value of 9 occurs when equality holds for AM-GM inequality and this requires $2a = b = 3c$. Substituting into $abc = 36$ yields $a(2a)\left(\dfrac{2}{3}a\right) = 36$ $a^3 = 27$ $a = 3$ $\therefore a = 3, b = 6, c = 2$ Check against original expression $\dfrac{4a^2}{b+3c} + \dfrac{b^2}{3c+2a} + \dfrac{9c^2}{2a+b} = \dfrac{4(3)^2}{6+3(2)} + \dfrac{(6)^2}{3(2)+2(3)} + \dfrac{9(2)^2}{2(3)+6} = 3+3+3 = 9$
(c)	Consider the substitution $a = \dfrac{1}{p}, b = \dfrac{1}{q}, c = \dfrac{1}{r}$ Then $abc = \dfrac{1}{p} \cdot \dfrac{1}{q} \cdot \dfrac{1}{r} = \dfrac{1}{pqr} = 36$

	$\frac{1}{9p^3(r+3q)}+\frac{1}{36q^3(3p+2r)}+\frac{1}{4r^3(2q+p)}$ $=\left[\frac{1}{9p^3(r+3q)}+\frac{1}{36q^3(3p+2r)}+\frac{1}{4r^3(2q+p)}\right]\times 36pqr$ $=\frac{4qr}{p^2r+3p^2q}+\frac{pr}{3q^2p+2q^2r}+\frac{9pq}{2r^2q+r^2p}$ $=\frac{4qr}{p^2r+3p^2q}\times\frac{\frac{1}{p^2qr}}{\frac{1}{p^2qr}}+\frac{pr}{3q^2p+2q^2r}\times\frac{\frac{1}{pq^2r}}{\frac{1}{pq^2r}}+\frac{9pq}{2r^2q+r^2p}\times\frac{\frac{1}{pqr^2}}{\frac{1}{pqr^2}}$ $=\frac{4\left(1/p\right)^2}{\left(1/q\right)+3\left(1/r\right)}+\frac{\left(1/q\right)^2}{3\left(1/r\right)+2\left(1/p\right)}+\frac{9\left(1/r\right)^2}{2\left(1/p\right)+\left(1/q\right)}$ $=\frac{4a^2}{b+3c}+\frac{b^2}{3c+2a}+\frac{9c^2}{2a+b}$ ≥ 9 by the result of the earlier part

2 **(a)** The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2xy}{x^2-2y^2}\ ,\ x\neq\sqrt{2}y \text{ for all } x,y\in\mathbb{R}.$$

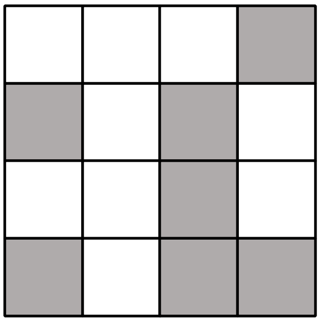
Using the substitution $y=ux$, find the general solution of the differential equation. [7]

(b) Describe geometrically the set of points that satisfy the general solution obtained in part **(a)**. [4]

2	Solutions [11]
(i)	Differentiating the substitution with respect to x, $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ Substituting $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ and $y=ux$ into the differential equation, we have $u+x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2x^2u}{x^2-2x^2u^2}$ $x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2u}{1-2u^2}-u$ $x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2u-u+2u^3}{1-2u^2}$ $x\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{u\left(2u^2+1\right)}{1-2u^2}$ Hence, $\int\frac{1-2u^2}{u\left(2u^2+1\right)}\ \mathrm{d}u=\int\frac{1}{x}\ \mathrm{d}x$ $\int\frac{1}{u}-\frac{4u}{2u^2+1}\ \mathrm{d}u=\int\frac{1}{x}\ \mathrm{d}x$ $\ln\left\|\frac{u}{2u^2+1}\right\|=\ln\|Ax\|$ $\frac{u}{2u^2+1}=Bx,$ where $B=\pm A$ $\frac{y}{x}=2B\frac{y^2}{x}+Bx$ $y=2By^2+Bx^2$
(ii)	Case 1: $B\neq 0$. $y=2By^2+Bx^2$ $\frac{x^2}{2}+\left(y-\frac{1}{4B}\right)^2=\frac{1}{16B^2}$ $\frac{(x-0)^2}{\left(\frac{1}{\sqrt{8}B}\right)^2}+\frac{\left(y-\frac{1}{4B}\right)^2}{\left(\frac{1}{4B}\right)^2}=1$

	The general solution represents an ellipse centre at $\left(0, \dfrac{1}{4B}\right)$ with horizontal radius $\dfrac{1}{\sqrt{8}B}$ and vertical radius $\dfrac{1}{4B}$ with $x \neq \sqrt{2}y$ for all $x, y \in \mathbb{R}$. Case 2: $B = 0$ The particular solution is the line $y = 0$.

