RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

<u>Year 5 H2 Math Timed Practice Topical Revision: Questions and Solutions</u>

You should check these solutions only after you have attempted the questions in this exercise.

<u>C2: Vectors</u>

1 ACJC Promo 9758/2022/Q7 modified

(a) Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non parallel vectors. Point C lies on OA, between O and A, such that $OC : CA = 2 : 1$. Point D lies on OB produced such that $BD = 5OB$.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines AB and CD meet. [3]

(b) The position vectors of the points D and E relative to the origin O are $\mathbf{d}$ and $\mathbf{e}$ respectively, where $\mathbf{d}$ and $\mathbf{e}$ are non-zero and non parallel vectors. It is given that the length OE is 2 units and $\mathbf{d}\cdot\mathbf{e} = 3$.

The point F, with position vector $\mathbf{f}$, is the reflection of the point D in the line OE.

(i) Express $\mathbf{f}$ in terms of vectors $\mathbf{d}$ and $\mathbf{e}$. [3]

(ii) Show that the area of triangle ODF can be expressed as $k\left|\mathbf{d}\times\mathbf{e}\right|$, where k is a constant to be determined. [2]

[(a) $\frac{5}{8}\mathbf{a}+\frac{3}{8}\mathbf{b}$ (bi) $\frac{3}{2}\mathbf{e}-\mathbf{d}$ (bii) $\frac{3}{4}$]

Solutions

(a)	$OC : CA = 2 : 1 \Rightarrow \overrightarrow{OC} = \frac{2}{3}\mathbf{a}$ $BD = 5OB \Rightarrow \overrightarrow{OD} = 6\mathbf{b}$ E on line AB $\Rightarrow \overrightarrow{OE} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ for some $\lambda \in \mathbb{R}$ $= \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a})$ $= (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$

	E on line CD $\overrightarrow{OE} = \overrightarrow{OC} + \mu\overrightarrow{CD}$ for some $\mu \in \mathbb{R}$ $= \frac{2}{3}\mathbf{a} + \mu\left(6\mathbf{b} - \frac{2}{3}\mathbf{a}\right)$ $= \frac{2}{3}(1-\mu)\mathbf{a} + 6\mu\mathbf{b}$ $(1-\lambda)\mathbf{a} + \lambda\mathbf{b} = \frac{2}{3}(1-\mu)\mathbf{a} + 6\mu\mathbf{b}$ Since **a** and **b** are non-zero and non-parallel, $(1-\lambda) = \frac{2}{3}(1-\mu)$ and $\lambda = 6\mu$ $\therefore \lambda = \frac{6}{16},\ \mu = \frac{1}{16}$ $\overrightarrow{OE} = \left(1 - \frac{6}{16}\right)\mathbf{a} + \frac{6}{16}\mathbf{b} = \frac{5}{8}\mathbf{a} + \frac{3}{8}\mathbf{b}$
(bi)	Let the foot of perpendicular from point *D* to line *OE* be *N*, with position vector **n**. *N* is on line *OE* $\Rightarrow \mathbf{n} = \lambda\mathbf{e}$ for some $\lambda \in \mathbb{R}$ *DN* is perpendicular to *OE* $\Rightarrow \overrightarrow{DN}\cdot\mathbf{e} = 0$ $(\lambda\mathbf{e} - \mathbf{d})\cdot\mathbf{e} = 0$ $\lambda\mathbf{e}\cdot\mathbf{e} - \mathbf{d}\cdot\mathbf{e} = 0$ $\lambda = \frac{\mathbf{d}\cdot\mathbf{e}}{\lvert\mathbf{e}\rvert^2} = \frac{3}{4}$ $\therefore \mathbf{n} = \frac{3}{4}\mathbf{e}$ $\mathbf{n} = \frac{1}{2}(\mathbf{d} + \mathbf{f})$ $\mathbf{f} = 2\mathbf{n} - \mathbf{d} = 2\frac{3}{4}\mathbf{e} - \mathbf{d} = \frac{3}{2}\mathbf{e} - \mathbf{d}$
(bii)	Area of triangle *ODF* $= \frac{1}{2}\left\lvert\mathbf{d}\times\left(\frac{3}{2}\mathbf{e} - \mathbf{d}\right)\right\rvert$ $= = \left\lvert\frac{3}{4}\mathbf{d}\times\mathbf{e} - \mathbf{d}\times\mathbf{d}\right\rvert$ $= \frac{3}{4}\lvert\mathbf{d}\times\mathbf{e}\rvert$ since $\mathbf{d}\times\mathbf{d} = \mathbf{0}$ **OR** $2\left(\frac{1}{2}\right)(ON)(DN) = 2\left(\frac{1}{2}\right)\left\lvert\frac{\mathbf{d}\cdot\mathbf{e}}{2}\right\rvert\left\lvert\frac{\mathbf{d}\times\mathbf{e}}{2}\right\rvert = \frac{3}{4}\lvert\mathbf{d}\times\mathbf{e}\rvert$

2 CJC Promo 9758/2022/Q9

Relative to the origin O, the position vectors of points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where $\mathbf{b}$ is a unit vector. It is given that $\mathbf{b}$ and $\mathbf{b}-\mathbf{a}$ are perpendicular and C lies on AB such that $AC : CB = 3:1$.

(i) Show that $|\mathbf{a}| = \sec\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

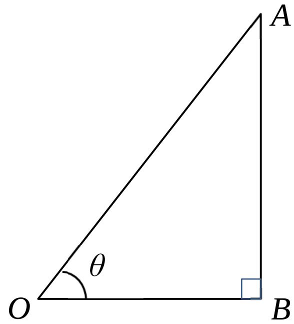
By expressing $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$,

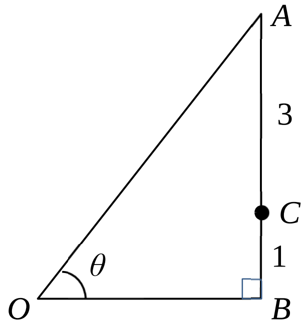
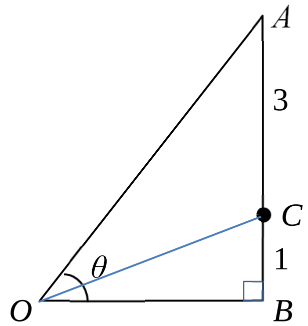
(ii) find the value of $|\mathbf{c}\cdot\mathbf{b}|$ and state the geometrical interpretation of $|\mathbf{c}\cdot\mathbf{b}|$, [4]

(iii) find the value of $\dfrac{|\mathbf{b}\times\mathbf{c}|}{|\mathbf{a}\times\mathbf{b}|}$. [2]

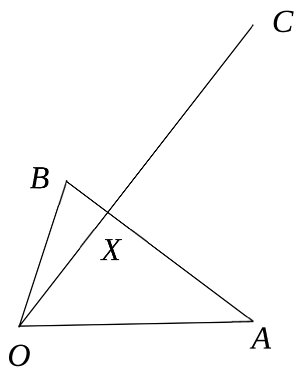
[(ii) 1 (iii) $\frac{1}{4}$]

Solutions

(i)	**<u>Method ①:</u>** Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Since $\underset{\sim}{b}$ and $\underset{\sim}{b}-\underset{\sim}{a}$ are perpendicular, $\underset{\sim}{b}\cdot(\underset{\sim}{b}-\underset{\sim}{a})=0$ $\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{b}\cdot\underset{\sim}{a}=0$ $\lvert\underset{\sim}{b}\rvert^2=\underset{\sim}{b}\cdot\underset{\sim}{a}=\underset{\sim}{a}\cdot\underset{\sim}{b}$ Since $\underset{\sim}{b}$ is a unit vector, $\underset{\sim}{a}\cdot\underset{\sim}{b}=1$ $\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta=1$ $\lvert\underset{\sim}{a}\rvert\cos\theta=1$ $\lvert\underset{\sim}{a}\rvert=\dfrac{1}{\cos\theta}=\sec\theta$ **<u>Method ②:</u>** $\cos\theta=\dfrac{OB}{OA}$ $=\dfrac{\lvert\underset{\sim}{b}\rvert}{\lvert\underset{\sim}{a}\rvert}$ $=\dfrac{1}{\lvert\underset{\sim}{a}\rvert}$ since $\underset{\sim}{b}$ is a unit vector $\lvert\underset{\sim}{a}\rvert=\dfrac{1}{\cos\theta}=\sec\theta$ (shown) 

(ii)	By Ratio Theorem, $\underset{\sim}{c}=\frac{1}{4}\underset{\sim}{a}+\frac{3}{4}\underset{\sim}{b}$  $\left\lvert\underset{\sim}{c}\cdot\underset{\sim}{b}\right\rvert=\left\lvert\left(\frac{1}{4}\underset{\sim}{a}+\frac{3}{4}\underset{\sim}{b}\right)\cdot\underset{\sim}{b}\right\rvert$ $=\left\lvert\frac{1}{4}\underset{\sim}{a}\cdot\underset{\sim}{b}+\frac{3}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}\right\rvert$ $=\left\lvert\frac{1}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}+\frac{3}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}\right\rvert$ since $\underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{b}$ $=\left\lvert\underset{\sim}{b}\right\rvert^2$ $=1$ $\left\lvert\underset{\sim}{c}\cdot\underset{\sim}{b}\right\rvert$ is the length of projection of $\underset{\sim}{c}$ on $\underset{\sim}{b}$.
(iii)	**<u>Method ①:</u>** $\frac{\left\lvert\underset{\sim}{b}\times\underset{\sim}{c}\right\rvert}{\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert}=\frac{\left\lvert\underset{\sim}{b}\times\left(\frac{1}{4}\underset{\sim}{a}+\frac{3}{4}\underset{\sim}{b}\right)\right\rvert}{\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert}$ $=\frac{\left\lvert\frac{1}{4}\left(\underset{\sim}{b}\times\underset{\sim}{a}\right)\right\rvert}{\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert}$, where $\underset{\sim}{b}\times\underset{\sim}{b}=\underset{\sim}{0}$ $=\frac{1}{4}\frac{\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert}{\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert}$, where $\left\lvert\underset{\sim}{b}\times\underset{\sim}{a}\right\rvert=\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert$ $=\frac{1}{4}$  **<u>Method ②:</u>** $\frac{\left\lvert\underset{\sim}{b}\times\underset{\sim}{c}\right\rvert}{\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert}=\frac{\frac{1}{2}\left\lvert\underset{\sim}{b}\times\underset{\sim}{c}\right\rvert}{\frac{1}{2}\left\lvert\underset{\sim}{a}\times\underset{\sim}{b}\right\rvert}$ $=\frac{\text{area of }\Delta OBC}{\text{area of }\Delta OAB}$ $=\frac{\frac{1}{2}\times BC\times h}{\frac{1}{2}\times AB\times h}$ $=\frac{1}{4}$

3 **EJC Promo 9758/2022/Q7**



Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. The lines AB and OC intersect at point X such that $AX : XB = 3:1$ and $OX : XC = 3:5$ (see diagram).

(a) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(b) Given that OC is perpendicular to AB and $OA:OB = 3:2$, show that $\mathbf{a}.\mathbf{b} = \frac{3}{8}|\mathbf{b}|^2$. [3]

(c) Hence, by considering $\left|\overrightarrow{OC}\right|^2$, find $OC : OB$. [3]

[(a) $\frac{2}{3}(\mathbf{a}+3\mathbf{b})$ (c) $\sqrt{6}:1$]

Solutions

(a)	By ratio theorem, $\overrightarrow{OX} = \frac{\mathbf{a}+3\mathbf{b}}{4}$. Since $\frac{OC}{OX} = \frac{8}{3}$, $\overrightarrow{OC} = \frac{8}{3}\left(\frac{\mathbf{a}+3\mathbf{b}}{4}\right) = \frac{2}{3}(\mathbf{a}+3\mathbf{b})$.
(b)	Since OC is perpendicular to AB, $\frac{2}{3}(\mathbf{a}+3\mathbf{b})\cdot(\mathbf{b}-\mathbf{a}) = 0$. $\mathbf{a}.\mathbf{b} - |\mathbf{a}|^2 + 3|\mathbf{b}|^2 - 3\mathbf{a}.\mathbf{b} = 0$. $-|\mathbf{a}|^2 + 3|\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b} = 0$ Since $|\mathbf{a}| = \frac{3}{2}|\mathbf{b}|$, $|\mathbf{a}|^2 = \frac{9}{4}|\mathbf{b}|^2$ We have $-\frac{9}{4}|\mathbf{b}|^2 + 3|\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b} = 0$ Rearranging, $\mathbf{a}.\mathbf{b} = \frac{3}{8}|\mathbf{b}|^2$. (shown)

(c)	Observe that $\left\|\overrightarrow{OC}\right\|^2 = \frac{2}{3}(\mathbf{a}+3\mathbf{b}).\frac{2}{3}(\mathbf{a}+3\mathbf{b})$ $= \frac{4}{9}\left(\|\mathbf{a}\|^2 + 6\mathbf{a}.\mathbf{b} + 9\|\mathbf{b}\|^2\right)$ $= \frac{4}{9}\left(\frac{9}{4}\|\mathbf{b}\|^2 + 6\left(\frac{3}{8}\|\mathbf{b}\|^2\right) + 9\|\mathbf{b}\|^2\right)$ $= \frac{4}{9}\left(\frac{27}{2}\|\mathbf{b}\|^2\right)$ $= 6\|\mathbf{b}\|^2$ So $OC{:}OB = \sqrt{6}:1$

4 **NYJC Promo 9758/2022/Q9**

The line l has equation $1-x=\frac{z-\alpha}{2}, y=1$, where α is a constant.

(i) Given that the point of intersection between the line l and the yz-plane is $(0,1,5)$, show that $\alpha = 3$. [2]

Referred to the origin, the points A and B have position vectors $-\mathbf{j}+3\mathbf{k}$ and $12\mathbf{i}+5\mathbf{j}-\mathbf{k}$ respectively.

(ii) Find the cartesian equation of the plane π_1 which contains the point A and the line l. [3]

(iii) Find the acute angle between the line AB and π_1. [3]

The plane π_2 has cartesian equation $x+2z=5$.

(iv) Find the cartesian equations of the planes such that the perpendicular distance from each plane to π_2 is $2\sqrt{5}$. [4]

[(ii) $2x-y+z=4$ (iii) 24.1° (iv) $x+2z=15$ or $x+2z=-5$]

Solutions

(i)	<u>Method 1 (using vector equation of *l*):</u> $l: \ \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ \alpha \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R}.$ $\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \alpha \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ $1-\mu = 0 \Rightarrow \mu = 1$ $5 = \alpha + 2\mu$ $\alpha = 3$ <u>Method 2 (using cartesian equation of *l*):</u> Since the point $(0,1,5)$ is on the line *l*, $1-0 = \frac{5-\alpha}{2},$ $\alpha = 3$ <u>Method 3:</u> Equation of *yz*- plane is $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0.$ [Note: *yz*-plane contains the *y* and *z* axes and the origin, with the *x* axis perpendicular to the plane] When ℓ meets the *yz*-plane, $\begin{pmatrix} 1-\mu \\ 1 \\ 3+2\mu \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$ $\Rightarrow 1-\mu = 0 \ \Rightarrow \ \mu = 1$ $\therefore \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1 \\ \alpha+2 \end{pmatrix}$ $\alpha = 3$
(ii)	$l: \ \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R}.$ Let *C* be the point (1, 1, 3) $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

	A vector normal to $\pi_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}4\\-2\\2\end{pmatrix}=2\begin{pmatrix}2\\-1\\1\end{pmatrix}$ Equation of π_1 : $\mathbf{r}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}$ $\mathbf{r}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}=4$ or $\mathbf{r}\bullet\begin{pmatrix}-2\\1\\-1\end{pmatrix}=-4$ Hence, Cartesian equation of π_1 is $2x-y+z=4$ or $-2x+y-z=-4$
(iii)	$\overrightarrow{AB}=\begin{pmatrix}12\\5\\-1\end{pmatrix}-\begin{pmatrix}0\\-1\\3\end{pmatrix}=\begin{pmatrix}12\\6\\-4\end{pmatrix}$ Let the angle between the line *AB* and π_1 be θ, $\sin\theta=\dfrac{\left\|\begin{pmatrix}12\\6\\-4\end{pmatrix}\bullet\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}12\\6\\-4\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\|}$ $\theta=24.1^\circ\ (1\text{dp})$
(iv)	Equation of π_2: $\mathbf{r}\bullet\begin{pmatrix}1\\0\\2\end{pmatrix}=5$ <u>Method 1:</u> Let a possible equation of a plane π_3 parallel to π_2 be $\mathbf{r}\bullet\begin{pmatrix}1\\0\\2\end{pmatrix}=D$. Note that point *E*(*D*,0,0) is on π_3 and point *F*(5,0,0) is on π_2. Distance between π_2 and π_3 is

	$\dfrac{\left\lvert \overrightarrow{DF} . \begin{pmatrix}1\\0\\2\end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix}1\\0\\2\end{pmatrix} \right\rvert} = \dfrac{\left\lvert \begin{pmatrix}5-D\\0\\0\end{pmatrix} . \begin{pmatrix}1\\0\\2\end{pmatrix} \right\rvert}{\sqrt{5}} = \dfrac{\lvert 5-D \rvert}{\sqrt{5}}$ $\left\lvert \dfrac{5-D}{\sqrt{5}} \right\rvert = 2\sqrt{5}$ $5-D=\pm 10$ $D=15 \quad \text{or} \quad -5$ <u>Method 2:</u> Let a possible equation of a plane π_3 parallel to π_2 be $\mathbf{r} \bullet \begin{pmatrix}1\\0\\2\end{pmatrix} = D$; Distance between π_3 and π_2 is $\left\lvert \dfrac{D-(5)}{\sqrt{5}} \right\rvert = 2\sqrt{5}$ $\therefore D=15 \quad \text{or} \quad -5$ Thus, possible equations of plane π_3 $\mathrm{r} \bullet \begin{pmatrix}1\\0\\2\end{pmatrix} = 15$ or $\mathrm{r} \bullet \begin{pmatrix}1\\0\\2\end{pmatrix} = -5$ Cartesian equations are: $x+2z=15$ or $x+2z=-5$

5 **HCI Promo 9758/2022/Q5**

Relative to the origin *O*, the points *M*, *S* and *T* have coordinates $(3, p, 0.03)$, $(1, 1, 0.02)$ and $(q, 0, 0.01)$ respectively, where *p* and *q* are real constants. The line l passes through the point *M* and is parallel to the vector $(\mathbf{i}+\mathbf{j})$.

