ANS

Name:		Index Number:		Class:	

MARKING SCHEME

CATHOLIC HIGH SCHOOL
Preliminary Examination
Secondary 4 (O-Level Programme)

A

Additional Mathematics

4049/01

Paper 1

14 September 2021
2 hours 15 minutes

Additional Materials: Answer Booklets A and B

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions in the space provided
If working is needed for any question, it must be shown with the answer.
Omission of essential working will result in loss of marks.
The use of an approved scientific calculator is expected, where appropriate.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer answer to **three significant figures**.
Give answers in **degrees to one decimal place**.
For π, use either your calculator value or 3.142, unless the question requires the answer in terms of π.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total of the marks for this paper is **90**.

This document consists of **18** printed pages, including 1 blank page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \tfrac{1}{2}ab\sin C$$

1. (a) Without using tables or calculator, find the value of k such that

$$\left(\frac{10}{\sqrt{15}}-\frac{\sqrt{243}}{3\sqrt{5}}+\frac{20}{\sqrt{180}}\right)\times\frac{3}{\sqrt{5}}=2+k\sqrt{3}.$$ [3]

1(a)	$\left(\frac{10}{\sqrt{15}}-\frac{\sqrt{243}}{3\sqrt{5}}+\frac{20}{\sqrt{180}}\right)\times\frac{3}{\sqrt{5}}$ $=\frac{30}{5\sqrt{3}}-\frac{27\sqrt{3}}{15}+\frac{60}{30}$ $=\frac{6}{\sqrt{3}}-\frac{9\sqrt{3}}{5}+2$ $=\frac{30-27+10\sqrt{3}}{5\sqrt{3}}$ $=\frac{3+10\sqrt{3}}{5\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$ $=\frac{3\sqrt{3}+30}{15}$ $=2+\frac{1}{5}\sqrt{3}$ $k=\frac{1}{5}$

(b) Given that $\log_4 xy = 7$ and $\dfrac{\log_4 x}{\log_4 y} = -8$, find the value of $\log_4 y$.

Hence, evaluate $\log_4 \dfrac{2x}{y^3}$. [6]

(b)	$\log_4 xy = 7$ $\log_4 x + \log_4 y = 7 \cdots (1)$ $\log_4 x = -8\log_4 y \cdots (2)$ Sub (2) into (1): $-8\log_4 y + \log_4 y = 7$ $\log_4 y = -1$ Hence from (2), $\log_4 x = 8$ $\log_4 \dfrac{2x}{y^3} = \log_4 2 + \log_4 x - 3\log_4 y$ $= \dfrac{1}{\log_2 4} + 8 - 3(-1)$ $= 11\dfrac{1}{2}$

2. The population of polar bears in the arctic is given by the formula

$N = 8000(2+3e^{-\frac{t}{50}})$, where t is measured in years. Find

(i) the initial population, [1]

(i)	When $t = 0$, $N = 8000\ (2 + 3e^{0})$ $= 40\ 000$

(ii) the population after 50 years, [1]

(ii)	When $t = 50$, $N = 8000(2+3e^{-\frac{t}{50}})$ $=8000(2+3e^{-\frac{50}{50}})$ $=8000(2+3e^{-1})$ $=24829.10$ $=24800$

(iii) the least number of years it would take the population to exceed 20 000, [3]

(iii)	$20000 = 8000(2+3e^{-\frac{t}{50}})$ $(2+3e^{-\frac{t}{50}}) = \frac{20000}{8000}$ $e^{-\frac{t}{50}} = \frac{1}{6}$ $-\frac{t}{50} = \ln\frac{1}{6}$ $t = -50\ln\frac{1}{6}$ $=89.587$ $=89.6$ Hence it will take at least 90 years.