3 Consider 4-by-4 unit-square grids where each unit-square is either shaded or white.
For example, the following diagram illustrates an example of such a grid.



(a) **(i)** How many possible 4-by-4 unit-square grids are there? [1]

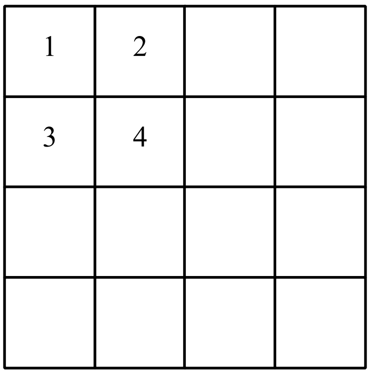
(ii) How many 4-by-4 unit-square grids are there which contain at least one entirely shaded 3-by-3 square? [4]

(b) Determine the minimum number of shaded unit-squares in a 4-by-4 unit-square grid to guarantee that it contains at least one entirely shaded 2-by-2 square. Justify your answer. [3]

(c) Define c_i to be the number of shaded unit squares in the ith column from the left of a given 4-by-4 unit-square grid.
For example, for the 4-by-4 unit-square above, $c_1 = 2, c_2 = 0, c_3 = 3, c_4 = 2$.
How many distinct possible tuples (c_1, c_2, c_3, c_4) are there if the given 4-by-4 unit-square grid has

(i) exactly 4 shaded unit-squares, [1]

(ii) at least 12 shaded unit-squares? [3]

3	Solutions [12] PIE, Pigeonhole Principle, Distribution Qns
(a)(i)	Total number of grids $= 2^{16} = 65536$
(a)(ii)	Label the squares as shown below  Let A_i be the number of grids with a 3-by-3 shaded square that's top-left square is labelled i. $\lvert A_i \rvert = 2^7$, $\lvert A_1 \cap A_2 \rvert = \lvert A_1 \cap A_3 \rvert = \lvert A_2 \cap A_4 \rvert = \lvert A_3 \cap A_4 \rvert = 2^4$,

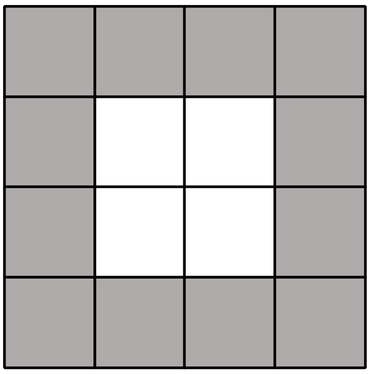
$\left|A_1 \cap A_4\right| = \left|A_2 \cap A_3\right| = 2^2$, $\left|A_i \cap A_j \cap A_k\right| = 2$ and

$\left|A_1 \cap A_2 \cap A_3 \cap A_4\right| = 1$.

By PIE,

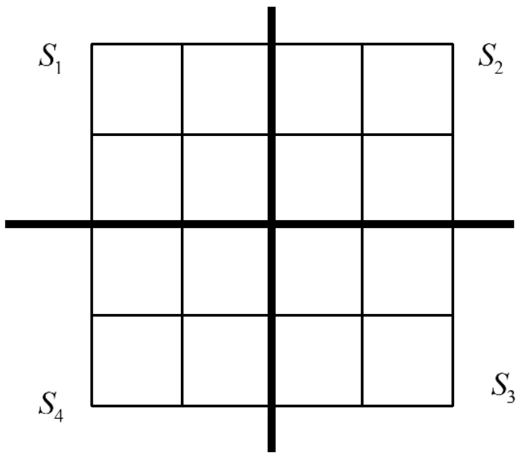
$$\begin{aligned}
&\left|A_1 \cup A_2 \cup A_3 \cup A_4\right| \\
&= \sum_{i=1}^{4}\left|A_i\right| - \sum_{1\le i<j\le 4}\left|A_i \cap A_i\right| \\
&\qquad + \sum_{1\le i<j<k\le 4}\left|A_i \cap A_i \cap A_k\right| - \left|A_1 \cap A_2 \cap A_3 \cap A_4\right| \\
&= 4\left(2^7\right) - \left(4\times 2^4 + 2\times 2^2\right) + \left(4\times 2\right) - 1 \\
&= 447.
\end{aligned}$$

(b) We require more than 12 from the following construct:



Now we show that every grid with 13 shaded unit square will satisfy the required condition.