(i) Given that l is perpendicular to the line passing through *S* and *T*, show that $q = 2$. [2]

(ii) Hence find the area of triangle OST, leaving your answer correct to 6 decimal places. [2]

(iii) It is further given that l and the line passing through *S* and *T* do not intersect. Determine the set of possible values of *p*. [3]

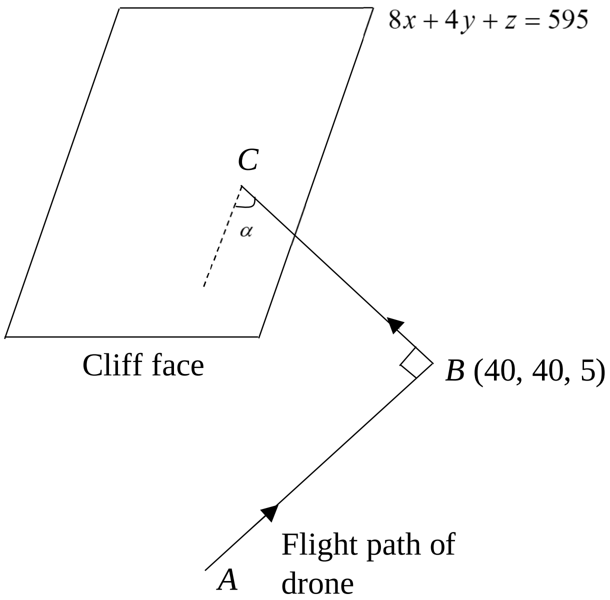
[(ii) 1.000125 units^2 (iii) $\{p \in \mathbb{R} \mid p \neq 5\}$]

Solutions

(i)	$\overrightarrow{ST} = \begin{pmatrix} q \\ 0 \\ 0.01 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0.02 \end{pmatrix} = \begin{pmatrix} q-1 \\ -1 \\ -0.01 \end{pmatrix}$ Since l is perpendicular to the line passing through S and T, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} q-1 \\ -1 \\ -0.01 \end{pmatrix} = 0$ $\Rightarrow q-1-1=0$ $\Rightarrow q=2$
(ii)	Area of triangle OST $= \frac{1}{2}\left\|\overrightarrow{OS} \times \overrightarrow{OT}\right\| = \frac{1}{2}\left\| \begin{pmatrix} 1 \\ 1 \\ 0.02 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 0.01 \end{pmatrix} \right\|$ $= \frac{1}{2}\left\| \begin{pmatrix} 0.01 \\ 0.03 \\ -2 \end{pmatrix} \right\|$ $= \frac{1}{2}\sqrt{0.01^2 + (0.03)^2 + (-2)^2}$ $= 1.000125 \text{ units}^2$ (6dp)
(iii)	Let l : $\mathbf{r} = \begin{pmatrix} 3 \\ p \\ 0.03 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\beta \in \mathbb{R}$ and

	$l_{ST}: \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0.02 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0.01 \end{pmatrix},\ \alpha \in \mathbb{R}$ If l and l_{ST} intersect each other, $\begin{pmatrix} 1 \\ 1 \\ 0.02 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0.01 \end{pmatrix} = \begin{pmatrix} 3 \\ p \\ 0.03 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ has solutions for α and β. $1-\alpha = 3+\beta$ ---(1) $1+\alpha = p+\beta$ ---- (2) $0.02+0.01\alpha = 0.03$ ---(3) From (3), $\alpha = 1$ From (1), $\beta = -3$ From (2), $p = 5$ $\therefore \{p \in \mathbb{R} \mid p \neq 5\}$ for the 2 lines to not intersect each other.

6 EJC Promo 9758/2022/Q12



A drone flies in a straight line from point A to point B with coordinates (40, 40, 5). At B, it makes a sharp 90° turn and thereafter flies in a straight line in the direction of $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$. Eventually, it crashes into a cliff face which is a part of the plane $8x+4y+z=595$, at point C (see diagram).

(a) Find the coordinates of C. [3]

(b) Determine the acute angle, α, that the path of the drone makes with the cliff face at C. [2]

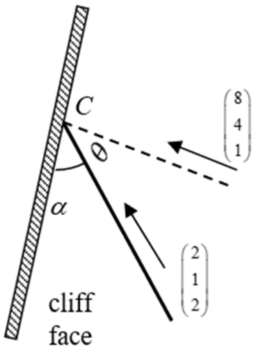
(c) Given that the coordinates of A are (q, 120, 1), find the value of q. [3]

(d) Max is trying to track the drone using his radar. From point D with coordinates (36, 36, 0), his radar emits constant signals in the directions of $\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}+\mathbf{k}$ for **all** values of θ. Assuming that signals travel indefinitely along straight lines, show that none of the signals emitted succeed in hitting the drone as it flies from B to C. [5]

[(a) $(50, 45, 15)$ (b) $54.6°$ (c) 4]

Solutions

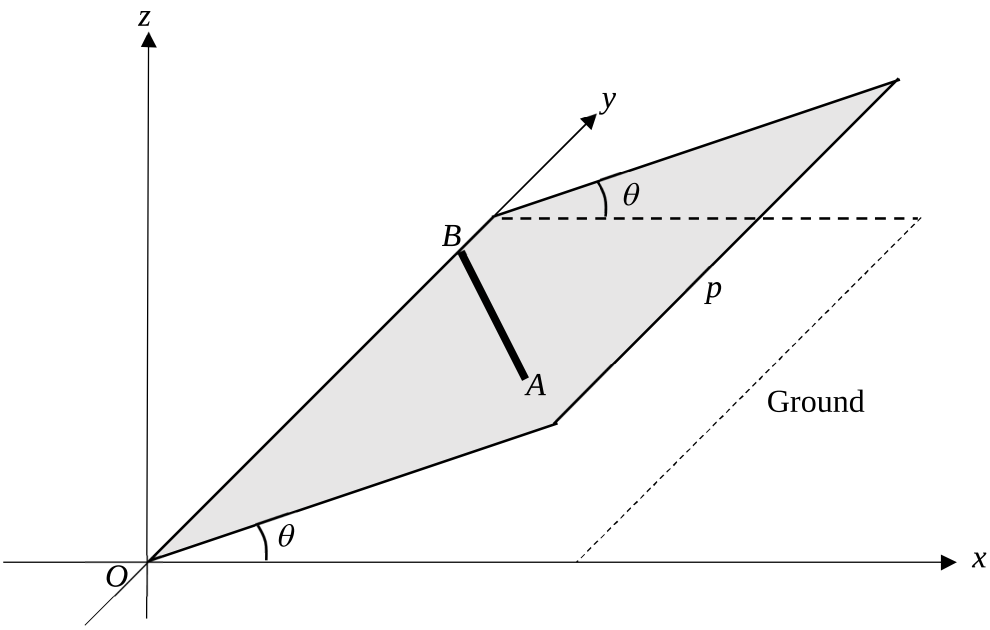
(a)	Equation of line BC is $\mathbf{r}=\begin{pmatrix}40\\40\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\2\end{pmatrix}, \lambda\in\mathbb{R}$. Since C lies on the line BC, $\overrightarrow{OC}=\begin{pmatrix}40\\40\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\2\end{pmatrix}$ for some real λ

	$\left[\begin{pmatrix}40\\40\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\2\end{pmatrix}\right].\begin{pmatrix}8\\4\\1\end{pmatrix}=595$ $\Rightarrow\begin{pmatrix}40+2\lambda\\40+\lambda\\5+2\lambda\end{pmatrix}.\begin{pmatrix}8\\4\\1\end{pmatrix}=595$ $\Rightarrow 8(40+2\lambda)+4(40+\lambda)+5+2\lambda=595$ $\Rightarrow \lambda=5$ Coordinates of $C=(50, 45, 15)$
(b)	$\cos(90°-\alpha)=\dfrac{\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}8\\4\\1\end{pmatrix}}{\sqrt{2^2+2^2+1^2}\sqrt{8^2+4^2+1^2}}=\dfrac{22}{27}$ $\sin\alpha=\dfrac{22}{27}$ So $\alpha=54.6°$ (1dp) 
(c)	Since AB and BC are perpendicular, $\overrightarrow{AB}\cdot\overrightarrow{BC}=0$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}40-q\\-80\\4\end{pmatrix}$ Since $\overrightarrow{BC}$ is parallel to $\begin{pmatrix}2\\1\\2\end{pmatrix}$, then $\begin{pmatrix}40-q\\-80\\4\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}=0$ $80-2q-80+8=0$ $\therefore q=4$
(d)	Equation of BC is $\mathbf{r}=\begin{pmatrix}40\\40\\5\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\2\end{pmatrix},\lambda\in\mathbb{R}$. Equation of any ray of signal is $\mathbf{r}=\begin{pmatrix}36\\36\\0\end{pmatrix}+\mu\begin{pmatrix}\cos\theta\\\sin\theta\\1\end{pmatrix},\mu\in\mathbb{R}$

	For radar to detect the drone, $\begin{pmatrix} 40+2\lambda \\ 40+\lambda \\ 5+2\lambda \end{pmatrix} = \begin{pmatrix} 36+\mu\cos\theta \\ 36+\mu\sin\theta \\ \mu \end{pmatrix}$ for some λ, μ . $40+2\lambda = 36+\mu\cos\theta ---(1)$ $40+\lambda = 36+\mu\sin\theta ---(2)$ $5+2\lambda = \mu ---------(3)$ From (1) and (2) : $\cos\theta = \frac{4+2\lambda}{\mu}$ and $\sin\theta = \frac{4+\lambda}{\mu}$ Using the identity $\sin^2\theta + \cos^2\theta = 1$, we have $(4+2\lambda)^2 + (4+\lambda)^2 = \mu^2$ From (3); we have $\mu = 5+2\lambda$. So $(4+2\lambda)^2 + (4+\lambda)^2 = (5+2\lambda)^2$. Simplifying, $\lambda^2 + 4\lambda + 7 = 0$ Since discriminant = $4^2 - 4(1)(7) < 0$, there are no real solutions for λ . So none of the signals hit the drone.

7 TJC Promo 9758/2022/Q12

As part of a structure for a marble race, a plane p is placed at an angle θ to the horizontal ground. The ground is taken to be the x-y plane and the line of intersection between p and the ground is the y-axis (see diagram).



(i) Given that a point A with coordinates (4, 1, 2) lies on p, find a vector equation of p in the scalar product form. Hence find the value of θ, leaving your answer in degrees correct to one decimal place. [4]

(ii) A straight rod is attached to p from A to a point B on the y-axis, as shown in the diagram. Given that AB = 6 units, find the position vector of B. [4]

A marble is released from rest at A and it moves along the rod AB towards B. It is observed that at time t seconds after the marble is released, the height of the marble above the ground is $2 - kt^2$ units where k is a positive constant less than 1.

(iii) Find in terms of k,

(a) the time taken for the marble to reach B. [1]

(b) the coordinates of the marble when t = 0.5. [4]

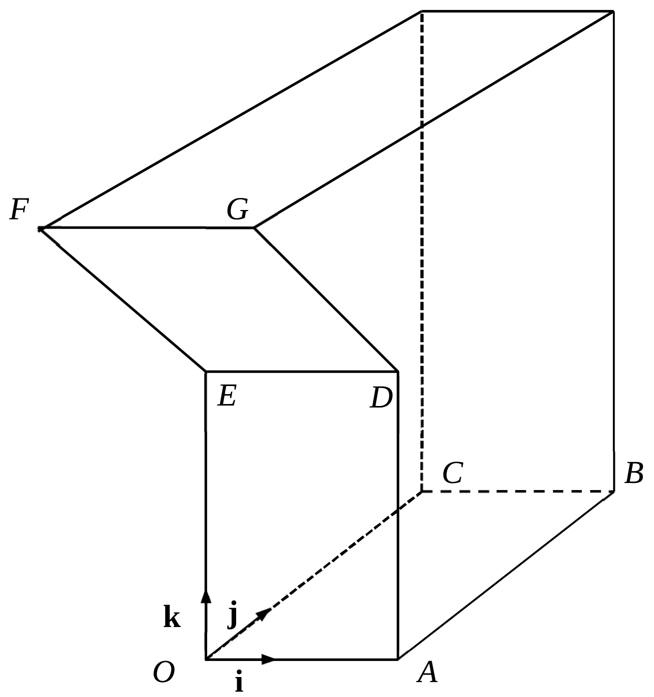
[(i) $\mathbf{r}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}=0$, $26.6°$ (ii) $\begin{pmatrix}0\\5\\0\end{pmatrix}$ (iiia) $\sqrt{\dfrac{2}{k}}$ s (iiib) $(4-\frac{k}{2}, 1+\frac{k}{2}, 2-\frac{k}{4})$]

Solutions

(i)	The plane is parallel to $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}4\\1\\2\end{pmatrix}$ Therefore a normal to p is $\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}2\\0\\-4\end{pmatrix}=2\begin{pmatrix}1\\0\\-2\end{pmatrix}$ Since the origin lies on p, we have equation of p is $\mathbf{r}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}=0$ $\cos\theta=\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}\right\|}{(1)\left(\sqrt{5}\right)}=\dfrac{2}{\sqrt{5}}\Rightarrow\theta=26.6^\circ\,(1\text{dp})$
(ii)	Since B is on the line of intersection, let $\overrightarrow{OB}=\begin{pmatrix}0\\b\\0\end{pmatrix}$. Since $AB=6$, we have $\left\|\overrightarrow{AB}\right\|=6$ $\Rightarrow\quad\left\|\begin{pmatrix}0\\b\\0\end{pmatrix}-\begin{pmatrix}4\\1\\2\end{pmatrix}\right\|=6$ $\Rightarrow\quad\sqrt{(-4)^2+(b-1)^2+(-2)^2}=6$ $\Rightarrow\quad 20+(b-1)^2=36$ $\Rightarrow\quad b-1=\pm4$ $\Rightarrow\quad b=5$ or $b=-3$ (reject since b>0 from diagram) $\therefore\quad\overrightarrow{OB}=\begin{pmatrix}0\\5\\0\end{pmatrix}$
(iiia)	For marble to reach B, height above the ground is zero, i.e. $2-kt^2=0\Rightarrow\quad t=\sqrt{\dfrac{2}{k}}$ or $-\sqrt{\dfrac{2}{k}}$ (reject since $t>0$) Marble took $\sqrt{\dfrac{2}{k}}$ s to reach B.