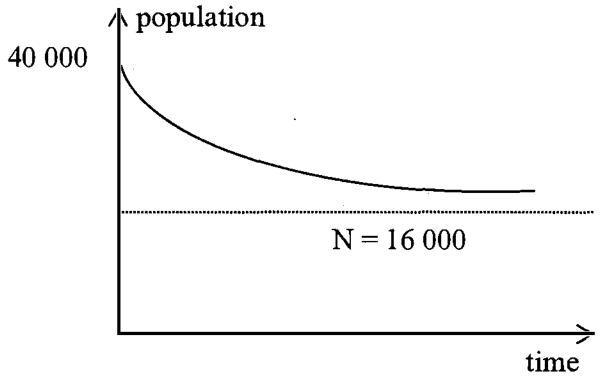
(iv) the rate at which the polar bears is decreasing when $t = 10$, [2]

(iv)	$N = 8000(2+3e^{-\frac{t}{50}})$ $\frac{dN}{dt} = 24000\left(-\frac{1}{50}\right)e^{-\frac{t}{50}}$ $= -480e^{-\frac{t}{50}}$ When $t = 10$ $\frac{dN}{dt} = -480e^{-\frac{10}{50}}$ $= -392.99$ $= -393$ The population of the polar bears is decreasing at a rate of 393 polar bears /year.

(v) From the formula $N = 8000(2+3e^{-\frac{t}{50}})$, explain why the population of the polar bears can never fall below 16000. [2]

(v)	As $t \to \infty$, $e^{-\frac{t}{50}} \to 0$ $\therefore N \to 8000(2) = 16000$ Hence the population of the polar bears will never fall below 16000. **Alternative Method** Use graph to explain

(vi) Sketch the population-time curve in the grid below. [2]



3. Given that the expression $2x^3 + ax^2 + bx - 6$ is divisible by $(x^2 - 2)$, find the value of a and of b. [4]

Let $f(x) = 2x^3 + ax^2 + bx - 6$

$$f(\sqrt{2}) = 2(\sqrt{2})^3 + a(\sqrt{2})^2 + b(\sqrt{2}) - 6 = 0$$

$$4(\sqrt{2}) + 2a + (\sqrt{2})b - 6 = 0 \cdots (1)$$

$$f(-\sqrt{2}) = -2(\sqrt{2})^3 + a(\sqrt{2})^2 - b(\sqrt{2}) - 6 = 0$$

$$-4(\sqrt{2}) + 2a - (\sqrt{2})b - 6 = 0 \cdots (2)$$

$(1) + (2)$

$$4a - 12 = 0$$

$$a = 3$$

$(1) - (2)$

$$8(\sqrt{2}) + 2(\sqrt{2})b = 0$$

$$b = \frac{-8(\sqrt{2})}{2(\sqrt{2})} = -4$$

	Alternative Method Let $f(x) = 2x^3 + ax^2 + bx - 6 = (x^2 - 2)(px + q)$ $2x^3 + ax^2 + bx - 6 = px^3 + qx^2 - 2px - 2q$ $\begin{Bmatrix} \text{Comparing coefficients of } x^3: p = 2 \\ \text{Comparing constant terms:} \quad q = 3 \end{Bmatrix}$ $\therefore a = 3$ $b = -2(2)$ $= -4$

Hence,

(i) solve the equation $2x^3 + ax^2 + bx - 6 = 0$, giving the exact value(s) of x. [3]

(i)	$f(x) = 2x^3 + 3x^2 - 4x - 6 = 0$ $(x^2 - 2)(2x + 3) = 0$ $\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)(2x + 3) = 0$ $x = -\sqrt{2}$ or $x = \sqrt{2}$ or $x = -\frac{3}{2}$ Alternative Method $2x^3 + 3x^2 - 4x - 6 = 0$ $(x^2 - 2)(px + q) = 0$ Comparing coefficients of x^3 : $p = 2$ Comparing constant terms : $-2q = -6$ $q = 3$ $f(x) = 2x^3 + 3x^2 - 4x - 6 = 0$ $(x^2 - 2)(2x + 3) = 0$ $\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)(2x + 3) = 0$ $x = -\sqrt{2}$ or $x = \sqrt{2}$ or $x = -\frac{3}{2}$

(ii) solve the equation $2 + ay + by^2 - 6y^3 = 0$. [2]

(ii)	$2+3y-4y^2-6y^3=0$ $2\left(\frac{1}{y}\right)^3+3\left(\frac{1}{y}\right)^2-4\left(\frac{1}{y}\right)-6=0$ $\therefore \frac{1}{y}=x$ $\frac{1}{y}=-\sqrt{2},\sqrt{2},-\frac{3}{2}$ $y=-\frac{1}{\sqrt{2}}$ or $y=\frac{1}{\sqrt{2}}$ or $y=-\frac{2}{3}$