Consider the partitioning for the grid by the four 2-by-2 squares as shown below:



	We have 13 shaded squares (pigeons) and four 2-by-2 squares (pigeonholes). By pigeonhole principle, there will be one 2-by-2 square with at least $\left\lceil \frac{13}{4} \right\rceil = 4$ shaded squares. This would be we have at least one shaded 2-by-2 square if we have 13 shaded square in the grid. Therefore, at minimum, we need 13 shaded squares.
(c)(i)	**<u>Method 1</u>:** This is equivalent to $c_1 + c_2 + c_3 + c_4 = 4$, where $c_i \in \mathbb{Z}_0^+$. Hence, $\binom{4+4-1}{4-1} = 35$. **<u>Method 2</u>:** By listing cases, $\underset{\text{1 column with 4}}{4} + \underbrace{{}^4C_3 \times 3}_{\text{2 columns with 1,1 with 2}} + \underset{\text{1 column with 3, 1 with 1}}{{}^4P_2} + \underset{\text{2 columns with 2}}{{}^4C_2} + \underset{\text{4 colums with 1}}{1} = 35$
(c)(ii)	As the number of white squares is $16 -$ the number of shaded squares. Counting this is equivalent to counting if the grids have at most 4 white squares. By symmetry this is the same as counting the grids that have at most 4 shaded squares. **<u>Method 1</u>:** So, we require, $\binom{4+4-1}{4-1} + \binom{3+4-1}{4-1} + \binom{2+4-1}{4-1} + \binom{1+4-1}{4-1} + \binom{0+4-1}{4-1}$ $= 70.$ **<u>Method 2</u>:** This is equivalent to $c_1 + c_2 + c_3 + c_4 + c_5 = 4$, where $c_i \in \mathbb{Z}_0^+$.

	We treat c_5 as the number of shaded squares not in the grid if there are less than 4 shaded squares in the grid. $\binom{4+5-1}{5-1}=70.$ **<u>Method 3</u>:** By listing cases: $\left(4+{}^4C_3\times 3+{}^4P_2+{}^4C_2+1\right)+\left(4+{}^4P_2+4\right)+\left(4+{}^4C_2\right)+(4)+(1)=70$

4 Let S be the set of all integers from 1 to n and T be the set of all subsets of S, including the empty set $\varnothing$ and S itself.

(a) Explain why $|T| = 2^n$. [1]

The sets A_1 and A_2, which are not necessarily distinct, are chosen randomly and independently from the 2^n subsets in T.

(b) In this part of the question, we will calculate $\mathrm{P}(A_1 \cap A_2 = \varnothing)$ in 2 ways.

(i) Let i be an integer such that $0 \le i \le n$. Show that $\mathrm{P}(i \notin A_1 \cap A_2) = \dfrac{3}{4}$. [2]

(ii) Hence, find $P(A_1 \cap A_2 = \varnothing)$. [1]

(iii) Let k be an integer such that $0 \le k \le n$. Find $\mathrm{P}(A_1 \cap A_2 = \varnothing \,|\, |A_1| = k)$. [1]

(iv) Deduce that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n+k}} = \left(\frac{3}{4}\right)^n$. [3]

(c) **(i)** Let k be an integer such that $0 \le k \le n$. Find $\mathrm{P}(|A_1 \cap A_2| = k)$. [2]

(ii) Find $\mathrm{E}(|A_1 \cap A_2|)$. [2]

4	**Solution[12]**
(a)	In constructing a subset of S, we have 2 choices for each of the n integers from 1 to n: include or exclude. By multiplication principle, there are 2^n subsets of S.
(b) (i)	Then, $P(i \notin A_1 \cap A_2) = 1 - P(i \in A_1 \cap A_2)$ $= 1 - P(i \in A_1)P(i \in A_2)$ (by independence) $= 1 - \left(\frac{1}{2}\right)^2$ $= \frac{3}{4}$ Alternatively, Consider the partition: $(A_1' \cap A_2') \cup (A_1' \cap A_2) \cup (A_1 \cap A_2') \cup (A_1 \cap A_2)$ Then i has equal probability to be in one of these four sets. Therefore, $P(i \notin A_1 \cap A_2) = \frac{3}{4}$.