(iiib)	Since $\overrightarrow{AB}=\begin{pmatrix}0\\5\\0\end{pmatrix}-\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}-4\\4\\-2\end{pmatrix}=2\begin{pmatrix}-2\\2\\-1\end{pmatrix}$ and A is a point on line AB, an equation of line AB is $$\mathbf{r}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\2\\-1\end{pmatrix},\quad \lambda\in\mathbb{R}$$ Let C be the position of the marble when $t=0.5$ Since C lies on the line AB, we have $\overrightarrow{OC}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\2\\-1\end{pmatrix}$ z-coordinate of C = height of marble $=2-k\left(\frac{1}{2}\right)^2=2-\frac{1}{4}k$ Therefore, comparing z-coordinate, we have $2-\lambda=2-\frac{1}{4}k\Rightarrow\lambda=\frac{1}{4}k$ Therefore, $\overrightarrow{OC}=\begin{pmatrix}4\\1\\2\end{pmatrix}+\frac{k}{4}\begin{pmatrix}-2\\2\\-1\end{pmatrix}=\begin{pmatrix}4-\frac{k}{2}\\1+\frac{k}{2}\\2-\frac{k}{4}\end{pmatrix}$ Therefore, coordinates of the marble at $t=0.5$ is $(4-\frac{k}{2},1+\frac{k}{2},2-\frac{k}{4})$

8 **YIJC Promo 9758/2022/Q11**



The diagram above shows part of the structure of a rock-climbing wall, with a horizontal base *OABC*, a vertical rectangular wall *OADE* and an inclined rectangular wall *DEFG*. With respect to the origin *O*, the vectors **i**, **j** and **k** each of length 1 m, are parallel to *OA*, *OC* and *OE* respectively. It is given that *OE* = 10 m.

The plane *OADE* has equation $\mathbf{r}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=p$, and the plane *DEFG* has equation

$\mathbf{r}=\begin{pmatrix}0\\0\\10\end{pmatrix}+\lambda\begin{pmatrix}3\\-2\\4\end{pmatrix}+\mu\begin{pmatrix}0\\-1\\q\end{pmatrix}$, where p and q are constants, λ and μ are parameters.

(a) Give a reason why $p=0$. [1]

(b) Write down a vector equation of the line passing through the points *D* and *E*. [1]

(c) Show that $q=2$. [2]

(d) Hence, find the obtuse angle between the walls *OADE* and *DEFG*. [3]

A rock climber is at the point $H(2,-1,12)$.

(e) Show that the climber is on the plane *DEFG*. [2]

The line l passing through the points F and G has equation $\mathbf{r}=\begin{pmatrix}3\\-2\\14\end{pmatrix}+\gamma\begin{pmatrix}1\\0\\0\end{pmatrix}$, where γ is a parameter.

(f) Find the exact shortest distance from the climber to l. [3]

[(b) $\mathbf{r}=\begin{pmatrix}0\\0\\10\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}, \lambda\in\mathbb{R}$ (d) 153.4° (f) $\sqrt{5}$ m]

Solutions

(a)	The origin lies on the wall *OADE*, so $\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}=p \quad\Rightarrow\quad p=0$
(b)	$\mathbf{r}=\begin{pmatrix}0\\0\\10\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}, \lambda\in\mathbb{R}$
(c)	$\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}3\\-2\\4\end{pmatrix}=\begin{pmatrix}0\\-4\\-2\end{pmatrix}$ $\begin{pmatrix}0\\-4\\-2\end{pmatrix}\bullet\begin{pmatrix}0\\-1\\q\end{pmatrix}=0$ $4-2q=0$ $q=2$
(d)	Let the acute angle between the walls be θ . Normal to wall *DEFG* is $\begin{pmatrix}0\\2\\1\end{pmatrix}$ $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}0\\1\\0\end{pmatrix}\bullet\begin{pmatrix}0\\2\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}0\\2\\1\end{pmatrix}\right\rvert}=\dfrac{2}{\sqrt{5}}$

	$\theta \approx 26.565°$ Hence the obtuse angle between the walls is $180° - 26.565° = 153.4°$ (1dp)
(e)	**Method 1** Equation of plane *DEFG* $\mathbf{r} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 10$ --- (1) Sub $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 12 \end{pmatrix}$ into (1) LHS= $\begin{pmatrix} 2 \\ -1 \\ 12 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = -2 + 12 = 10 = \text{RHS}$ Hence point *H* lies on the plane *DEFG*. **Method 2** When $\lambda = \frac{2}{3}, \mu = -\frac{1}{3}$, $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 12 \end{pmatrix}$ Hence point *H* lies on the plane *DEFG*.
(f)	$\begin{pmatrix} 3 \\ -2 \\ 14 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

	Shortest distance $= \dfrac{\left\| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\|} = \left\| \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\| = \sqrt{5}$ metres Shortest distance = $\sqrt{5}$ metres

9 2018/9758/RVHS/Prelim/01/Q4

Relative to the origin O, the points A, B and C are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$ respectively, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors which are mutually non-parallel. The plane Π and the line l have the following equations

$$\Pi : \mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b} \quad \text{where } \lambda, \mu \in \mathbb{R} \quad \text{and}$$
$$l : \mathbf{r} = \mathbf{a} + \beta\mathbf{c} \quad \text{where } \beta \in \mathbb{R}$$

respectively.

(i) Find the point(s) of intersection between Π and l given that the points O, A, B and C are

(a) coplanar,

(b) not coplanar [3]

(ii) State the geometrical meaning of $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$. [1]

In the rest of the question, O, A, B and C are not coplanar.

(iii) The vector $\mathbf{p}$ is a unit vector in the direction of $\mathbf{a} \times \mathbf{b}$. State the geometrical meaning of $|\mathbf{c} \cdot \mathbf{p}|$. [1]

(iv) It is given that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$. Find the volume of the pyramid $OABC$. [3]

[(i) (a) line l, (b) point A (iv) $\frac{1}{3}$ unit^3]

Solution	
(ia)	Note that origin is on plane Π. When O, A, B and C are coplanar, the line l lies on the plane Π, therefore the line l and plane Π intersect along the line l. (Points of intersection are points on l.)
(ib)	When O, A, B and C are not coplanar, the line l and plane Π intersect at the point A.
(ii)	$\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ gives the area of a triangle formed with two sides parallel and equal in magnitude to $\mathbf{a}$ and $\mathbf{b}$.

(iii)	$\lvert \mathbf{c}\cdot\mathbf{p} \rvert$ gives the length of projection of $\mathbf{c}$ on $\mathbf{p}$ (<u>or</u> on $\mathbf{a}\times\mathbf{b}$, <u>or</u> on normal of Π).
(iv)	Area of triangle $OAB = \dfrac{1}{2}\left\lvert \begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}\right\rvert = \dfrac{1}{2}\left\lvert \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rvert = \dfrac{\sqrt{3}}{2}$ units^2 Using **(iii)** result, height of pyramid = height of point C relative to plane OAB $= \lvert \mathbf{c}\cdot\mathbf{p} \rvert$ where $\mathbf{p}$ is a unit vector in the direction of $\mathbf{a}\times\mathbf{b}$. $= \left\lvert \begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rvert = \dfrac{2}{\sqrt{3}}$ Volume of the pyramid $OABC$ $= \dfrac{1}{3}\times\dfrac{\sqrt{3}}{2}\times\dfrac{2}{\sqrt{3}} = \dfrac{1}{3}$ unit^3

<u>C3: Equations and Inequalities</u>

1 DHS Prelim 9758/2022/01/Q1

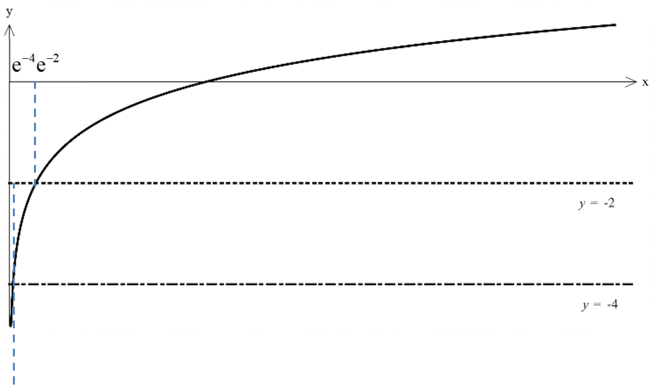
(a) Show that $-x^2+6x-14$ is always negative. [1]

(b) Without using a calculator, solve the inequality $\dfrac{x-5}{x-2} \geq \dfrac{3}{4-x}$, giving your answer in exact form. [3]

(c) Hence find the solution for the inequality $\dfrac{\ln x+5}{\ln x+2} \geq \dfrac{3}{4+\ln x}$. [3]

[(b) $x<2$ or $x>4$ (c) $x>\mathrm{e}^{-2}$ or $0<x<\mathrm{e}^{-4}$]

Solutions

(a)	$-x^2+6x-14=-\left(x^2-6x+14\right)=-\left[(x-3)^2+5\right]$ $=-(x-3)^2-5<0$ for all real values of x since $(x-3)^2 \geq 0$ **<u>Alternative</u>** Discriminant of $-x^2+6x-14=6^2-4(-1)(-14)$ $=36-56$ $=-20<0$ Since coeff of $x^2<0 \Rightarrow -x^2+6x-14<0$ for all real values of x
(b)	$\dfrac{x-5}{x-2} \geq \dfrac{3}{4-x}$ $\dfrac{(x-5)(4-x)-3(x-2)}{(x-2)(4-x)} \geq 0$ $\dfrac{-x^2+6x-14}{(x-2)(4-x)} \geq 0$ Since $-x^2+6x-14<0$ for all real x, $(x-2)(4-x)<0$ $x<2$ or $x>4$
(c)	$\dfrac{x-5}{x-2} > \dfrac{3}{4-x}$, Replace x with $-\ln x$. $\dfrac{\ln x+5}{\ln x+2} > \dfrac{3}{4+\ln x}$. $-\ln x<2$ or $-\ln x>4$ $\ln x>-2$ or $\ln x<-4$ $x>\mathrm{e}^{-2}$ or $0<x<\mathrm{e}^{-4}$ 

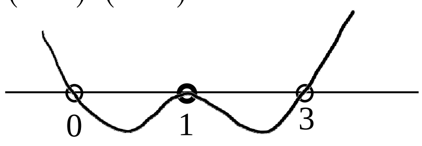
2 HCI Promo 9758/2022/Q1

(i) Solve algebraically the inequality $x^2 + x > \dfrac{4x}{3-x}$. [3]

(ii) Hence solve the inequality $x - x^2 < \dfrac{4x}{3+x}$. [2]

[(i) $x<0$ or $x>3$ (ii) $x>0$ or $x<-3$]

Solutions

(i)	$x^2 + x - \dfrac{4x}{3-x} > 0$ $\dfrac{x\left((x+1)(x-3)+4\right)}{x-3} > 0$ $\dfrac{x\left(x^2-2x+1\right)}{x-3} > 0$ $\dfrac{x(x-1)^2}{x-3} > 0$ $x(x-1)^2(x-3) > 0$  $x<0$ or $x>3$
(ii)	$x - x^2 < \dfrac{4x}{3+x}$ $x^2 - x > \dfrac{-4x}{3+x}$ $(-x)^2 + (-x) > \dfrac{4(-x)}{3-(-x)}$ Replace x with $-x$. $-x<0$ or $-x>3$ $x>0$ or $x<-3$

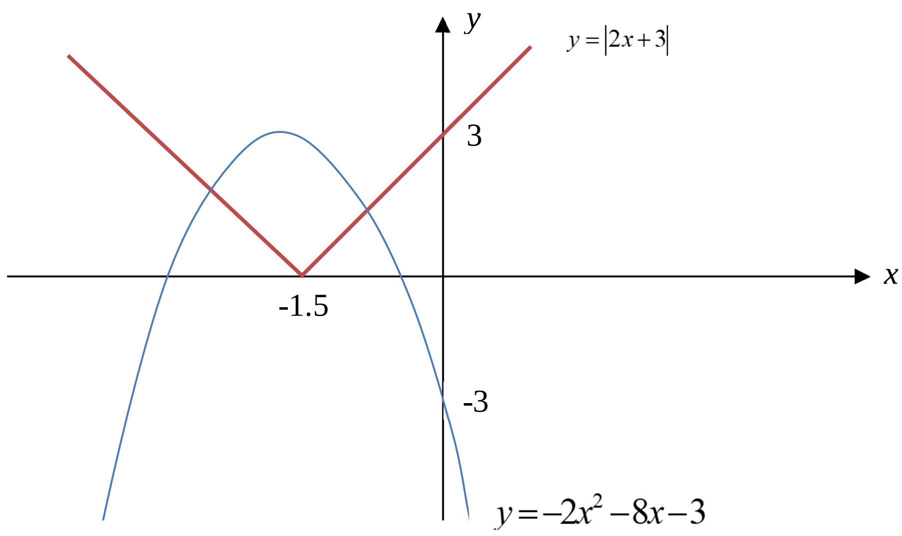
3 **CJC Promo 9758/2022/Q1**

(i) Sketch, on the same diagram, the graphs of $y=|2x+3|$ and $y=-2x^2-8x-3$.
(You are not required to label any axial intercepts and stationary points.) [2]

(ii) Solve exactly the inequality $|2x+3|>-2x^2-8x-3$. [3]

[$x>\frac{-5+\sqrt{13}}{2}$ or $x<-3$]

Solution

<table>
<tr><td>(i)</td><td>
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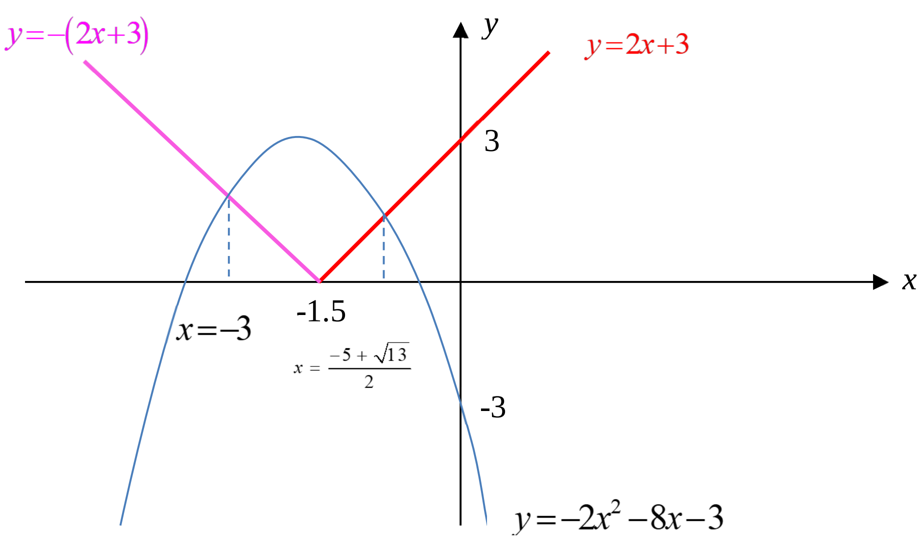
<tr><td>(ii)</td><td>

For exact points of intersection, let
<table><tr><td>$2x+3=-2x^2-8x-3$</td><td>$-(2x+3)=-2x^2-8x-3$</td></tr></table></td></tr>
</table>

$2x^2+10x+6=0$ $x^2+5x+3=0$ $x=\frac{-5\pm\sqrt{5^2-4(1)(3)}}{2(1)}$ $x=\frac{-5+\sqrt{13}}{2}$ or $x=\frac{-5-\sqrt{13}}{2}$ $\therefore x=\frac{-5+\sqrt{13}}{2}$ (reject $x=\frac{-5-\sqrt{13}}{2}$ since from graph, $x>-1.5$)	$-2x-3=-2x^2-8x-3$ $2x^2+6x=0$ $x(x+3)=0$ $x=0$ or $x=-3$ $\therefore x=-3$ (reject $x=0$, since from graph, $x<-1.5$)

Therefore, for $\lvert 2x+3\rvert>-2x^2-8x-3$,

$x>\frac{-5+\sqrt{13}}{2}$ or $x<-3$

4 **EJC Promo 9758/2022/Q4**

(a) Sketch, on the same axes, the graphs of $y=\ln(4x)$ and $y=\left|x^3-2\right|$ for $x>0$.
Hence, solve the inequality $\left|x^3-2\right|>\ln(4x)$. [4]

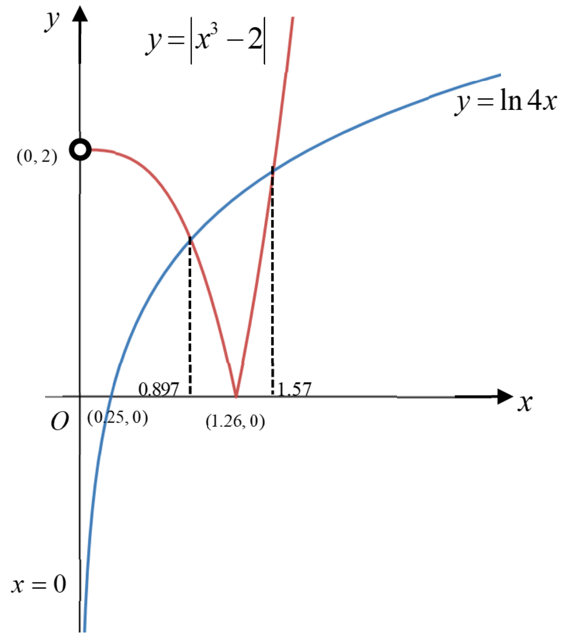
(b) Using your answer in part **(a)**, find the solution of $\left|\dfrac{1-2x^3}{x^3}\right|+\ln x>\ln 4$ for $x>0$. [3]