4. (a) Solve $2\sin x = \tan x$ for which $0^\circ \le x \le 360^\circ$. [4]

(a)	$2\sin x = \tan x$ $2\sin x = \dfrac{\sin x}{\cos x}$ $2\sin x\cos x - \sin x = 0$ $\sin x(2\cos x - 1) = 0$ $\sin x = 0$ or $\cos x = \dfrac{1}{2}$ $x = 0^\circ, 180^\circ, 360^\circ$ or $x = 60^\circ, 300^\circ$ $x = 0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ$

(b) Find all the angles between 0 and 6 for which
$2\tan y(\tan y + 1) = 3(2 - \sec^2 y)$. [4]

(b)	$2\tan y(\tan y + 1) = 3(2 - \sec^2 y)$ $2\tan^2 y + 2\tan y = 6 - 3(1 + \tan^2 y)$ $5\tan^2 y + 2\tan y - 3 = 0$ $(5\tan y - 3)(\tan y + 1) = 0$ $\tan y = \dfrac{3}{5}$ or $\tan y = -1$ $BA = 0.5404195$ $BA = \dfrac{\pi}{4}$ or 0.785398 $y = 0.540, \pi + 0.5404195$ $y = \pi - \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$ $= 0.540, 3.68$ $= \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ or $2.36, 5.50$ $y = 0.540, \dfrac{3\pi}{4}, 3.68, \dfrac{7\pi}{4}$ or $y = 0.540, 2.36, 3.68, 5.50$

5. (a) Given that $\cos A = -\frac{3}{5}$ where $\pi < A < \frac{3\pi}{2}$. Find , without using a calculator, the value of

(i) $\sin(\pi - A)$, [1]

(i)	$\sin(\pi - A) = \sin A$ $= -\frac{4}{5}$

(ii) $\cot A$, [1]

(ii)	$\cot A = \frac{1}{\tan A}$ $= \frac{3}{4}$

(iii) $\cos\frac{A}{2}$. [2]

(iii)	$\cos A = 2\cos^2\frac{A}{2} - 1$ $\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$ $= -\sqrt{\frac{1-\frac{3}{5}}{2}}$ $= -\sqrt{\frac{1}{5}}$ $= -\frac{\sqrt{5}}{5}$

(b) Prove the identity

$$\frac{2-\sec^2 x}{\sec^2 x + 2\tan x} \equiv \frac{\cos x - \sin x}{\cos x + \sin x}.$$ [4]

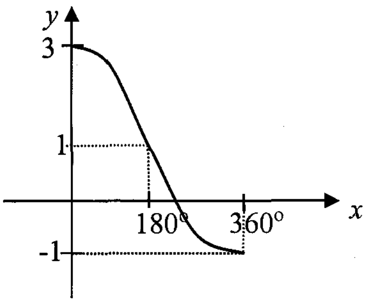
(b)	$LHS \equiv \dfrac{2-\sec^2 x}{\sec^2 x+2\tan x}$ $=\dfrac{2-\dfrac{1}{\cos^2 x}}{\dfrac{1}{\cos^2 x}+\dfrac{2\sin x}{\cos x}}$ $=\dfrac{2\cos^2 x-1}{1+2\sin x\cos x}$ $=\dfrac{\cos^2 x-\sin^2 x}{\sin^2 x+\cos^2 x+2\sin x\cos x}$ $=\dfrac{(\cos x+\sin x)(\cos x-\sin x)}{(\cos x+\sin x)^2}$ $=\dfrac{\cos x-\sin x}{\cos x+\sin x} \equiv RHS$ Alternative Method $\dfrac{2-(1+\tan^2 x)}{1+\tan^2 x+2\tan x}$ $=\dfrac{1-\tan^2 x}{(1+\tan x)^2}$ $=\dfrac{(1-\tan x)(1+\tan x)}{(1+\tan x)^2}$ $=\dfrac{(1-\tan x)}{(1+\tan x)}$ $=\dfrac{1-\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin x}{\cos x}}$ $=\dfrac{\dfrac{\cos x-\sin x}{\cos x}}{\dfrac{\cos x+\sin x}{\cos x}}$ $=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