(b) (ii)	$P\left(A_1 \cap A_2 = \varnothing\right) = \prod_{i=1}^{n} P\left(i \notin A_1 \cap A_2\right)$ $= \left(\frac{3}{4}\right)^n$
(b) (iii)	Let the k elements of A_1 be $a_1, \ldots, a_k$. $P\left(A_1 \cap A_2 = \varnothing \mid \lvert A_1 \rvert = k\right) = \prod_{i=1}^{k} P\left(a_i \notin A_2\right)$ $= \frac{1}{2^k}$ Alternatively, $P\left(A_1 \cap A_2 = \varnothing \mid \lvert A_1 \rvert = k\right) = \frac{P\left(A_1 \cap A_2 = \varnothing \text{ and } \lvert A_1 \rvert = k\right)}{P\left(\lvert A_1 \rvert = k\right)}$ $= \frac{\left[\frac{\binom{n}{k} 2^{n-k}}{4^n}\right]}{\left[\frac{\binom{n}{k}}{2^n}\right]}$ $= \frac{1}{2^k}$
(b) (iv)	$P\left(A_1 \cap A_2 = \varnothing\right) = \sum_{k=0}^{n} P\left(\lvert A_1 \rvert = k\right) P\left(A_1 \cap A_2 = \varnothing \mid \lvert A_1 \rvert = k\right)$ $= \sum_{k=0}^{n} \left[\binom{n}{k} \frac{1}{2^n} \times \frac{1}{2^k}\right]$ (there are $\binom{n}{k}$ subsets with k elements; part (bii)) $= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n+k}}$ From (bi), we have $\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n+k}} = \left(\frac{3}{4}\right)^n$.
(c) (i)	WLOG, let the k elements of their intersection be 1, …, k. Then, we must have $i \in A_1 \cap A_2$ for $1 \le i \le k$ and $j \notin A_1 \cap A_2$ for $k+1 \le j \le n$. By (bi), $P\left(i \in A_1 \cap A_2\right) = \frac{1}{4}$ and $P\left(i \notin A_1 \cap A_2\right) = \frac{3}{4}$. Moreover, there are $\binom{n}{k}$ ways to pick the k elements.

	So $P\left(\lvert A_1 \cap A_2 \rvert = k\right) = \binom{n}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{n-k}$
(c) (ii)	By (ci), $\lvert A_1 \cap A_2 \rvert \sim B(n, \frac{1}{4})$. So $E\left(\lvert A_1 \cap A_2 \rvert\right) = n \times \frac{1}{4} = \frac{n}{4}$.

5 **(a)** Let $\mathrm{f}(x)$ be a polynomial with integer coefficients.

(i) For integers α and β show that $\mathrm{f}(x)-\mathrm{f}(\alpha)$ has a factor of $x-\alpha$ and hence, or otherwise, show that

$$\mathrm{f}(\beta)-\mathrm{f}(\alpha)\equiv 0 \text{ (modulo } \beta-\alpha \text{)}.$$ [2]

(ii) Deduce that there does not exist a polynomial f with integer coefficients such that $\mathrm{f}(1)=2,\ \mathrm{f}(3)=5$. [1]

(iii) Explain why there does not exist a polynomial f with integer coefficients such that $\mathrm{f}(1)=0,\ 0<\mathrm{f}(2025)<2024$. [2]

(b) A monic degree *n* polynomial is a polynomial where the coefficient of x^n is 1.
Let p be an odd prime. Let $\mathrm{h}(x)$ be a monic degree *n* polynomial with integer coefficients.
We call an integer γ a *root modulo p* of h(*x*) if $\mathrm{h}(\gamma)\equiv 0$ (modulo *p*).
Moreover, if $\alpha\not\equiv\beta$ (modulo *p*), we call integers α,β *incongruent*.

(i) Find two incongruent roots modulo 7 for x^2+x+1. [1]

(ii) For $n\geq 1$, show that if γ is a root modulo p of h(*x*), then there exists a monic degree $n-1$ polynomial $\mathrm{h}_0(x)$ with integer coefficients, such that

$$\mathrm{h}(x)\equiv(x-\gamma)\mathrm{h}_0(x) \text{ (modulo } p\text{)}.$$ [3]

(iii) Prove, by induction, that any monic degree *n* polynomial $\mathrm{h}(x)$ has at most *n* incongruent roots modulo *p* for $n\geq 0$. [5]