(c) State the solution of the inequality $\left|x^3-2\right|<-x$ for $x>0$. [1]

[(a) $0<x<0.897$ or $x>1.57$ (b) $0<x<0.639$ or $x>1.11$ (c) $\varnothing$]

Solution

(a)



Using GC, the intersection points are: (0.89719, 1.2778) and (1.5652, 1.8343) (to 5 sf)

Hence, the solution to the inequality $\left|x^3-2\right|>\ln 4x$ is $0<x<0.897$ or $x>1.57$

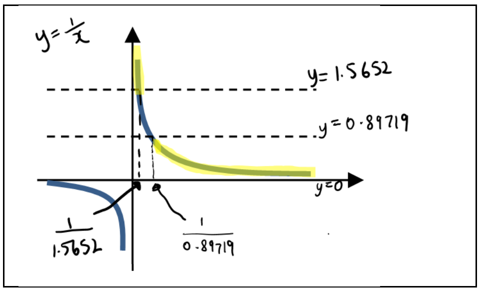
(b)

Replace x with $\dfrac{1}{x}$

$$\left|\left(\frac{1}{x}\right)^3-2\right|>\ln\frac{4}{x}$$

$$\left|\frac{1}{x^3}-2\right|>\ln 4-\ln x$$

$$\left|\frac{1-2x^3}{x^3}\right|+\ln x>\ln 4$$

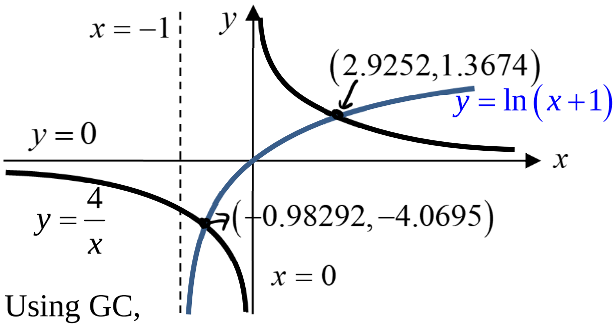
	$0<\frac{1}{x}<0.89719$ or $\frac{1}{x}>1.5652$  $x>\frac{1}{0.89719}$ or $0<x<\frac{1}{1.5652}$ $\Rightarrow x>1.11$ or $0<x<0.639$
(c)	Since $\left\|x^3-2\right\|$ is non-negative for all values of x and $-x$ is negative for $x>0$, there is <u>no solution</u>.

5 **NYJC Promo 9758/2022/Q1**

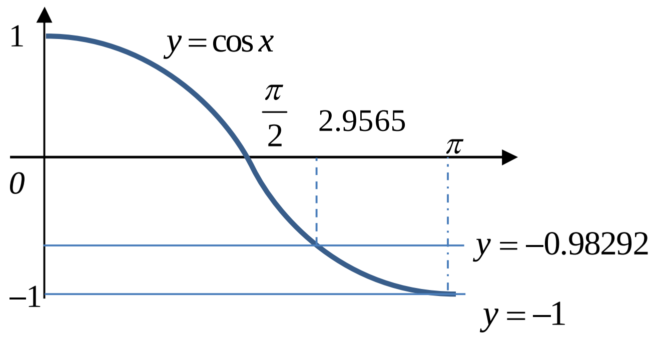
(i) Solve the inequality $\ln(x+1) \le \frac{4}{x}$. [3]

(ii) Hence solve the inequality $\cos x + 1 \le e^{4\sec x}$ for $0 < x < \pi$. [3]

Solutions

<table>
<tr><td>(i)</td><td>


Using GC,
Intersection occurs at $x = -0.98292$ and $x = 2.9252$
For $\ln(x+1) \le \frac{4}{x}$, $-1 < x \le -0.983$ or $0 < x \le 2.93$ (3sf)
</td></tr>
<tr><td>(ii)</td><td>
From (i), $(x+1) \le e^{\frac{4}{x}}$
Replacing x with $\cos x$,
$\cos x + 1 \le e^{\frac{4}{\cos x}}$
$\cos x + 1 \le e^{4\sec x}$

$\therefore$ $-1 < \cos x \le -0.98292$ or $0 < \cos x \le 2.9252$


From GC,
$2.96 \le x < \pi$ or $0 < x < \frac{\pi}{2}$
</td></tr>
</table>

6 NJC Promo 9758/2022/Q1

Ben, Caleb and Dylan went to the supermarket in the morning to buy salmon, tuna and swordfish. Ben paid \$246 for 2 kg of salmon, 1 kg of tuna and 3 kg of swordfish. Caleb bought 1 kg of salmon and 1 kg of swordfish. Caleb paid \$18 more than Dylan who bought 1 kg of tuna.

After 12pm, a discount of 15%, 10% and 5% is given for salmon, tuna and swordfish respectively such that the total cost of 1 kg of salmon, 1 kg of tuna and 1 kg of swordfish after discount is \$123.30.

Find the selling price of 1 kg of salmon, tuna and swordfish respectively before discount. [3]

[\$48, \$60 and \$30]

Solutions

Let \$*A*, \$*B* and \$*C* be the selling price of 1 kg of salmon, 1 kg of tuna and 1 kg of swordfish respectively before discount.

$2A+B+3C=246 \quad \cdots(1)$

$A+C=B+18$

$\therefore A-B+C=18 \quad \cdots(2)$

$0.85A+0.9B+0.95C=123.3 \quad \cdots(3)$

By GC, $A=48, B=60, C=30$

The selling price of 1 kg of salmon, tuna and swordfish before discount is \$48, \$60 and \$30 respectively.

7 **VJC Promo 9758/2022/Q1**

When the polynomial $\mathrm{P}(x)=ax^3+bx^2+cx-1$ is divided by $(x+1)$ and $(x-3)$, the remainders are -12 and 140 respectively. It is given that $(2x-1)$ is a factor of $\mathrm{P}(x)$.

Find the values of a, b and c. [4]

[$a=6$, $b=-3$, $c=2$]

Solutions

Let $\mathrm{P}(x)=ax^3+bx^2+cx-1$

$$\mathrm{P}(-1)=-12 \quad \Rightarrow (-1)^3a+(-1)^2b-c-1=-12$$
$$\Rightarrow -a+b-c=-11 \qquad \text{---- (1)}$$
$$\mathrm{P}(3)=140 \quad \Rightarrow (3)^3a+(3)^2b+(3)c-1=140$$
$$\Rightarrow 27a+9b+3c=141 \qquad \text{---- (2)}$$
$$\mathrm{P}\left(\frac{1}{2}\right)=0 \quad \Rightarrow \left(\frac{1}{2}\right)^3a+\left(\frac{1}{2}\right)^2b+\left(\frac{1}{2}\right)c-1=0$$
$$\Rightarrow \frac{1}{8}a+\frac{1}{4}b+\frac{1}{2}c=1 \qquad \text{---- (3)}$$

From GC, $a=6$, $b=-3$, $c=2$

8 **TJC Prelim 9758/2022/02/Q1**

A movie theatre supports a charity event by providing discounted movie tickets at the following prices based on age group.

Age group	Discounted ticket price
≤ 12 years	\$6
13 to 49 years	\$9
≥ 50 years (senior citizen)	\$4

The event organiser plans to spend \$700 to sponsor 100 participants of which at least 35 of them are senior citizens. Find the possible number of participants for each age group. [4]

Solutions

Let x, y and z denote the number of participants in the age group ≤ 12 years, 13 to 49 years and ≥ 50 years respectively.

$$x+y+z=100 \quad \text{--- (1)}$$
$$6x+9y+4z=700 \quad \text{--- (2)}$$
$$z\ge 35$$

From GC,

$x=\frac{200}{3}-\frac{5}{3}z\geq 0 \quad \Rightarrow \quad z\leq 40 \quad \therefore \quad 35\leq z\leq 40$

$y=\frac{100}{3}+\frac{2}{3}z$

<u>Method 1</u> Using GC table of values:

z	x	y
37	5	58
40	0	60

<u>Method 2</u>

Since $x=\frac{200}{3}-\frac{5}{3}z=\frac{5}{3}(40-z)$ **must be a positive integer**,

$z=40$ or $z=37$

Thus the possible number of participants are $\begin{cases} x=5 \\ y=58 \\ z=37 \end{cases}$ or $\begin{cases} x=0 \\ y=60 \\ z=40 \end{cases}$

9 RVHS Prelim 9758/2022/01/Q1

In a game show, each contestant earns points by performing consecutive basketball free-throws within a time limit. If the basketball goes through the hoop, the contestant earns x points; if the basketball merely hits the hoop without going through it, the contestant earns y points. Otherwise, the contestant earns no point. The number of basketballs that went through the loop and that which hit the loop without going through it for contestants A, B and C are shown in the following table.

Contestant	Number of basketballs that went through the loop	Number of basketballs that hit the loop without going through it
A	4	9
B	2	10
C	9	8

For the second part of the game, each of the contestants takes turn to choose from 2 envelopes, one with the number '*n*' written on it while the other with nothing written on it. The contestant who chooses the envelope with '*n*' written on it will have his total points earned increased by n times. Otherwise, the points remain the same. Both contestants A and B chose the envelope with 'n' written on it, while contestant C chose the one with nothing written on it.

The final points earned by contestants A, B and C are 762, 480 and 498 respectively.

Find x, y and n. [4]

Using the value of n found above and suppose contestant C also chooses the envelope with 'n' written on it, how many more points will he have in comparison to the combined points of contestants A and B? [2]

[$x = 50$, $y = 6$ and $n = 2$; 252 more points]

Solutions

Contestant A: $4(n+1)x + 9(n+1)y = 762 ---(1)$

Contestant B: $2(n+1)x + 10(n+1)y = 480 ---(2)$

Contestant C: $9x + 8y = 498 ---(3)$

From (1): $4x + 9y - 762\left(\frac{1}{n+1}\right) = 0 ---(4)$

From (2): $2x + 10y - 480\left(\frac{1}{n+1}\right) = 0 ---(5)$

From (3): $9x + 8y + 0\left(\frac{1}{n+1}\right) = 498 ---(6)$

Solving with GC: $x = 50$, $y = 6$ and $\frac{1}{n+1} = \frac{1}{3} \Rightarrow n = 2$

Alternatively,

$(1) \div (2)$: $\frac{4x+9y}{2x+10y} = \frac{762}{480} \Rightarrow 99x - 825y = 0$,

and solve with (3) above.

Suppose contestant C chooses the envelope with 'n',

Number of points = $498(n + 1) = 1494$

Hence, he will have 1494 – (762 + 480) = **252** more points.

C4: Functions

1 **YIJC Prelim 9758/2022/02/Q3 modified**

Functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2+3x-1,\ x \in \mathbb{R},\ x \le k,$$

$$\mathrm{g}: x \mapsto \sqrt{x+5},\ x \in \mathbb{R},\ x \ge -5.$$

(a) Given that f^{-1} exists, state the largest possible value of k. Using this value of k, find $\mathrm{f}^{-1}(x)$. [3]

For the rest of this question, let $k=-2$.

(b) Determine whether the composite functions fg and gf exist. If the composite function exists, give a definition (including the domain) of the function. [3]

(c) Hence find the exact range of the composite function that exists. [1]

[(a) -1.5, $\mathrm{f}^{-1}(x)=-1.5-\sqrt{x+3.25}$ (b) gf $(x)=\sqrt{x^2+3x+4}$, $(-\infty,-2]$ (c) $[\sqrt{2},\infty)$]]

Solutions

(a)	Largest possible $k=-1.5$ $y=x^2+3x-1=(x+1.5)^2-2.25-1$ $(x+1.5)^2=y+3.25$ $x=-1.5\pm\sqrt{y+3.25}$ since $x\le-1.5$ $x=-1.5-\sqrt{y+3.25}$ $\mathrm{f}^{-1}(x)=-1.5-\sqrt{x+3.25}$
(b)	fg does not exist as $\mathrm{R_g}=[0,\infty)\not\subseteq \mathrm{D_f}=(-\infty,-2]$ gf exists as $\mathrm{R_f}=[-3,\infty)\subseteq \mathrm{D_g}=[-5,\infty)$ $\mathrm{gf}(x)=\mathrm{g}(x^2+3x-1)$ $=\sqrt{x^2+3x-1+5}$ $=\sqrt{x^2+3x+4}$ Domain of gf is $\mathrm{D_f}$, ie $(-\infty,-2]$
(c)	$\mathrm{R_f}=[-3,\infty)\Rightarrow \mathrm{R_{gf}}=$ Range of g with domain restricted to $[-3,\infty)=[\sqrt{2},\infty)$

2 **VJC Prelim 9758/2022/02/Q2**

(a) Functions f and g are defined by

$$\mathrm{f}: x \mapsto 2-\mathrm{e}^{x+a}, \quad \text{for } x \in \mathbb{R},\ x>-2,$$

$$\mathrm{g}: x \mapsto x^2+2x, \quad \text{for } x \in \mathbb{R},\ x<-1,$$

where a is a constant.

(i) Find $\mathrm{g}^{-1}(x)$. [2]

(ii) Explain why the composite function fg exists. [2]

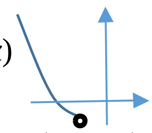
(iii) Find, in terms of a, an expression for $\mathrm{fg}(x)$ and write down the domain of fg. [2]

(iv) Find the range of fg, giving your answer in terms of e and a. [2]

(b) The function h is defined by $\mathrm{h}: x \mapsto \mathrm{e}^{|x+\lambda|}$, $x \in \mathbb{R}$, where λ is a constant. Does h have an inverse? Justify your answer. [2]

[(ai) $\mathrm{g}^{-1}(x)=-1-\sqrt{1+x}$ (iii) $\mathrm{fg}(x)=2-\mathrm{e}^{x^2+2x+a}$, $\mathrm{D_{fg}}=(-\infty,-1)$
(iv) $\left(-\infty, 2-\mathrm{e}^{a-1}\right)$]

Solutions

(ai)	Let $y=\mathrm{g}(x)=x^2+2x$ $x^2+2x-y=0$ $x=\dfrac{-2\pm\sqrt{4-4(-y)}}{2}=-1\pm\sqrt{1+y}=-1-\sqrt{1+y} \quad (\because x<-1)$ $\therefore \mathrm{g}^{-1}(x)=-1-\sqrt{1+x}$
(aii)	Range of g = $(-1,\infty)$. Domain of f = $(-2,\infty)$. Since range of g $\subseteq$ domain of f, fg exists. 
(aiii)	$\mathrm{fg}(x)=\mathrm{f}\left(x^2+2x\right)$ $=2-\mathrm{e}^{x^2+2x+a}$ $\mathrm{D_{fg}}=\mathrm{D_g}=(-\infty,-1)$.
(aiv)	Range of g $=(-1,\infty)$. Range of fg = $\left(-\infty, 2-\mathrm{e}^{a-1}\right)$
(b)	$\mathrm{h}(-\lambda+1)=\mathrm{e}^{\lvert-\lambda+1+\lambda\rvert}=\mathrm{e}$, $\mathrm{h}(-\lambda-1)=\mathrm{e}^{\lvert-\lambda-1+\lambda\rvert}=\mathrm{e}$ Hence, h is not one-one and h does not have an inverse.

3 SAJC Prelim 9758/2022/01/Q3 modified

The function f is defined by $\mathrm{f} : x \mapsto ax^3 + bx^2 + cx + d$, where $x \in \mathbb{R}$ and a, b, c and d are constants.

The graph of f intersects the y-axis at $y = -3$ and passes through the points $(-1, 0)$ and $(2, 0)$.

(i) Explain why f does not have an inverse. [1]

(ii) Given also that the tangent to the graph of f at $x = 1$ is a horizontal line, find $\mathrm{f}(x)$. [3]

(iii) Given that the function f has an inverse if its domain is restricted to $x \geq k$, state the smallest possible value of k. [1]

For the rest of the question, use the domain given and value of *k* found in part (iii).