6 The function h is defined, for $0° \leq x \leq 360°$, by $h(x) = a\cos(bx) + c$, where a, b and c are positive integers. Given that the greatest and least values of h are 3 and -1 respectively and the period of h is $720°$,

(i) state the value of a, of b and of c. [3]

(i)	$a = 2$ $c = 1$ Period, $720° = \dfrac{360°}{b}$ $b = \dfrac{1}{2}$

(ii) Sketch the graph of h. [2]



7 Express $\dfrac{4x^3+7x^2+49x+14}{(2x+1)(x^2+9)}$ in partial fractions. [6]

	$$\begin{array}{r} 2 \\ 2x^3+x^2+18x+9\overline{)4x^3+7x^2+49x+14} \\ 4x^3+2x^2+36x+18 \end{array}$$ $$\frac{4x^3+7x^2+49x+14}{(2x+1)(x^2+9)}=2+\frac{5x^2+13x-4}{(2x+1)(x^2+9)}$$ $$\frac{5x^2+13x-4}{(2x+1)(x^2+9)}=\frac{A}{2x+1}+\frac{Bx+C}{x^2+9}$$ $$5x^2+13x-4=A(x^2+9)+(Bx+C)(2x+1)$$ $$=(A+2B)x^2+(B+2C)x+9A+C$$ Comparing coefficients: $A+2B=5\cdots(1)$ $B+2C=13\cdots(2)$ $9A+C=-4\cdots(3)$ $A=-1$ $B=3$ $C=5$ $$\frac{4x^3+7x^2+49x+14}{(2x+1)(x^2+9)}=2-\frac{1}{2x+1}+\frac{3x+5}{x^2+9}$$

8 Two points A and B have coordinates (1, 2) and $(-3,6)$ respectively. A point $P\,(x\,,y)$ is such that AP and BP are perpendicular.

(a) Show that P lies on the circumference of a circle. [2]

(a)	Since $AP \perp PB$ (given), $\angle APB = 90^\circ$ Hence P lies on the circumference of a semi-circle (Right angle in semi-circle)

Find

(b) the coordinates of the centre of the circle and the radius of the circle, [4]

(b)	From (a), we know AB is a diameter of the circle Centre of circle, $C\left(\frac{1-3}{2},\frac{2+6}{2}\right)$ $=(-1,4)$ Radius, $r=\sqrt{(1+1)^2+(2-4)^2}$ $=\sqrt{8}$ $=2\sqrt{2}$

(c) the equation of the circle. [1]

(c)	$(x+1)^2+(y-4)^2=\left(\sqrt{8}\right)^2$ $(x+1)^2+(y-4)^2=8$

(d) Explain why the tangents to the circle at A and B are parallel. [2]

(d)	Since AB is a diameter of the circle, both tangents to the circle at A and B are perpendicular to AB (radius is perpendicular to the tangent) Hence the tangents to the circle at A and B are parallel.