Q5	Solutions [14]
(a)(i)	$\mathrm{f}(x)-\mathrm{f}(\alpha)$ is a polynomial. When $x=\alpha$, $\mathrm{f}(\alpha)-\mathrm{f}(\alpha)=0$. By factor theorem, $x-\alpha$ is a factor of $\mathrm{f}(x)-\mathrm{f}(\alpha)$. i.e. $\mathrm{f}(x)-\mathrm{f}(\alpha)=(x-\alpha)\mathrm{f}_0(x)$ for some integer polynomial $\mathrm{f}_0(x)$. When $x=\beta$, $\mathrm{f}(\beta)-\mathrm{f}(\alpha)=(\beta-\alpha)\mathrm{f}_0(\beta)\equiv 0$ (modulo $\beta-\alpha$). <u>Alternatively</u> By remainder theorem, $\mathrm{f}(x)=(x-\alpha)\mathrm{g}(x)+\mathrm{f}(\alpha)$, where g is a polynomial with integer coefficients. Sub $x=\beta$: $\mathrm{f}(\beta)\equiv\mathrm{f}(\alpha)$ (modulo $\beta-\alpha$).
(a)(ii)	Suppose there exists such a polynomial. $\mathrm{f}(3)-\mathrm{f}(1)\equiv 0$ (modulo 2) from (a)(i). But $\mathrm{f}(3)-\mathrm{f}(1)=5-2=3\equiv 1$ (modulo 2) a contradiction.
(a)(iii)	$\mathrm{f}(2025)-\mathrm{f}(1)\equiv 0$ (modulo 2024) $\mathrm{f}(2025)\equiv 0$ (modulo 2024) Since f(2025) is a multiple of 2024, it cannot be strictly between $0\times 2024=0$ and $1\times 2024=2024$.]

(b)(i)	$2 \not\equiv 4$ (modulo 7) $2^2+2+1=7\equiv 0$ (modulo 7) And $4^2+4+1=21\equiv 0$ (modulo 7) So, 2 and 4 are incongruent roots modulo 7 of x^2+x+1.
(b)(ii)	Consider $\mathrm{h}(x)-\mathrm{h}(\gamma)$. From (a)(i) we have that $\mathrm{h}(x)-\mathrm{h}(\gamma)=(x-\gamma)\mathrm{h}_0(x)$ for some integer polynomial $\mathrm{h}_0(x)$ with degree $n-1$. We note that x^{n-1} coefficient of $\mathrm{h}_0(x)$ must be 1 when we compare the x^n coefficient in the equation $\mathrm{h}(x)-\mathrm{h}(\gamma)=(x-\gamma)\mathrm{h}_0(x)$. $\mathrm{h}(x)-\mathrm{h}(\gamma)=(x-\gamma)\mathrm{h}_0(x)$ $\mathrm{h}(x)-0\equiv(x-\gamma)\mathrm{h}_0(x) \qquad (\text{modulo } p)$ $\mathrm{h}(x)\equiv(x-\gamma)\mathrm{h}_0(x) \qquad (\text{modulo } p)$ <u>Alternatively</u> $\mathrm{h}(x)=(x-\gamma)\mathrm{h}_0(x)+\mathrm{h}(\gamma)$ by Remainder Theorem, where h is a degree $n-1$ polynomial with integer coefficients. h is monic by comparing coefficients of x^n. $\mathrm{h}(x)=(x-\gamma)\mathrm{h}_0(x)+\mathrm{h}(\gamma)$ $\equiv(x-\gamma)\mathrm{h}_0(x) \qquad (\text{modulo } p)$ since γ is a root modulo *p*.
(b)(iii)	Let P(*n*) be the statement that any monic degree *n* polynomial $\mathrm{h}(x)$ has at most *n* incongruent roots modulo *p* for $n\geq 0$. Base case: When $n=0, \mathrm{h}(x)=1$ for all *x* and has no solution for that $\mathrm{h}(x)\equiv 0$ (modulo *p*). Hence it has at most 0 roots. Inductive step: Assume P(*k*) is true any monic degree *k* polynomial $\mathrm{h}(x)$ has at most *k* incongruent roots modulo *p* for some $k\geq 0$. Now suppose h(*x*) is a degree $k+1$ polynomial. If it has no roots we are done. If it has a root, say *q* (modulo *p*), then $\mathrm{h}(q)\equiv 0$ (modulo *p*). From (b)(i) we have that there exist monic degree $n-1$ integer polynomial $\mathrm{h}_0(x)$ such that $\mathrm{h}(x)\equiv(x-q)\mathrm{h}_0(x) \qquad (\text{modulo } p)$. This implies that h will have at most *k* + 1 roots modulo *p* that are incongruent modulo *p* (at most *k* from $\mathrm{h}_0(x)$ and additionally *q*). Hence, P(*k* + 1) is true. Conclusion: Since P(0) is true and P(*k*) is true $\Rightarrow \mathrm{P}(k+1)$ is true, by Principle of Mathematical Induction, P(*n*) is true for all $n\geq 0$.

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

On the irrationality of π

We denote $f^{(k)}(x)$ to be the *k*th derivative of $f(x)$ with respective to *x* and $f^{(0)}(x)=f(x)$.