(iv) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram. [3]

(v) Show that the solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ satisfies the equation $3x^3 - 11x - 6 = 0$. Hence, find the solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

(vi) It is given that $\mathrm{g}(x) = \ln(x+5)$, where $x > -5$. A student attempts to find the composite function gf. The student's solution is shown below:

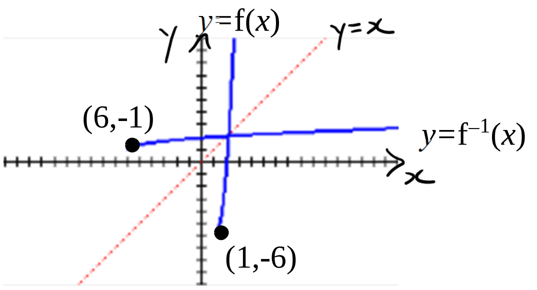
$\mathrm{g}(x) = \ln(x+5)$ $\mathrm{D_{gf}} = \mathrm{D_f} = [k, \infty)$ $\therefore \mathrm{gf}(x) = \ln\left(ax^3 + bx^2 + cx + d + 5\right),\ x \in \mathbb{R},\ x \geq k.$

Comment on the validity of the student's solution. [1]

[(ii) $\mathrm{f}(x) = \dfrac{3}{2}x^3 - \dfrac{9}{2}x - 3$ (iii) k=1 (v) 2.14]

Solutions

(i)	Given $\mathrm{f}(-1) = \mathrm{f}(2) = 0$ but $-1 \neq 2$ and $-1, 2 \in \mathrm{D_f}$, f is not a one-to-one function. Hence, f does not have an inverse.
(ii)	Let $y = \mathrm{f}(x) = ax^3 + bx^2 + cx + d$ Curve passes through $(0, -3)$

	$d=-3$ $y=\mathrm{f}(x)=ax^3+bx^2+cx-3$ Curve passes through $(-1,0)$ $-a+b-c=3$ ----------- (1) Curve passes through $(2,0)$ $8a+4b+2c=3$ ----------- (2) $\frac{\mathrm{d}y}{\mathrm{d}x}=3ax^2+2bx+c$ Tangent to the curve at $x=1$ is a horizontal line, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $3a+2b+c=0$ ------------ (3) Solving (1), (2) and (3) using GC, $a=\frac{3}{2}$, $b=0$, $c=-\frac{9}{2}$ $\therefore\ \mathrm{f}(x)=\frac{3}{2}x^3-\frac{9}{2}x-3$
(iii)	Smallest k = 1
(iv)	
(v)	Since the graphs $y=\mathrm{f}(x)$, $y=\mathrm{f}^{-1}(x)$ and $y=x$ intersect at the same point, the solution of $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$ is the same as the solution of $\mathrm{f}(x)=x$. $\Rightarrow \frac{3}{2}x^3-\frac{9}{2}x-3=x$ $\Rightarrow 3x^3-11x-6=0$ (shown) Solving the equation using GC, $x=2.14$(3 s.f.) since $x\geq 1$
(vi)	$\mathrm{R_f}=[-6,\infty)\not\subseteq \mathrm{D_g}=(-5,\infty)$ Hence gf does not exist, so the student's solution is not valid.

4 **MI PU2 P1 Promo 9758/2022/Q6**

The function f is defined by

$$\mathrm{f} : x \mapsto x^2 - 4x + 5 \quad \text{for} \quad x \in \mathbb{R}, \quad x \le 3.$$

(i) Find the range of f. [1]

The function g is defined by

$$\mathrm{g} : x \mapsto \ln(2x - a) \quad \text{for} \quad x \in \mathbb{R}, \quad x > \frac{a}{2}.$$

(ii) Given that $a < 2$, explain why gf exists. [2]

Use $a = 1$ for the rest of this question.

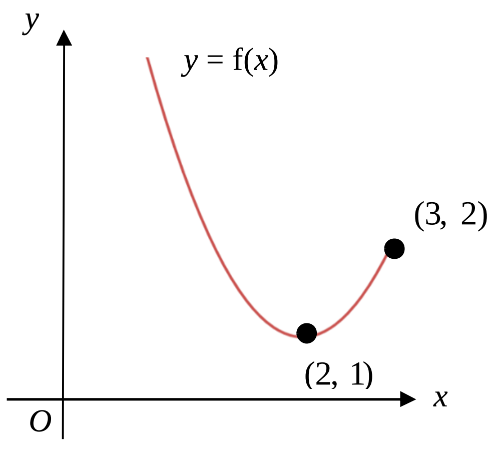
(iii) Find $\mathrm{gf}(x)$ and state the domain of gf. [2]

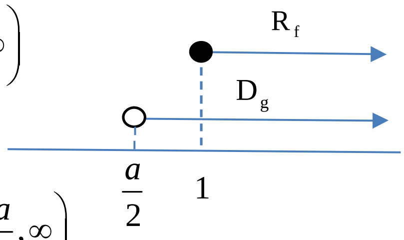
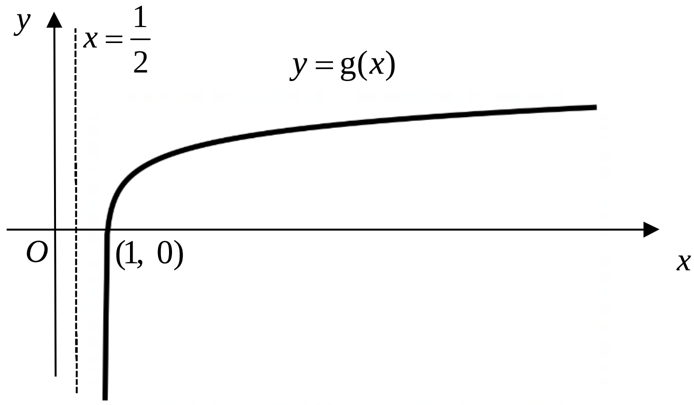
(iv) Find the range of gf. [2]

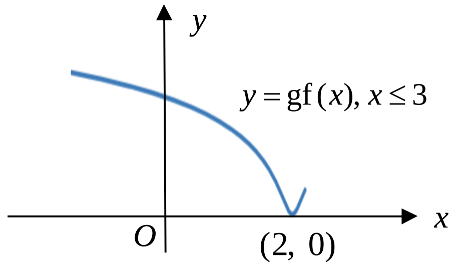
(v) A function h is said to be self-inverse if $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$ for all x in the domain of h.
A self-inverse function h is such that the composite function $\mathrm{h}^{-1}\mathrm{g}^{-1}$ exists and $\mathrm{h}^{-1}(1) = 2$. Without finding an expression for $\mathrm{g}^{-1}(x)$, find the exact value of the constant k such that $\mathrm{h}^{-1}\mathrm{g}^{-1}(k) = 1$. [3]

[(i) $[1, \infty)$ (iii) $\ln\left(2x^2 - 8x + 9\right); (-\infty, 3]$ (iv) $[0, \infty)$ (v) $\ln 3$]

Solutions

(i)	 The range of f, $\mathrm{R_f} = [1, \infty)$

(ii)	For gf to exist, $\mathrm{R_f} \subseteq \mathrm{D_g}$. $\mathrm{R_f} = [1, \infty)$, $\mathrm{D_g} = \left(\frac{a}{2}, \infty\right)$  Since $a < 2$, $\frac{a}{2} < 1$, $\mathrm{R_f} = [1, \infty) \subseteq \mathrm{D_g} = \left(\frac{a}{2}, \infty\right)$ $\therefore$ gf exists.
(iii)	$\mathrm{gf}(x) = \mathrm{g}[\mathrm{f}(x)]$ $= \ln\left[2\left(x^2 - 4x + 5\right) - 1\right]$ $= \ln\left(2x^2 - 8x + 9\right)$ $\mathrm{D_{gf}} = \mathrm{D_f} = (-\infty, 3]$
(iv)	**<u>Method 1</u>**: To find $\mathrm{R_{gf}}$: $\underbrace{(-\infty, 3]}_{\mathrm{D_f}} \xrightarrow{\mathrm{f}} \underbrace{[1, \infty)}_{\mathrm{R_f}} \xrightarrow{\mathrm{g}} \underbrace{?}_{\mathrm{R_{gf}}}$ Sketch the graph of $y = \mathrm{g}(x) = \ln(2x - 1)$:  From the graph, $\mathrm{R_{gf}} = [0, \infty)$. **<u>Method 2</u>** Sketch the graph of $y = \mathrm{gf}(x) = \ln\left(2x^2 - 8x + 9\right)$, $x \le 3$:

	 From the graph, $\mathrm{R}_{\mathrm{gf}} = [0, \infty)$.
(v)	$\mathrm{h}^{-1}\left(\mathrm{g}^{-1}(k)\right) = 1$ $\mathrm{g}^{-1}(k) = \mathrm{h}(1)$ $= 2$ since $\mathrm{h}(1) = \mathrm{h}^{-1}(1) = 2$ $k = \mathrm{g}(2)$ $k = \ln\left(2(2) - 1\right)$ $k = \ln 3$

5 **RVHS Prelim 9758/2022/01/Q6**

The function f is defined by

$$\mathrm{f}: x \mapsto \begin{cases} \mathrm{e}^{x}-1, & x \in \mathbb{R}, x \leq 0, \\ 1-\mathrm{e}^{-x}, & x \in \mathbb{R}, x > 0. \end{cases}$$

(i) Sketch the graph of $y=\mathrm{f}(x)$. [2]

(ii) Explain why f^{-1} exists and define f^{-1} in a similar form. [4]

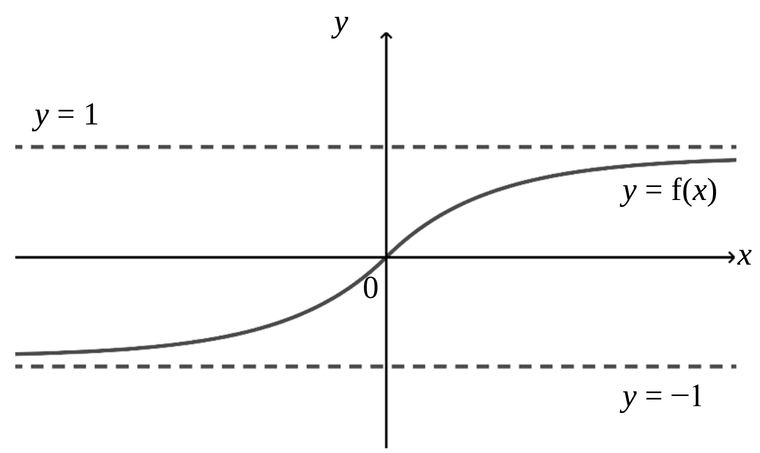
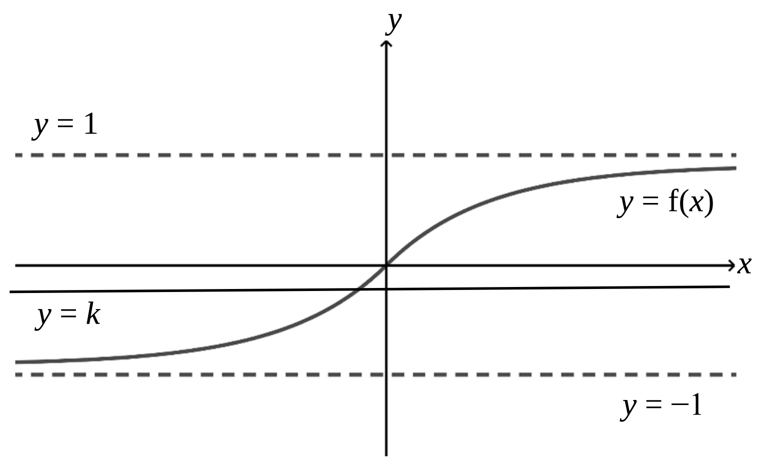
The function g is defined for $\mathbb{R}$ such that its domain and range are as follows.

Function	Domain	Range
g	$(-4,\infty)$	$(2,\infty)$

Find the exact domain and range of the function fg. [3]

[(ii) $\mathrm{f}^{-1}: x \mapsto \begin{cases} \ln(x+1), x \in \mathbb{R}, -1 < x \leq 0 \\ -\ln(1-x), x \in \mathbb{R}, 0 < x < 1 \end{cases}$; $\mathrm{D}_{\mathrm{fg}} = (-4,\infty)$, $\mathrm{R}_{\mathrm{fg}} = (1-\mathrm{e}^{-2}, 1)$]

Solutions

(i)	
(ii)	

	From the graph, the line $y = k$ where $-1<k<1$ cuts the graph of $y=\mathrm{f}(x)$ at one and only one point. Therefore f is a one to one function thus f^{-1} exists. For $x \le 0$, $y=\mathrm{e}^{x}-1$, $y+1=\mathrm{e}^{x}$, $x=\ln(y+1)$ For $x>0$, $y=1-\mathrm{e}^{-x}$, $\mathrm{e}^{-x}=1-y$, $x=-\ln(1-y)$ $\therefore \mathrm{f}^{-1}: x \mapsto \begin{cases}\ln(x+1), x\in\mathbb{R}, -1<x\le 0\\ -\ln(1-x), x\in\mathbb{R}, 0<x<1\end{cases}$
(iii)	$\mathrm{D_{fg}}=\mathrm{D_g}=(-4,\infty)$ $\mathrm{R_g}=\mathrm{D_{g^{-1}}}=(2,\infty)$ $\mathrm{D_{fg}} \xrightarrow{\mathrm{g}} \mathrm{R_g}=(2,\infty) \xrightarrow{\mathrm{f}} \mathrm{R_{fg}}=\left(1-\mathrm{e}^{-2},1\right)$

6 **HCI Prelim 9758/2022/01/Q6**

The function f is defined by

$$\mathrm{f}: x \mapsto \begin{cases} 5+\ln x & \text{for } 0<x<1, \\ -x^2+6x & \text{for } 1 \le x \le a, \end{cases}$$

where a is a constant.

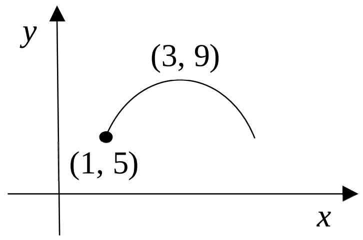
(i) Given that f has an inverse, determine the range of values of a. [2]

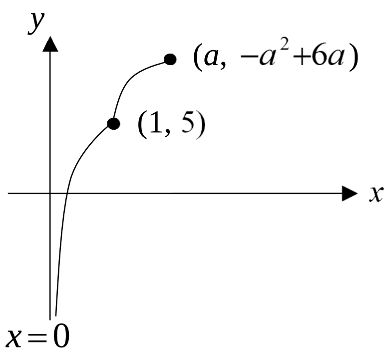
(ii) Given that a satisfies the range of values found in part **(i),** sketch the graph of f, indicating clearly the coordinates of the end point(s) in terms of a. [2]

(iii) Find f^{-1} in similar form. [3]

[(i) $1 \le a \le 3$ (iii) $\mathrm{f}^{-1}: x \mapsto \begin{cases} \mathrm{e}^{x-5}, & x<5 \\ 3-\sqrt{9-x}, & 5 \le x \le -a^2+6a \end{cases}$]

Solutions

<table>
<tr><td>(i)</td><td>$-x^2+6x=-[x^2-6x]=-(x-3)^2+9$


For f to be 1-1 on $[1, a]$, $a \le 3$
$\therefore$ the range of values of a is $1 \le a \le 3$</td></tr>
<tr><td>(ii)</td><td>
</td></tr>
<tr><td>(iii)</td><td>Let $y=\mathrm{f}(x) \Leftrightarrow x=\mathrm{f}^{-1}(y)$

For $0<x<1$

$5+\ln x=y$</td></tr>
</table>

	$\ln x = y - 5$ $x = \mathrm{e}^{y-5}$, $y < 5$ For $0 \le x \le a \le 3$ $y = -(x-3)^2 + 9$ $(x-3)^2 = 9 - y$ $x = 3 \pm \sqrt{9-y}$ Since $x \le 3$ $x = 3 - \sqrt{9-y}$, $5 \le y \le -a^2 + 6a$ $\therefore \mathrm{f}^{-1} : x \mapsto \begin{cases} \mathrm{e}^{x-5}, & x < 5 \\ 3 - \sqrt{9-x}, & 5 \le x \le -a^2 + 6a \end{cases}$

C5: Complex Numbers

1 **2018/9758/EJC/Prelim/01/Q5(a)**

Showing your working clearly, find the complex numbers z and w which satisfy the simultaneous equations

$$4\mathrm{i}z - w = 9\mathrm{i} - 13 \ ,$$
$$(4+2\mathrm{i})w^* = z + 3\mathrm{i}.$$ [4]

[(a) $w = 1 - \mathrm{i}$, z=2+3i]