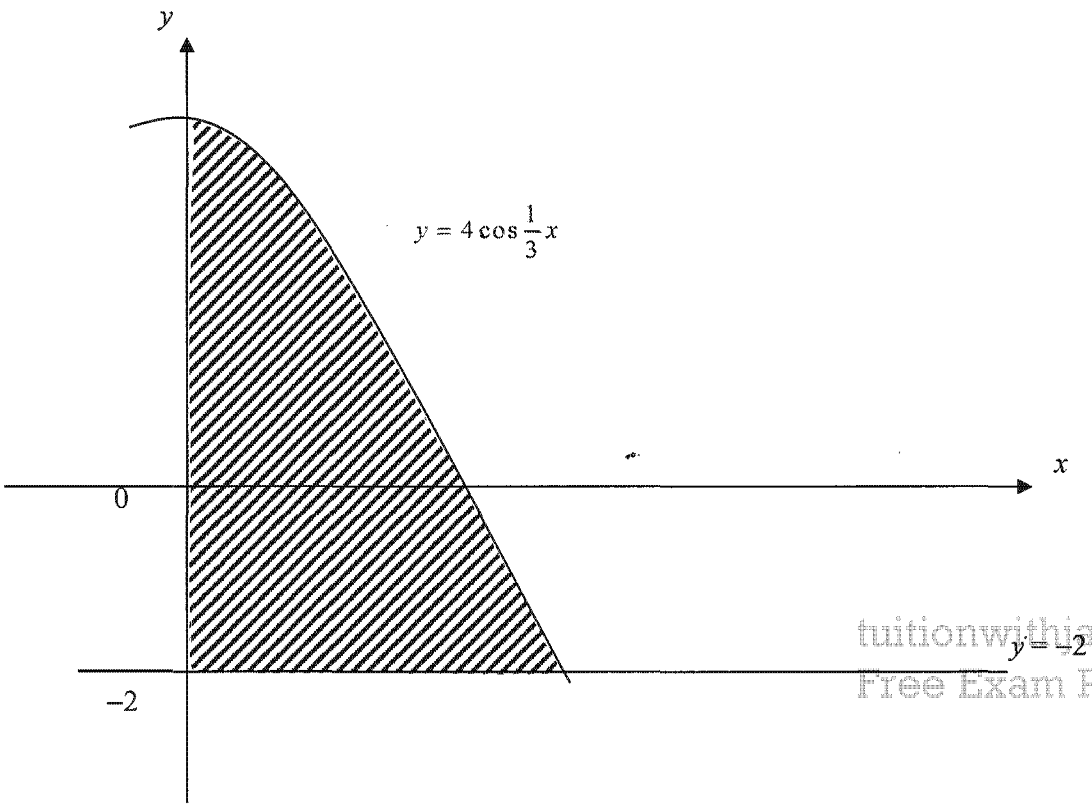
9 (a) A curve has the equation $y = \dfrac{2x+11}{x+1}, x \neq -1$. Show that y is a decreasing function. [3]

(a)	$y = \dfrac{2x+11}{x+1}$ $\dfrac{dy}{dx} = \dfrac{(x+1)2-(2x+11)}{(x+1)^2}$ $= \dfrac{-9}{(x+1)^2}$ Since $(x+1)^2 > 0$ and -9 is a negative constant, $\dfrac{dy}{dx} < 0$, hence y is a decreasing function

(b) Find the coordinates of the stationary point(s) of the curve $y = xe^{-2x}$ and determine the nature of the stationary point(s). [6]

(b)	$y = xe^{-2x}$ $\frac{dy}{dx} = -2xe^{-2x} + e^{-2x}$ $= e^{-2x}(1-2x)$ For stationary points, $\frac{dy}{dx} = 0$ $e^{-2x}(1-2x) = 0$ $x = \frac{1}{2}$ $y = \frac{1}{2e}$ $\therefore$ the stationary point is $\left(\frac{1}{2}, \frac{1}{2e}\right)$. $\frac{d^2y}{dx^2} = -2e^{-2x} - 2(1-2x)e^{-2x}$ $= 4e^{-2x}(x-1)$ When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 4e^{-1}\left(\left(\frac{1}{2}\right) - 1\right) < 0$ $\left(\frac{1}{2}, \frac{1}{2e}\right)$ is a maximum point.

10 The diagram shows part of the curve $y = 4\cos\frac{1}{3}x$. Find the area of the shaded region bounded by the curve, the y-axis and the line $y = -2$. [6]



$y = 4\cos\frac{1}{3}x$

$4\cos\frac{1}{3}x = -2$ and $2\cos\frac{1}{3}x = 0$

$\cos\frac{1}{3}x = -\frac{1}{2}$ $\qquad \frac{1}{3}x = \frac{\pi}{2}$

$BA = \frac{\pi}{3}$ $\qquad x = \frac{3\pi}{2}$

$\frac{1}{3}x = \frac{2\pi}{3}$

$x = 2\pi$

Area of the shaded region $= \int_0^{2\pi}\left(4\cos\frac{1}{3}x + 2\right)dx$

$= \left[12\sin\frac{1}{3}x + 2x\right]_0^{2\pi}$

$= \left(12\sin\frac{2\pi}{3} + 2(2\pi) - 0\right)$

$= 12\frac{\sqrt{3}}{2} + 4\pi$

$= (6\sqrt{3} + 4\pi)$ units^2

$= 22.95$

$= 23.0$ units^2

Alternative Method

Area of the shaded region

$= \int_0^{\frac{3\pi}{2}}\left(4\cos\frac{1}{3}x\right)dx + 2(2\pi) - \left[-\int_{\frac{3\pi}{2}}^{2\pi}\left(4\cos\frac{1}{3}x\right)dx\right]$