A number is called *rational* if it can be simplified to the form $\frac{a}{b}$ where *a* is an integer and *b* is a positive integer where *a* and *b* do not share any prime factors. A number is called irrational if it is not a rational number. For example, the numbers $0.1, \frac{4}{3}, -\frac{39}{13}=-3$ are rational numbers but $\sqrt{2}$ and π are not.

However, showing that π is irrational, is not trivial.
In 1947, the Canadian-American mathematician Ivan Morton Niven produced an elementary yet elegant proof that π is irrational. His methods could be traced back to French mathematician Charles Hermite's 1873 paper on proving irrationality of e^r, where *r* is a rational number. We will investigate a modified version of Niven's proof of the irrationality of π.

We suppose that $\pi=\frac{a}{b}$ for positive integers *a*, *b*, for contradiction.
In his proof, for each positive integer *n*, Niven considered the polynomial

$$f(x)=\frac{x^n(a-bx)^n}{n!}=\frac{b^n\left[x(\pi-x)\right]^n}{n!}.$$

Using binomial expansion, we can write this polynomial in the form

$$f(x)=\frac{1}{n!}\sum_{r=n}^{2n}c_r x^r,$$

where c_r are integers for all *r*.
By considering the expansion, we can show that $f^{(k)}(0)$ is an integer, for $0\le k\le 2n$. This would then imply that $f^{(k)}(\pi)$ is an integer, for $0\le k\le 2n$, as well.

Niven also define another function $F(x)$ as

$$F(x)=f(x)-f^{(2)}(x)+f^{(4)}(x)-...+(-1)^n f^{(2n)}(x).$$

This is carefully chosen so that the identity $\int_0^{\pi} f(x)\sin(x)\,dx=F(0)+F(\pi)$ holds for all *n*.

For $0<x<\pi$, we have $0<f(x)<\frac{\pi^n m^n}{n!}$ for some positive real constant *m*.
We can then develop the bound

$$0<\int_0^{\pi} f(x)\sin(x)\,dx<\frac{2\pi^n m^n}{n!} \text{ for all } n\ge 1.$$

It is a known fact that if $\lim_{n\to\infty} u_n=0$, this implies that for sufficiently large *n*, we have $u_n<1$.

By selecting a sufficiently large *n*, we will attain the required contradiction to prove that π is irrational.

6 In this question you are asked to prove that π is irrational.

For the entirety of the question, we will assume that $\pi = \frac{a}{b}$, where a, b are positive integers, for contradiction.

(a) **(i)** Consider any c_r in the sum $\frac{1}{n!}\sum_{r=n}^{2n} c_r x^r$.

Briefly explain why c_r is an integer. [2]

(ii) Hence, explain why $\mathrm{f}^{(k)}(0)$ is an integer for $0 \le k \le 2n$. [2]

(b) By showing that $\mathrm{f}(x) = \mathrm{f}(\pi - x)$, prove that $\mathrm{f}^{(k)}(\pi)$ is an integer for $0 \le k \le 2n$. [3]

(c) **(i)** Prove that $\mathrm{F}''(x) = \mathrm{f}(x) - \mathrm{F}(x)$. [2]

(ii) Deduce that $\int_0^{\pi} \mathrm{f}(x)\sin(x)\,\mathrm{d}x = \mathrm{F}(0) + \mathrm{F}(\pi)$. [3]

(d) **(i)** Prove that for some positive real constant m, $0 < \mathrm{f}(x) < \frac{\pi^n m^n}{n!}$ for $0 < x < \pi$. [2]

(ii) Hence, show that $0 < \int_0^{\pi} \mathrm{f}(x)\sin(x)\,\mathrm{d}x < \frac{2\pi^n m^n}{n!}$. [1]

(iii) By considering the series expansion of e^x, or otherwise, explain why $\lim_{n\to\infty} \frac{2\pi^n m^n}{n!} = 0$. [2]

(e) Prove that π is irrational. [3]

6	Solution [20] Irrationality, Differentiation, Integration, Functions, Inequality, Limits.
(a)(i)	$$\begin{aligned} \frac{x^n(a-bx)^n}{n!} &= \frac{x^n}{n!}\sum_{r=0}^{n}\binom{n}{r}(-1)^r a^{n-r}b^r x^r \\ &= \frac{1}{n!}\sum_{r=0}^{n}\binom{n}{r}(-1)^r a^{n-r}b^r x^{n+r} \\ &= \frac{1}{n!}\sum_{r=n}^{2n}(-1)^{r-n}\binom{n}{r-n}a^{2n-r}b^{r-n}x^r \\ &= \frac{1}{n!}\sum_{r=n}^{2n}c_r x^r, \quad c_r = (-1)^{r-n}\binom{n}{r-n}a^{2n-r}b^{r-n} \end{aligned}$$ We note that $\binom{n}{r-n} \in \mathbb{Z}$ and $a^{2n-r}b^{r-n} \in \mathbb{Z}$ as $n \le r \le 2n$. Therefore, $c_r = (-1)^{r-n}\binom{n}{r-n}a^{2n-r}b^{r-n} \in \mathbb{Z}$.