Solution

1	$4\mathrm{i}z - w = 9\mathrm{i} - 13$ ------- (1) $(4+2\mathrm{i})w^* = z + 3\mathrm{i}$ $z = (4+2\mathrm{i})w^* - 3\mathrm{i}$ -----(2) Sub (2) into (1), we have $4\mathrm{i}\left[(4+2\mathrm{i})w^* - 3\mathrm{i}\right] - w = 9\mathrm{i} - 13$ Let $w = x + \mathrm{i}y$ $4\mathrm{i}[(4+2\mathrm{i})(x - \mathrm{i}y) - 3\mathrm{i}] - (x + \mathrm{i}y) = 9\mathrm{i} - 13$ $4\mathrm{i}[(4x+2x\mathrm{i} - 4\mathrm{i}y + 2y - 3\mathrm{i}] - x - \mathrm{i}y) = 9\mathrm{i} - 13$ $16\mathrm{i}x - 8x + 16y + 8y\mathrm{i} + 12 - x - \mathrm{i}y = 9\mathrm{i} - 13$ $\left(-9x + 16y\right) + \left(16x + 7y\right)\mathrm{i} = 9\mathrm{i} - 25$ Compare real and imaginary parts $16y - 9x = -25$ ------- (3) $16y + 8y - y = 9$ $16x + 7y = 9$ ---------- (4) Using GC, $x = 1, \ \ y = -1$ $\therefore w = 1 - \mathrm{i}$ $z = (4+2\mathrm{i})(1+\mathrm{i}) - 3\mathrm{i} \ = 2+3\mathrm{i}$

2 **2018/9758/NJC/Prelim/02/Q4(b)**

The complex variables *u* and *v* satisfy the equations

$$\mathrm{i}u - v = 3 \quad \text{and} \quad u^* + (1-\mathrm{i})v = 7 + 4\mathrm{i}.$$

Find the values of *u* and *v*, giving your answers in the form $x + \mathrm{i}y$. [4]

[$u = 1 - 8\mathrm{i}$, $v = 5 + \mathrm{i}$ **]**

Solutions

<table>
<tr><td>2</td><td>
$\mathrm{i}u - v = 3 \Rightarrow v = \mathrm{i}u - 3$
Then substituting $v = \mathrm{i}u - 3$ into the other equation,
$u^* + (1-\mathrm{i})(\mathrm{i}u - 3) = 7 + 4\mathrm{i}$
$u^* + \mathrm{i}u - 3 - \mathrm{i}^2 u + 3\mathrm{i} = 7 + 4\mathrm{i}$
$u^* + \mathrm{i}u + u = 10 + \mathrm{i}$
$2a + \mathrm{i}(a + \mathrm{i}b) = 10 + \mathrm{i}$, where $u = a + \mathrm{i}b$
$2a - b + \mathrm{i}a = 10 + \mathrm{i}$
Comparing the real and imaginary parts, we get
$a = 1$ and $2a - b = 10 \Rightarrow 2 - b = 10 \Rightarrow b = -8$.

Therefore $u = 1 - 8\mathrm{i}$ and $v = \mathrm{i}(1 - 8\mathrm{i}) - 3 = 5 + \mathrm{i}$
</td></tr>
<tr><td></td><td>
Alternatively,
$\mathrm{i}u - v = 3$ (1)
$u^* + (1-\mathrm{i})v = 7 + 4\mathrm{i}$
Let $u = a + b\mathrm{i}$ and $v = c + d\mathrm{i}$ where $a, b, c, d \in \mathbb{R}$.
Equation (1) becomes $\mathrm{i}(a + b\mathrm{i}) - (c + d\mathrm{i}) = 3$
$(-b - c) + \mathrm{i}(a - d) = 3 + 0\mathrm{i}$
Comparing the real and imaginary parts,
$-b - c = 3$ (3)
$a - d = 0$ (4)
Equation (2) becomes $(a - b\mathrm{i}) + (1-\mathrm{i})(c + d\mathrm{i}) = 7 + 4\mathrm{i}$
$(a + c + d) + \mathrm{i}(-b + d - c) = 7 + 4\mathrm{i}$
Comparing the real and imaginary parts,
$a + c + d = 7$ (5)
$-b - c + d = 4$ (6)
Using GC to solve (3), (4), (5) and (6) simultaneously,
$a = 1,\ b = -8,\ c = 5$ and $d = 1$
Therefore, $u = 1 - 8\mathrm{i}$ and $v = 5 + \mathrm{i}$.
</td></tr>
</table>

3 2018/9758/RVHS/Prelim/01/Q8 (modified)

Given that $\mathrm{f}(z)=z^4+2z^3+(\pi^2+\mathrm{e}^2)z^2+pz+q$, where $p\geq 0$ and $q\geq 0$, and that $\mathrm{i}\pi$ satisfies the equation $\mathrm{f}(z)=0$.

(i) Find the exact values of p and q. [2]

(ii) Hence find the remaining roots of $\mathrm{f}(z)=0$ in the form $\alpha+\beta\mathrm{i}$, where α and β are exact real numbers. [4]

[(i) $q=\mathrm{e}^2\pi^2$ and $p=2\pi^2$ (ii) $-\mathrm{i}\pi$, $-1+\mathrm{i}\sqrt{\mathrm{e}^2-1}$, $-1+\mathrm{i}\sqrt{\mathrm{e}^2-1}$ **]**

Solutions

(i)	Since $i\pi$ is a root, $(i\pi)^4+2(i\pi)^3+(\pi^2+e^2)(i\pi)^2+p(i\pi)+q=0$ $\pi^4+2\pi^3(-i)+(\pi^2+e^2)(-\pi^2)+i(p\pi)+q=0$ $(q-e^2\pi^2)+i(p\pi-2\pi^3)=0$ Equating real and imaginary parts, $q=e^2\pi^2$ and $p=2\pi^2$
(ii)	Now, $\mathrm{f}(z)=z^4+2z^3+(\pi^2+\mathrm{e}^2)z^2+2\pi^2z+\mathrm{e}^2\pi^2$ As the polynomial f(z) has real coefficients, $-i\pi$ is also a root. Thus $(z-i\pi)$ and $(z+i\pi)$ are factors of f(z) i.e., $z^2+\pi^2$ is a factor $\mathrm{f}(z)=\left(z^2+\pi^2\right)\left(z^2+mz+e^2\right)$ where m is to be determined. Equating coeff of z^3 or z in f(z), $m=2$ $\mathrm{f}(z)=\left(z^2+\pi^2\right)\left(z^2+2z+e^2\right)$ for $\mathrm{f}(z)=0$ $\left(z^2+\pi^2\right)\left(z^2+2z+\mathrm{e}^2\right)=0$ $z=\pm i\pi$ or $z=\dfrac{-2\pm\sqrt{4-4\mathrm{e}^2}}{2}$ $=-1\pm\sqrt{1-\mathrm{e}^2}$ $=-1\pm i\sqrt{\mathrm{e}^2-1}$ $(\mathit{NOTE}: 1-\mathrm{e}^2$ is negative) Hence the remaining roots are $-i\pi$, $-1+i\sqrt{\mathrm{e}^2-1}$ or $-1+i\sqrt{\mathrm{e}^2-1}$

4 2018/9758/SRJC/Prelim/02/Q4

Given that the equation $\mathrm{f}(z)=2z^3+az^2+(2a-4)z+(2a-8)=0$ has no real solution, explain clearly why a is not a real number. [2]

(i) It is known that f has a factor $(2z+\mathrm{i})$, find a. Hence find all the roots of $\mathrm{f}(z)=0$, showing your workings clearly. [4]

(ii) Deduce the roots of the equation $2-\dfrac{a}{w}+\dfrac{(2a-4)}{w^2}-\dfrac{2a-8}{w^3}=0$, where a takes the value obtained in **(i)**. [2]

[(i) $4+\mathrm{i}$; $-1+\mathrm{i}$, $-1-\mathrm{i}$ and $-\dfrac{\mathrm{i}}{2}$ (ii) $1-\mathrm{i}$, $1+\mathrm{i}$ and $\dfrac{\mathrm{i}}{2}$ **]**

Solution	
If all coefficients are real, then non-real roots will occur in conjugate pairs. As f is a polynomial of degree 3, f(z)=0 have either 3 real roots or 1 real root with a pair of conjugate roots. But it is given that f(z) = 0 has no real solution and so it must therefore means that at least one of the coefficients is non-real. Hence a is not a real number.	
(i)	$\mathrm{f}\left(-\frac{\mathrm{i}}{2}\right)=2\left(-\frac{\mathrm{i}}{2}\right)^3+a\left(-\frac{\mathrm{i}}{2}\right)^2+(2a-4)\left(-\frac{\mathrm{i}}{2}\right)+2a-8=0$ $\frac{\mathrm{i}}{4}-\frac{a}{4}-a\mathrm{i}+2\mathrm{i}+2a-8=0$ $8-\frac{9}{4}\mathrm{i}=a\left(\frac{7}{4}-\mathrm{i}\right)$ $a=4+\mathrm{i}$ So $\mathrm{f}(z)=2z^3+(4+\mathrm{i})z^2+(4+2\mathrm{i})z+2\mathrm{i}=(2z+\mathrm{i})(z^2+2z+2)$ For $\mathrm{f}(z)=0 \Rightarrow (2z+\mathrm{i})(z^2+2z+2)=0$ Consider $z^2+2z+2=0$, $z=\frac{-2\pm\sqrt{4-8}}{2}$ $z=-1\pm\mathrm{i}$ Hence the roots are $-1+\mathrm{i}$, $-1-\mathrm{i}$ and $-\frac{\mathrm{i}}{2}$
(ii)	Since $2z^3+az^2+(2a-4)z+2a-8=0$, $\Rightarrow 2+\frac{a}{z}+\frac{2a-4}{z^2}+\frac{2a-8}{z^3}=0$ Replace z by $-w$ $\Rightarrow 2-\frac{a}{w}+\frac{2a-4}{w^2}-\frac{2a-8}{w^3}=0$ So the roots of this equations are $1-\mathrm{i}$, $1+\mathrm{i}$ and $\frac{\mathrm{i}}{2}$

5 EJC JC2 Prelim 9758/2019/02/Q1

(a) Find the complex numbers z and w that satisfy the equations

$$\frac{z}{w} = 2 + 2\text{i},$$
$$(1-2\text{i})z = 39 - (11\text{i})w. \qquad [3]$$

(b) It is given that $(1+\text{i}c)^3$ is real, where c is also real. By first expressing $(1+\text{i}c)^3$ in Cartesian form, find all possible values of c. [3]

[(a) $z = 10 - 2\text{i}, w = 2 - 3\text{i}$ (b) 0, $\pm\sqrt{3}$]

Solutions	
(a)	Sub $z = (2+2\text{i})w$ into the other equation $\Rightarrow (1-2\text{i})(2+2\text{i})w = 39 - 11w\text{i}$ $\Rightarrow w = \dfrac{39}{(1-2\text{i})(2+2\text{i})+11\text{i}} = 2 - 3\text{i}$ (using GC) Thus, $z = (2+2\text{i})(2-3\text{i}) = 10 - 2\text{i}$
(b)	$(1+\text{i}c)^3 = 1 + 3\text{i}c + 3(\text{i}c)^2 + (\text{i}c)^3$ $= 1 + 3\text{i}c - 3c^2 - \text{i}c^3$ $= 1 - 3c^2 + \text{i}(3c - c^3)$ Since $(1+\text{i}c)^3$ is real, $3c - c^3 = 0$ $c(3 - c^2) = 0$ $c = 0, \pm\sqrt{3}$

6 ASRJC JC2 Prelim 9758/2019/01/Q7

Do not use a calculator in answering this question.

The roots of the equation $z^2 + (2 - 2i)z = -3 - 2i$ are z_1 and z_2.

(i) Find z_1 and z_2 in cartesian form $x + iy$, showing clearly your working. [5]

(ii) The complex numbers z_1 and z_2 are also roots of the equation $z^4 + 4z^3 + 14z^2 + 4z + 13 = 0$. Find the other roots of the equation, explaining clearly how the answers are obtained. [2]

(iii) Using your answer in part **(i)**, solve $z^2 + (2 + 2i)z = 3 + 2i$. [2]

[(i) –i , –2 + 3i (ii) i, –2 – 3i (iii) 1, –3 – 2i]

Solutions	
(i)	Let $z = x + iy$, where $x, y \in \mathbb{R}$ $\therefore (x + iy)^2 + (2 - 2i)(x + iy) = -3 - 2i$ $x^2 + 2xyi - y^2 + 2x + 2yi - 2xi + 2y = -3 - 2i$ $(x^2 + 2x - y^2 + 2y) + i(2xy + 2y - 2x) = -3 - 2i$ By comparing real and imaginary parts, Real: $x^2 + 2x - y^2 + 2y = -3$ …(1) Imaginary: $2xy + 2y - 2x = -2$ $y(x + 1) = x - 1$ $y = \dfrac{x-1}{x+1}$ …(2)
	Subt (2) into (1): $x^2 + 2x - \left(\dfrac{x-1}{x+1}\right)^2 + 2\left(\dfrac{x-1}{x+1}\right) = -3$ $x^2(x+1)^2 + 2x(x+1)^2 - (x-1)^2 + 2(x-1)(x+1) = -3(x+1)^2$ $x^2(x^2+2x+1) + 2x(x^2+2x+1) - (x^2-2x+1) + 2(x^2-1) = -3(x^2+2x+1)$ $x^4 + 4x^3 + 9x^2 + 10x = 0$ Let $f(x) = x(x^3 + 4x^2 + 9x + 10)$ $f(0) = 0$ and $f(-2) = -2(-8 + 16 - 18 + 10) = 0$ $\Rightarrow x = 0, -2$ are roots of $x^4 + 4x^3 + 9x^2 + 10x = 0$ When $x = 0, y = -1$ and when $x = -2, y = 3$ $\therefore z_1 = -i$ and $z_2 = -2 + 3i$
(ii)	Since all the coefficients of $z^4 + 4z^3 + 14z^2 + 4z + 13 = 0$ are real, the other roots of the equation must be complex conjugates of –i and –2 + 3i. Hence the other two roots are i and –2 – 3i.
(iii)	$z^2 + (2 - 2i)z = -3 - 2i$ $-z^2 + (-2 + 2i)z = 3 + 2i$ $i^2z^2 + (2i^2 + 2i)z = 3 + 2i$ $(iz)^2 + (2i + 2)(iz) = 3 + 2i$ which is equivalent in form to the new equation given in **(iii)** Hence from part **(i)**, roots of $z^2 + (2 + 2i)z = 3 + 2i$ are i(–i) and i(–2 + 3i), ie 1 and –3 – 2i

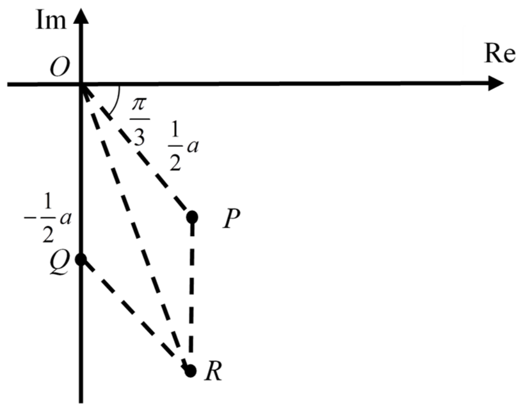
7 HCI JC2 Prelim 9758/2019/02/Q3

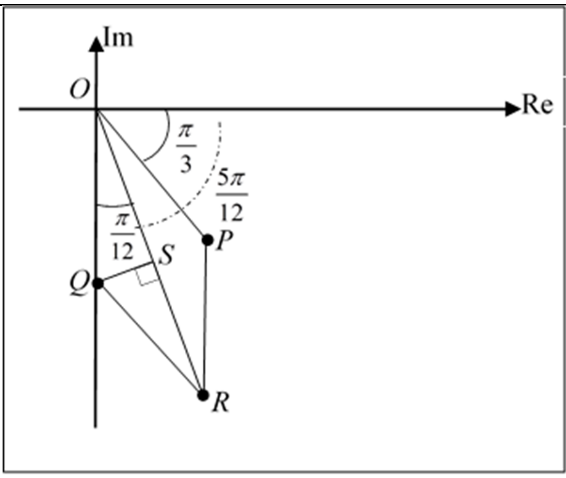
The complex numbers p and q are given by $\dfrac{a}{1+\sqrt{3}\mathrm{i}}$ and $-\dfrac{a}{2}\mathrm{i}$ respectively, where a is a positive real constant.