$= \left[12\sin\frac{1}{3}x\right]_0^{\frac{3\pi}{2}} + 4\pi + \left[12\sin\frac{1}{3}x\right]_{\frac{3\pi}{2}}^{2\pi}$

$= 12\sin\frac{\pi}{2} - 0 + 4\pi + 12\sin\frac{2\pi}{3} - 12\sin\frac{\pi}{2}$

$= 4\pi + 12\frac{\sqrt{3}}{2}$

$= (6\sqrt{3} + 4\pi)$ units^2

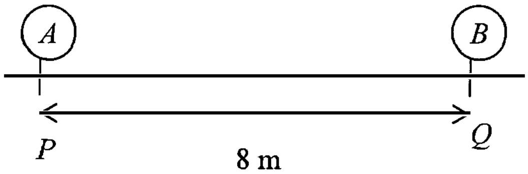
$= 23.0$ units^2

11 A particle A moves in a straight line, so that, t seconds after leaving a fixed point P, its velocity, v_A ms^{-1}, is given by $v_A = 2t - 6$.

(a) Find the distance travelled by the particle A before it comes to instantaneous rest. [3]

(a)	At rest, $V_A = 2t - 6 = 0$ [M1] $V_A = 0$ $\therefore t = 3$ $s_A = \int (2t-6)dt$ or $s_A = \int_0^3 (2t-6)dt$ $= t^2 - 6t + c$ $= \left[t^2 - 6t\right]_0^3$ $t = 0, s = 0 \Rightarrow c = 0$ $= 3^2 - 6(3) = -9$ $\therefore s_A = t^2 - 6t$ When $t = 3, s_A = 3^2 - 6(3) = -9m$ Hence total distance travelled is 9 m

A second particle B moves along the same horizontal line as A, and at the same instant that A begins to move. B starts from Q, a point 8 m away from P, with a velocity of 5 ms^{-1} and decelerates at 1 ms^{-2}.



(b) Calculate the distance between particle A and particle B when A comes to instantaneous rest. [3]

(b)	$s_B = \int (5 - t)dt$ $= 5t - \frac{t^2}{2} + c$ $t = 0, s = 8 \Rightarrow c = 8$ $\therefore s_B = 5t - \frac{t^2}{2} + 8$ When $t = 3, s_B = 15 - \frac{9}{2} + 8 = 18\frac{1}{2} m$ Distance between A and B = $18\frac{1}{2} + 9$ $= 27.5m$ or $s_B = \int_0^3 (5 - t)dt$ $= \left[5t - \frac{t^2}{2}\right]_0^3$ $=5(3) - \frac{9}{2}$ $=10\frac{1}{2}$ Distance between A and B = $10\frac{1}{2} + 8 + 9$ $= 27.5m$

(c) Find the time during the interval $0 \le t \le 5$ when the distance between particle A and particle B is at its maximum. Calculate this maximum distance. [3]

(c)	$s_B = 5t - \frac{t^2}{2} + 8$ $s_{AB} = 11t - \frac{3t^2}{2} + 8$ $\frac{ds}{dt} = 11 - 3t$ For max distance, $\frac{ds}{dt} = 0$ $\therefore t = \frac{11}{3}$ When $t = \frac{11}{3}$, $s = 11\left(\frac{11}{3}\right) - \frac{3}{2}\left(\frac{11}{3}\right)^2 + 8$ Max distance is $= 28\frac{1}{6} m$

(d) Find the range of values of t for which both particles are moving in the same direction. [1]

(d)	Hence both particles are moving in the same direction for the period $3 < t < 5$.

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