(a)(ii)	For $0 \le k \le n-1$, $\mathrm{f}^{(k)}(x)$ is a polynomial with no constant term and hence that $\mathrm{f}^{(k)}(0)=0$. For $n \le k \le 2n$, $\mathrm{f}^{(k)}(x)$ will have a constant term $\frac{k!c_k}{n!}$ which will be an integer as $n \le k \le 2n$.
(b)	$\mathrm{f}(\pi - x)$ $= \frac{b^n\left[(\pi - x)(\pi-(\pi-x))\right]^n}{n!}$ $= \frac{b^n\left[(\pi - x)x\right]^n}{n!}$ $= \mathrm{f}(x)$. (Shown) $\mathrm{f}(\pi - x) = \mathrm{f}(x)$ $(-1)\mathrm{f}'(\pi - x) = \mathrm{f}'(x)$ $(-1)^2\mathrm{f}''(\pi - x) = \mathrm{f}''(x)$ ... $(-1)^k\mathrm{f}^{(k)}(\pi - x) = \mathrm{f}^{(k)}(x)$ Let $x=0$: $(-1)^k\mathrm{f}^{(k)}(\pi) = \mathrm{f}^{(k)}(0) \Rightarrow \mathrm{f}^{(k)}(\pi) = (-1)^k\mathrm{f}^{(k)}(0) \in \mathbb{Z}$.

(c)(i)	$\mathrm{F}(x)=\sum_{r=0}^{n}\left[(-1)^r \mathrm{f}^{(2r)}(x)\right]$ $\mathrm{F}'(x)=\sum_{r=0}^{n}\left[(-1)^r \mathrm{f}^{(2r+1)}(x)\right]$ $\mathrm{F}''(x)=\sum_{r=0}^{n}\left[(-1)^r \mathrm{f}^{(2r+2)}(x)\right]$ $\mathrm{F}''(x)=\sum_{r=0}^{n-1}\left[(-1)^r \mathrm{f}^{(2r+2)}(x)\right]+\underbrace{\mathrm{f}^{(2n+2)}(x)}_{=0}$ $\mathrm{F}''(x)=\sum_{r=1}^{n}\left[(-1)^{r-1} \mathrm{f}^{(2r)}(x)\right](\text{replace } r \text{ with } r-1)$ $\mathrm{F}''(x)=\mathrm{f}(x)-\mathrm{f}(x)-\sum_{r=1}^{n}\left[(-1)^r \mathrm{f}^{(2r)}(x)\right]$ $\mathrm{F}''(x)=\mathrm{f}(x)-\left(\sum_{r=0}^{n}\left[(-1)^r \mathrm{f}^{(2r)}(x)\right]\right)$ $\mathrm{F}''(x)=\mathrm{f}(x)-\mathrm{F}(x) \qquad (\text{Shown})$
(c)(ii)	Thus by (c)(i), $\int_0^\pi \mathrm{f}(x)\sin(x)\,\mathrm{d}x$ $=\int_0^\pi \mathrm{F}''(x)\sin(x)+\mathrm{F}(x)\sin(x)\,\mathrm{d}x$ $=\left[\sin(x)\mathrm{F}'(x)\right]_0^\pi-\int_0^\pi \mathrm{F}'(x)\cos(x)\,\mathrm{d}x$ $\qquad -\left[\cos(x)\mathrm{F}(x)\right]_0^\pi+\int_0^\pi \mathrm{F}'(x)\cos(x)\,\mathrm{d}x$ $=(0-0)+\left(-\mathrm{F}(\pi)\cos(\pi)+\mathrm{F}(0)\cos(0)\right)$ $=\mathrm{F}(0)+\mathrm{F}(\pi).\,(\text{Shown})$
(d)(i)	<u>**Method 1**</u>: If $0<x<\pi$, then $0<\pi-x<\pi$. And $b^n\pi^n=a^n$ as $\pi=\dfrac{a}{b}$. Hence,