(i) Find the modulus and argument of p . [2]

(ii) Illustrate on an Argand diagram, the points P, Q and R representing the complex numbers p, q and $p+q$ respectively. State the shape of $OPRQ$.
Hence, find the argument of $p+q$ in terms of π and the modulus of $p+q$ in exact trigonometrical form. [6]

[(i) $\dfrac{a}{2}$, $-\dfrac{\pi}{3}$ (ii) $-\dfrac{5\pi}{12}, a\cos\dfrac{\pi}{12}$]

Solutions	
(i)	$p=\dfrac{a}{1+\sqrt{3}\mathrm{i}}\times\dfrac{1-\sqrt{3}\mathrm{i}}{1-\sqrt{3}\mathrm{i}}=\dfrac{a}{4}\left(1-\sqrt{3}\mathrm{i}\right)$ $\lvert p\rvert=\left\lvert\dfrac{a}{4}\left(1-\sqrt{3}\mathrm{i}\right)\right\rvert=\dfrac{a}{4}\sqrt{1+3}=\dfrac{a}{2}p$ $\arg(p)=\arg\left(1-\sqrt{3}\mathrm{i}\right)=-\dfrac{\pi}{3}$, since $a>0$ is a constant
(ii)	 The shape is a rhombus.
	$\measuredangle QOR=\dfrac{\frac{\pi}{2}-\frac{\pi}{3}}{2}=\dfrac{\pi}{12}$ $\arg(p+q)=-(\dfrac{\pi}{3}+\dfrac{\pi}{12})=-\dfrac{5\pi}{12}$

	 $\lvert p+q\rvert = 2\times OS = 2\times\dfrac{a}{2}\cos\dfrac{\pi}{12} = a\cos\dfrac{\pi}{12}$

C6: Differentiation Techniques and Applications

1 **DHS Prelim 9758/2022/01/Q5**

A curve C has equation $y^2 = 2\sin x + 2xy$, where $0 \le x \le 2\pi$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos x + y}{y - x}$. [2]

(b) Hence find the coordinates of the stationary points and use the second derivative test to determine its nature. [6]

Solutions

1(a)	$y^2 = 2\sin x + 2xy$ ... (1) Differentiate with respect to x, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x + 2\left(y + x\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$ $2y\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x + 2y$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos x + y}{y - x}$ (shown)
(b)	For stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos x + y}{y - x} = 0$ $\cos x + y = 0$ $y = -\cos x$ Substitute $y = -\cos x$ into (1) $(-\cos x)^2 = 2\sin x + 2x(-\cos x)$ $\cos^2 x - 2\sin x + 2x\cos x = 0$ From GC : $x = 0.87394$ or 4.4877 $y = -\cos(0.87394)$ or $-\cos(4.4877)$ $= -0.64181$ or 0.22280 Coordinates : $P(0.874, -0.642)$ & $Q(4.49, 0.223)$ $(y - x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x + y$ $(y - x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x} - 1\right) = -\sin x + \dfrac{\mathrm{d}y}{\mathrm{d}x}$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, $(y - x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\sin x$ At $x = 0.87394$ and $y = -0.64181$,

	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.50593 > 0$ $\therefore (0.874, -0.642)$ is a minimum point. At $x = 4.4877$ and $y = 0.22280$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -0.22858 < 0$ $\therefore (4.49, 0.223)$ is a maximum point.

2 **JPJC Prelim 9758/2022/01/Q7**

The equation of a curve C is $2x^3 + y^3 - 3xy = k$, where k is a constant.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [2]

It is given that C has a tangent which is parallel to the y-axis.

(ii) Show that the y-coordinate of the point of contact of the tangent with C must satisfy $ay^6 + by^3 - k = 0$, where the constants a and b are to be determined. [3]

(iii) Hence, find the values of k when the line $x = 1$ is a tangent to the curve C. [3]

Solution	
(i)	$2x^3 + y^3 - 3xy = k$ Differentiate w.r.t. x, $6x^2 + 3y^2\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3y - 6x^2}{3y^2 - 3x} = \dfrac{y - 2x^2}{y^2 - x}$
(ii)	When tangent is parallel to the y-axis, $y^2 - x = 0$
	Sub $x = y^2$ into $2x^3 + y^3 - 3xy = k$ $2(y^2)^3 + y^3 - 3(y^2)y = k$ $2y^6 - 2y^3 - k = 0$ i.e. $a = 2$, $b = -2$
(iii)	line $x = 1$ is a tangent to the curve C $y^2 = 1$ $y = \pm 1$
	When $y = 1$, $2 - 2 - k = 0$ $k = 0$
	When $y = -1$, $2 + 2 - k = 0$ $k = 4$

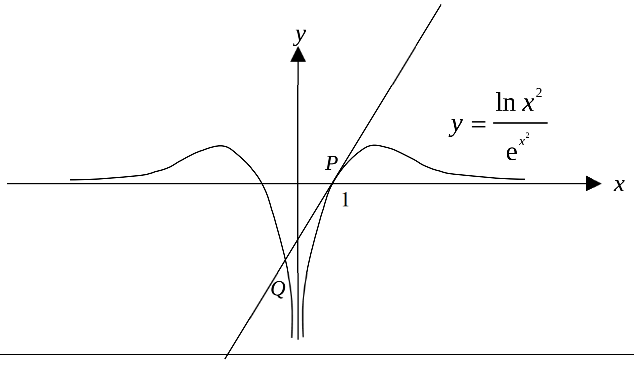
3 **HCI Prelim 9758/2022/01/Q9**

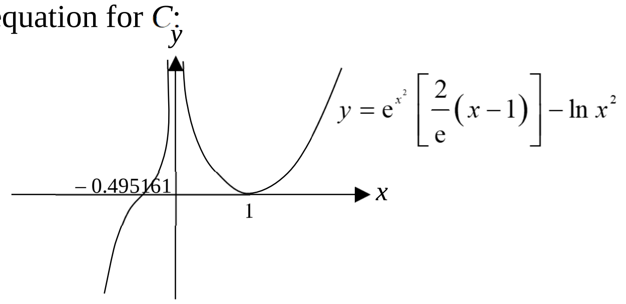
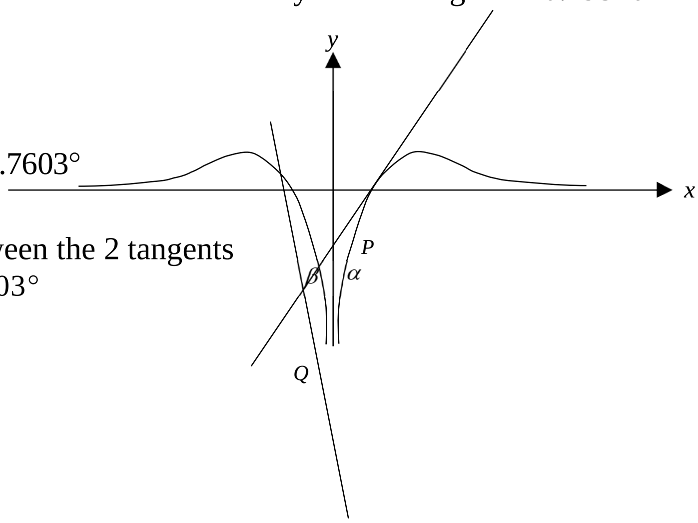
A curve C is given by the equation $e^{x^2}y = \ln x^2$, where $x \neq 0$.

(i) Show that $2x\ln x^2 + e^{x^2}\dfrac{dy}{dx} = \dfrac{2}{x}$. [2]

(ii) Find the equation of the tangent to C at the point P where $x = 1$. Given that the tangent at P meets C again at the point Q, find the coordinates of Q, leaving your answer correct to 5 significant figures. [4]

(iii) Determine the acute angle, in degrees, between the tangents to C at the points P and Q. [3]

Solutions	
3(i)	$e^{x^2}y = \ln x^2$ Differentiating wrt x both sides, $2xe^{x^2}y + e^{x^2}\dfrac{dy}{dx} = \dfrac{1}{x^2}(2x)$ $2x\ln x^2 + e^{x^2}\dfrac{dy}{dx} = \dfrac{2}{x}$ (Shown) $\left(\because e^{x^2}y = \ln x^2\right)$
(ii)	When $x = 1$, $y = \dfrac{\ln 1^2}{e^{1^2}} = 0$ $\dfrac{dy}{dx} = \dfrac{\frac{2}{1} - 2(1)\ln 1^2}{e^{1^2}} = \dfrac{2}{e}$ (from **(i)**) Hence equation of tangent at P: $y = \dfrac{2}{e}(x-1)$ **<u>Method 1</u>** $\dfrac{2}{e}(x-1) = \dfrac{\ln x^2}{e^{x^2}}$ Using GC, another point of intersection between the tangent $y = \dfrac{2}{e}(x-1)$ and C: $y = \dfrac{\ln x^2}{e^{x^2}}$ is $Q\ (-0.49516, -1.1001)$ (to 5 sf). 

	<u>Method 2</u> Substitute $y=\frac{2}{\mathrm{e}}(x-1)$ into the equation for C: $\mathrm{e}^{x^2}\left[\frac{2}{\mathrm{e}}(x-1)\right]=\ln x^2$ ..... (1) $\mathrm{e}^{x^2}\left[\frac{2}{\mathrm{e}}(x-1)\right]-\ln x^2=0$  At Q, $x=-0.495161$ $y=\frac{2}{\mathrm{e}}(-0.495161-1)=-1.100078256$ Hence, $Q\ (-0.49516,-1.1001)\ \ (\text{to 5 sf})$.
(iii)	At Q, $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\ln(x^2)}{\mathrm{e}^{x^2}}\right]\Bigg\vert_{x=-0.495161}=-4.2502646$ (from GC) (Alternatively, the above value can be obtained by substituting $x=-0.495161$ into equation in (i)) $\alpha=\tan^{-1}\left(\frac{2}{\mathrm{e}}\right)=36.3441°$ $\beta=\tan^{-1}(4.2502646)=76.7603°$  Required acute angle between the 2 tangents $=180°-36.3441°-76.7603°$ $=66.9°$ (1 d.p.)

4 EJC Promo 9758/2022/Q3

A curve C_1 has parametric equations

$$x = at(t+2), \qquad y = a(t^2+1)$$

where a is a positive constant.

(a) Find, in terms of a, the equation of the normal to the curve at the point where $t = \dfrac{1}{2}$. [3]

Another curve C_2 has equation $e^{3x} + e^{y} = 4$.

(b) Given that the normal in part **(a)** is also tangential to C_2 at a point, find the exact value of a. [4]

[(a) $y = -3x + 5a$ (b) $\dfrac{2\ln 2}{5}$]

Solutions

(a)	$\dfrac{dx}{dt} = 2at + 2a = 2a(t+1)$ $\dfrac{dy}{dt} = 2at$ $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{2at}{2a(t+1)} = \dfrac{t}{t+1}$ Gradient of Normal = $\dfrac{-1}{\left(\dfrac{t}{t+1}\right)} = -\dfrac{t+1}{t}$ When $t = \dfrac{1}{2}$, $x = \dfrac{5}{4}a$, $y = \dfrac{5}{4}a$ and gradient of normal = -3 Equation of Normal: $\left(y - \dfrac{5}{4}a\right) = -3\left(x - \dfrac{5}{4}a\right)$ $y = -3x + 5a$
(b)	Differentiating C_2 with respect to x $3e^{3x} + e^{y}\left(\dfrac{dy}{dx}\right) = 0$ $\dfrac{dy}{dx} = -\dfrac{3e^{3x}}{e^{y}} = -3e^{3x-y}$ Let the point of the tangent at C_2 be P. Since P lies on the normal found in part (a), the gradient of the tangent at P must be equal to -3.

	$\frac{dy}{dx} = -3e^{3x-y} = -3$ $e^{3x-y} = 1$ $x = \frac{y}{3} ---(1)$ Substituting this relationship into the equation of C_2, $e^y + e^y = 4$ $\Rightarrow 2e^y = 4$ $\Rightarrow y = \ln 2$ From (1), we have $x = \frac{\ln 2}{3}$. Hence, $y = -3x + 5a$ is a tangent to C_2 at the point $\left(\frac{\ln 2}{3}, \ln 2\right)$. Substituting this point into the equation of the normal, $\ln 2 = -3\left(\frac{\ln 2}{3}\right) + 5a$ $5a = 2\ln 2$ $a = \frac{2\ln 2}{5}$

5 **JPJC Promo 9758/2022/Q11 modified**

A curve C has parametric equations

$$x = 3t^2, \qquad y = 6t^3, \quad \text{where } t \text{ is a real parameter.}$$

(i) Find the exact coordinates of the point P on C where the tangent is parallel to the line $y = 4 - 2x$. [3]

(ii) Show that the equation of the tangent to C at the point $Q\left(\frac{4}{3}, \frac{16}{9}\right)$ is $9y = 18x - 8$. [2]

(iii) The tangent at Q cuts the x-axis at the point R. Find the area of triangle PQR. [2]

(iv) The tangent at Q cuts C again at the point S. Find the coordinates of S. [3]

[(i) $\left(\frac{4}{3}, -\frac{16}{9}\right)$ (iii) $\frac{128}{81}$ (iv) $\left(\frac{1}{3}, -\frac{2}{9}\right)$]

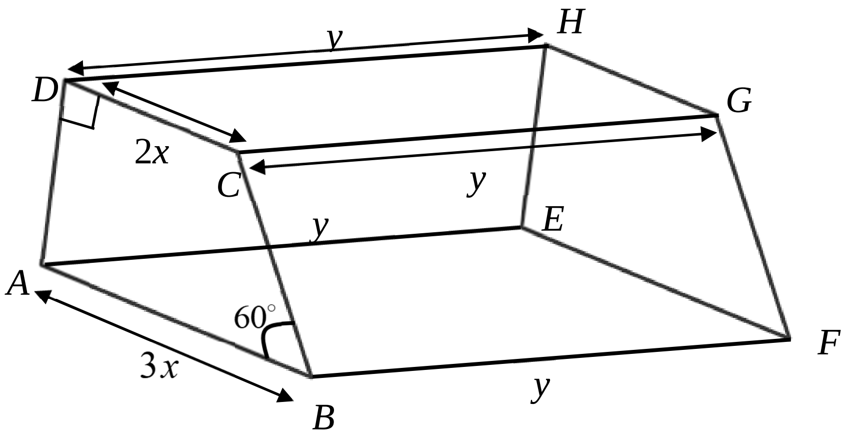
Solutions

(i)	$x = 3t^2 \qquad y = 6t^3$ $\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 18t^2$ $\frac{dy}{dx} = 3t$ Since tangent is parallel to the line $y = 4 - 2x$, $\frac{dy}{dx} = 3t = -2$ $t = -\frac{2}{3}$ Hence, $x = 3\left(-\frac{2}{3}\right)^2 = \frac{4}{3}, \qquad y = 6\left(-\frac{2}{3}\right)^3 = -\frac{16}{9}$ Coordinates of P are $\left(\frac{4}{3}, -\frac{16}{9}\right)$
(iii)	At $Q\left(\frac{4}{3}, \frac{16}{9}\right)$, $y = 6t^3 = \frac{16}{9} \Rightarrow t = \frac{2}{3}$ $\frac{dy}{dx} = 3t = 3\left(\frac{2}{3}\right) = 2$ Equation of tangent at Q :

	$y-\frac{16}{9}=2\left(x-\frac{4}{3}\right)$ $y=2x-\frac{8}{9}$ $9y=18x-8$ (shown)
(iii)	At R: $y=0$ $0=18x-8 \Rightarrow x=\frac{4}{9}$ $R\left(\frac{4}{9},0\right)$ Area of triangle PQR $=\frac{1}{2}\left(\frac{4}{3}-\frac{4}{9}\right)\left[\frac{16}{9}-(-\frac{16}{9})\right]$ $=\frac{128}{81}$ units^2
(iv)	$9y=18x-8$ $9\left(6t^3\right)=18\left(3t^2\right)-8$ $27t^3-27t^2+4=0$ $t=\frac{2}{3}$ (N.A as this gives point Q) or $t=-\frac{1}{3}$ $x=3\left(-\frac{1}{3}\right)^2=\frac{1}{3}$, $y=6\left(-\frac{1}{3}\right)^3=-\frac{2}{9}$ Coordinates of S are $\left(\frac{1}{3},-\frac{2}{9}\right)$

6 **ACJC Promo 9758/2022/Q12**

Mr See wants to build a fish tank with a fixed capacity $\left(5\sqrt{3}\right)k$ m^3. The following diagram shows the dimensions of the fish tank he desires, with variables x and y.



The cross-section of the fish tank $ABCD$ is a right trapezoid with AB being $3x$ m long, DC being $2x$ m long, and $\angle ABC = 60^\circ$. The cross-section $ABCD$ is perpendicular to both the rectangular base $ABFE$ and the rectangular opening $DCGH$, with $AE = BF = DH = CG = y$ m.