	$\frac{(0)(0)}{n!} < \frac{b^n\left[(\pi - x)x\right]^n}{n!} < \frac{b^n\left[(\pi)^n(\pi)^n\right]}{n!}$ $0 < \mathrm{f}(x) = \frac{b^n\left[(\pi - x)x\right]^n}{n!} < \frac{\pi^n a^n}{n!} = \frac{\pi^n m^n}{n!}$ (Shown) Where $m = a$. (or $m = b\pi$.) **<u>Method 2</u>**: Note that $\mathrm{f}(x) = \frac{b^n}{n!}\left[x(\pi - x)\right]^n > 0$ for all $x \in (0, \pi)$. Furthermore, the graph of $y = x(\pi - x)$ is concave downwards on the interval $[0, \pi]$ and $\mathrm{f}(x)$ is a polynomial in x and so is continuous (and infinitely differentiable) on $\mathbb{R} \supset (0, \pi)$. Hence, $x(\pi - x) \le \frac{\pi}{2}$ for all $x \in (0, \pi)$. Then $\mathrm{f}(x) = \frac{b^n}{n!}\left[x(\pi - x)\right]^n$ $\le \frac{b^n}{n!} \cdot \frac{\pi^n}{2^n}$ $= \frac{\pi^n}{n!} \cdot \frac{b^n}{2^n} = \frac{\pi^n m^n}{n!}$ Where $m = \frac{b}{2}$.
(d)(ii)	As $0 < \mathrm{f}(x) = \frac{b^n\left[(\pi - x)x\right]^n}{n!} < \frac{\pi^n m^n}{n!}$ (by (d)(i)), we have: $0 < \int_0^\pi \mathrm{f}(x)\sin(x)\,\mathrm{d}x < \frac{\pi^n m^n}{n!}\int_0^\pi \sin x\,\mathrm{d}x$ $0 < \int_0^\pi \mathrm{f}(x)\sin(x)\,\mathrm{d}x < \frac{\pi^n m^n}{n!}(-\cos\pi + \cos 0)$ $0 < \int_0^\pi \mathrm{f}(x)\sin(x)\,\mathrm{d}x < \frac{2\pi^n m^n}{n!}$. (Shown)
(d)(iii)	**<u>Method 1</u>:**

	As the Maclaurin series of e^x converges we see that the general term of the series must converge to 0, i.e. $\lim\limits_{n\to\infty}\dfrac{x^n}{n!}=0$. Let $x=\pi m$, we have $\lim\limits_{n\to\infty}\dfrac{(\pi m)^n}{n!}=0$. $\lim\limits_{n\to\infty}\dfrac{2\pi^n m^n}{n!}=2\lim\limits_{n\to\infty}\dfrac{(\pi m)^n}{n!}=0.$ **<u>Method 2</u>:** For large enough *n*, say when $n>m$, $0<\dfrac{\pi a}{n}<\dfrac{1}{2}$. Let $\dfrac{2\pi^{n_0}m^{n_0}}{n_0!}=M$. Then $\lim\limits_{n\to\infty}\dfrac{2\pi^n a^n}{n!}=\lim\limits_{k\to\infty}\dfrac{2\pi^{n_0+k}a^{n_0+k}}{(n_0+k)!}\le M\lim\limits_{k\to\infty}\left(\dfrac{1}{2}\right)^k=0$.
(e)	By (d)(ii), $0<\displaystyle\int_0^\pi \mathrm{f}(x)\sin(x)\,\mathrm{d}x<\dfrac{2\pi^n m^n}{n!}$. By (d)(iii), as $\lim\limits_{n\to\infty}\dfrac{2\pi^n m^n}{n!}=0$, by the text, there is sufficiently large *n* such that $\dfrac{2\pi^n m^n}{n!}<1$. Thus, we select a sufficiently large *n* to get: $0<\displaystyle\int_0^\pi \mathrm{f}(x)\sin(x)\,\mathrm{d}x<1$. Moreover, by (a), $\mathrm{f}^{(2r)}(0),\mathrm{f}^{(2r)}(\pi)\in\mathbb{Z}$ for $0\le r\le n$, thus, $\displaystyle\int_0^\pi \mathrm{f}(x)\sin(x)\,\mathrm{d}x=\mathrm{F}(0)+\mathrm{F}(\pi)$ $\displaystyle=\sum_{r=0}^{n}\left[(-1)^r\,\mathrm{f}^{(2r)}(0)+(-1)^r\,\mathrm{f}^{(2r)}(\pi)\right]\in\mathbb{Z}$

	However, coupled with $0 < \int_0^\pi \mathrm{f}\left(x\right)\sin\left(x\right)\,\mathrm{d}x < 1$, we arrived at a contradiction as there is no integer between 0 to 1. Therefore, π is irrational.