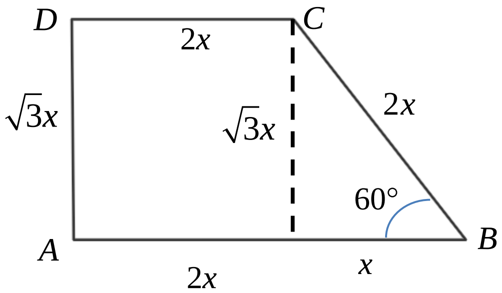
(i) Show that the total surface area A m^2 of the fish tank in contact with water when it is fully filled is $A = 5\sqrt{3}x^2 + \dfrac{2\left(5+\sqrt{3}\right)k}{x}$. [4]

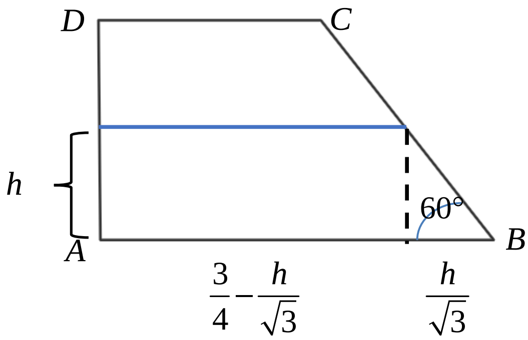
(ii) Mr See would like to minimise the amount of material required to construct the walls and base of the fish tank. Using differentiation, find the value of x in terms of k when A is minimum. [4]

(iii) Mr See decided to build his tank with $k = \dfrac{3}{160}$ and $y = 0.6$. After which, he used a hose to fill it with water at a constant rate of 0.015 m^3 per minute. Find the rate at which the depth of the water is increasing when the fish tank is 50% filled with water. [5]

[(ii) $\left(\dfrac{\left(5+\sqrt{3}\right)k}{5\sqrt{3}}\right)^{\frac{1}{3}}$ (iii) 0.0392 m per minute]

Solutions

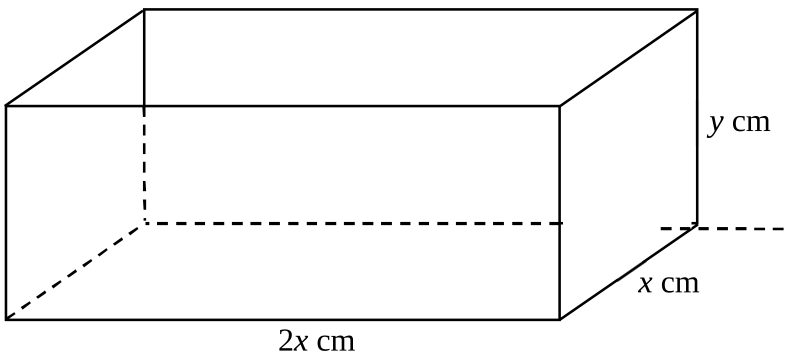
(i)	 $AD = x\tan 60^\circ = \sqrt{3}x$ $BC = \dfrac{x}{\cos 60^\circ} = 2x$ $\text{Volume} = \dfrac{1}{2}(2x+3x)\times\sqrt{3}x\times y$ $5\sqrt{3}k = \dfrac{5\sqrt{3}x^2}{2}y$ $y = \dfrac{2k}{x^2}$ $A = \dfrac{1}{2}(2x+3x)\times\sqrt{3}x\times 2 + 3xy + 2xy + \sqrt{3}xy$ $= 5\sqrt{3}x^2 + \left(5+\sqrt{3}\right)xy$ $= 5\sqrt{3}x^2 + \left(5+\sqrt{3}\right)x\left(\dfrac{2k}{x^2}\right)$ $= 5\sqrt{3}x^2 + \dfrac{2\left(5+\sqrt{3}\right)k}{x}$ (Shown)
(ii)	$\dfrac{\mathrm{d}A}{\mathrm{d}x} = 10\sqrt{3}x - \dfrac{2\left(5+\sqrt{3}\right)k}{x^2}$ For minimum A, $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$

	$10\sqrt{3}x-\dfrac{2\left(5+\sqrt{3}\right)k}{x^2}=0$ $5\sqrt{3}x=\dfrac{\left(5+\sqrt{3}\right)k}{x^2}$ $x^3=\dfrac{\left(5+\sqrt{3}\right)k}{5\sqrt{3}}$ $x=\left(\dfrac{\left(5+\sqrt{3}\right)k}{5\sqrt{3}}\right)^{\frac{1}{3}}$ $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}=10\sqrt{3}+\dfrac{4\left(5+\sqrt{3}\right)k}{x^3}$ Since x>0 which implies $x^3>0$ and k>0 (since full capacity of tank is $\left(5\sqrt{3}\right)k$), $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}>0$ Alternatively, When $x=\left(\dfrac{\left(5+\sqrt{3}\right)k}{5\sqrt{3}}\right)^{\frac{1}{3}}$, $\dfrac{\mathrm{d}^2A}{\mathrm{d}x^2}=10\sqrt{3}+\dfrac{4\left(5+\sqrt{3}\right)k}{\left(\dfrac{\left(5+\sqrt{3}\right)k}{5\sqrt{3}}\right)}=30\sqrt{3}>0$ Hence, A is minimum when $x=\left(\dfrac{\left(5+\sqrt{3}\right)k}{5\sqrt{3}}\right)^{\frac{1}{3}}$.
(iii)	When $k=\dfrac{3}{160}$ and $y=0.6$, $0.6=\dfrac{2}{x^2}\times\dfrac{3}{160}$ => $x=0.25$ Hence, $AB=0.25\times3=0.75=\dfrac{3}{4}$ and $AD=\sqrt{3}\times0.25=\dfrac{\sqrt{3}}{4}$  Let V' denote the volume of water in the fish tank.

	$V' = \frac{1}{2}\left(\frac{3}{4} - \frac{h}{\sqrt{3}} + \frac{3}{4}\right) \times h \times 0.6$ $= 0.3h\left(\frac{3}{2} - \frac{h}{\sqrt{3}}\right)$ $= 0.45h - \frac{0.3}{\sqrt{3}}h^2$ When the fish tank is half filled, $V' = \frac{1}{2}\left(5\sqrt{3} \times \frac{3}{160}\right) = \frac{3\sqrt{3}}{64}$ $0.45h - \frac{0.3}{\sqrt{3}}h^2 = \frac{3\sqrt{3}}{64}$ $\frac{0.3}{\sqrt{3}}h^2 - 0.45h + \frac{3\sqrt{3}}{64} = 0$ $h = 0.19507$ *or* $h = 2.40301$ (rejected since $h < AD \approx 0.433$) $\frac{dV'}{dh} = 0.45 - \frac{0.6}{\sqrt{3}}h$ $\frac{dV'}{dt} = \frac{dV'}{dh} \times \frac{dh}{dt}$ $0.015 = \left(0.45 - \frac{0.6}{\sqrt{3}} \times 0.19507\right) \times \frac{dh}{dt}$ $\frac{dh}{dt} = 0.0392$ (3sf) Hence the rate of change of height is 0.0392 m per minute.

7 **TJC Promo 9758/2022/Q11**

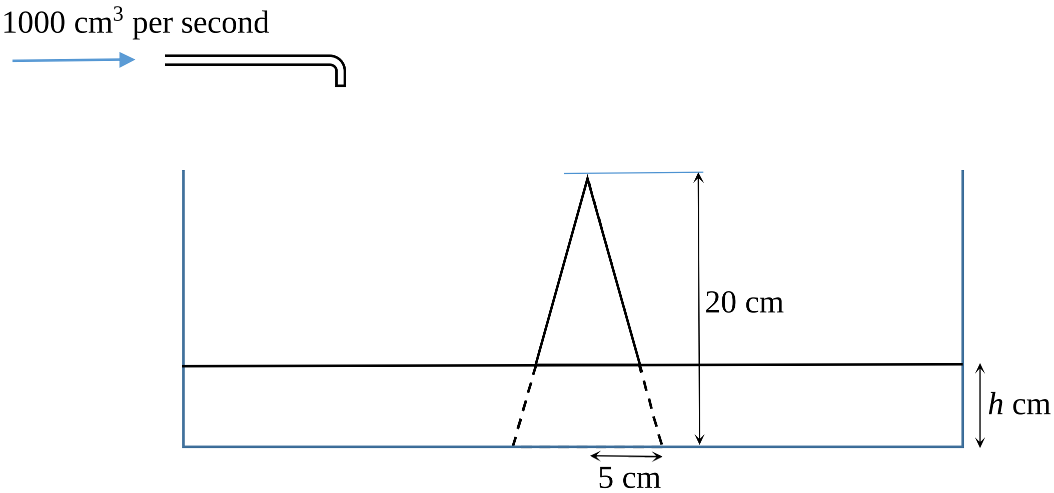
A fish tank manufacturer makes an open fish tank in a shape of a cuboid with internal breadth, length and height of x cm, $2x$ cm and y cm respectively (see diagram).



The manufacturer determines that the internal volume of the tank must be 36 000 cm^3 and the total internal surface area of the tank, A cm^2, should be as small as possible.

(i) Find a formula for A in terms of x. Hence show that for A to be minimum, the breadth, length and height of the tank are 30 cm, 60 cm and 20 cm respectively. [6]

(ii) A fish tank of dimension found in part **(i)** is built and is placed on a horizontal shelf. A solid cone with height 20 cm has its circular base with radius 5cm attached to the internal rectangular base of the fish tank (see diagram).



Water is poured at a constant rate of 1000 cm^3 per second into the tank. At time t seconds after the start, the height of the water level in the tank is h cm and the volume of water is V cm^3.

[The volume of a cone of base radius r and height h is given by $V=\frac{1}{3}\pi r^2 h$.]

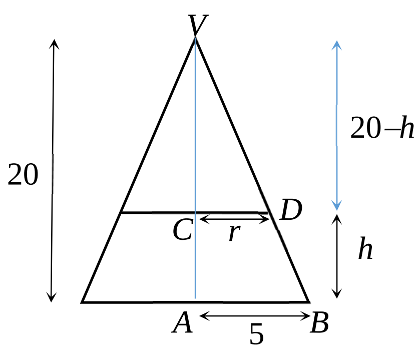
(a) Show that for $h < 20$,

$$V=1800h-\frac{\pi}{48}\left[8000-(20-h)^3\right] \qquad [3]$$

(b) Find the rate of increase of the water level when it is at a height of 10 cm. [3]

[(i) $\frac{108000}{x}+2x$ (iib) 0.562 cm/s]

Solutions

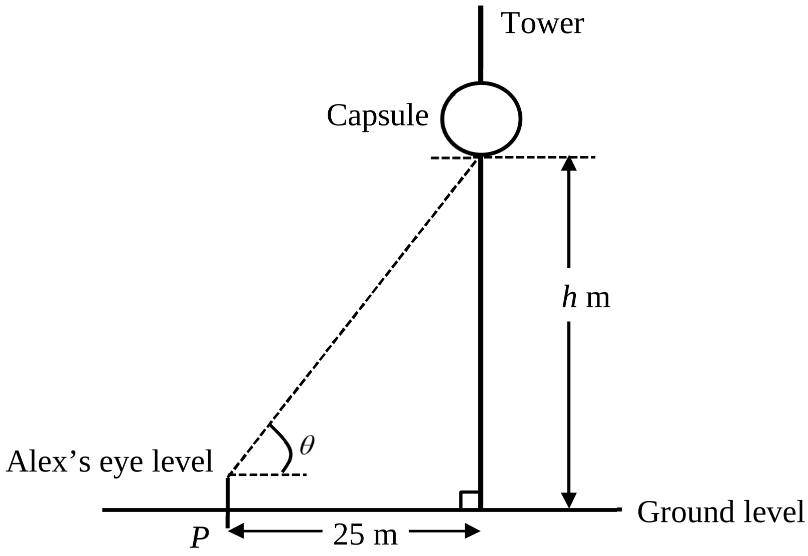
(i)	$2x^2y=36000 \quad \Rightarrow \quad y=\frac{18000}{x^2}$ $A=6xy+2x^2=6x\left(\frac{18000}{x^2}\right)+2x^2=\frac{108000}{x}+2x^2$ $\frac{\mathrm{d}A}{\mathrm{d}x}=-\frac{108000}{x^2}+4x$ Set $\frac{\mathrm{d}A}{\mathrm{d}x}=0$, we have $\frac{108000}{x^2}=4x$ $x=30$ and $y=\frac{18000}{30^2}=20$ $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\frac{216000}{x^3}+4$ and $\left.\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\right\|_{x=30}=12>0$ (minimum value of A occurs at $x=30$) Therefore, breadth, length and height of the tank are 30 cm, 60 cm and 20 cm respectively.
(iia)	$\Delta VAB \sim \Delta VCD$ $\therefore \frac{r}{5}=\frac{20-h}{20}$ $\therefore r=\frac{20-h}{4}$  Vol of cone submerged in water $=\frac{1}{3}\pi(5)^2(20)-\frac{1}{3}\pi r(20-h)$ $=\frac{500}{3}\pi-\frac{1}{3}\pi\left(\frac{20-h}{4}\right)^2(20-h)$

	$=\frac{500}{3}\pi-\frac{1}{48}\pi(20-h)^3$ $=\frac{\pi}{48}\left[8000-(20-h)^3\right]$ Therefore, $V=30\times60\times h-\frac{\pi}{48}\left[8000-(20-h)^3\right]$ $=1800h-\frac{\pi}{48}\left[8000-(20-h)^3\right]$
(iib)	$\frac{\mathrm{d}V}{\mathrm{d}h}=1800-\frac{\pi}{48}\left[3(20-h)^2\right]=1800-\frac{\pi}{16}(20-h)^2$ When $h=10$, we have $\frac{\mathrm{d}V}{\mathrm{d}h}=1800-\frac{\pi}{16}(100)$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}h}{\mathrm{d}V}\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{1000}{1800-\frac{\pi}{16}(100)}=0.562$ (3 s.f.) Therefore, the rate of increase of water level is 0.562 cm/s

8 **MI PU2 P1 Promo 9758/2022/Q11**

A tourist attraction, Sky Lift, has just opened for business. Visitors can enjoy panoramic views of Singapore as they sit in a spherical capsule and travel up and down a tower. The ride starts and ends at the ground level. The duration of the ride is α minutes. At t minutes after the start of the ride, the height, h m, from the horizontal ground to the bottom of the capsule is given by

$$h = -\frac{1}{6}t^3 + \frac{1}{2}t^2 + 8t, \ 0 \le t \le \alpha .$$



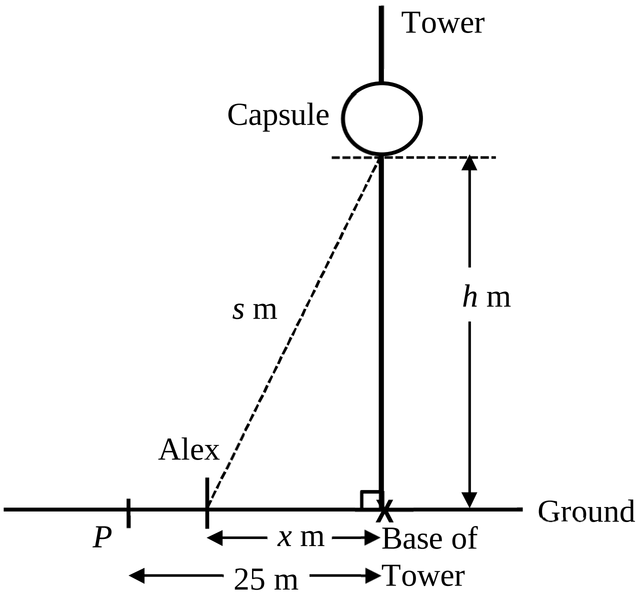
(i) Find the value of α. [1]

(ii) Find the time taken for the capsule to reach its highest point. [1]

Alex stood at the point P which was situated 25 m away from the base of the tower and looked up at capsule as it travelled upwards. His eye level was 1.6 m above the ground. The angle of elevation from his eye level to the bottom of the capsule is denoted by θ radians.

(iii) By expressing $\tan\theta$ in terms of h, find the rate of increase of θ when t =4. [5]

As soon as the capsule started to descend from the highest point, Alex started walking in a straight line from point P towards the base of the tower, at a constant speed of 20 m/min. Let x m be Alex's distance away from the base of the tower and let s m be the distance between Alex and the bottom of the capsule.



(iv) Form an equation involving s, x and h and show that

$$s\frac{\mathrm{d}s}{\mathrm{d}t} = x\frac{\mathrm{d}x}{\mathrm{d}t} + h\frac{\mathrm{d}h}{\mathrm{d}t}.$$ [2]

(v) Using the results in parts **(ii)** and **(iv)**, find $\frac{\mathrm{d}s}{\mathrm{d}t}$ at the instant when Alex had walked for 1 minute. [3]

[(i) 8.59 (ii) 5.12 minutes (iii) 0.0717 rad/min (v) -7.90 m / min]

Solutions

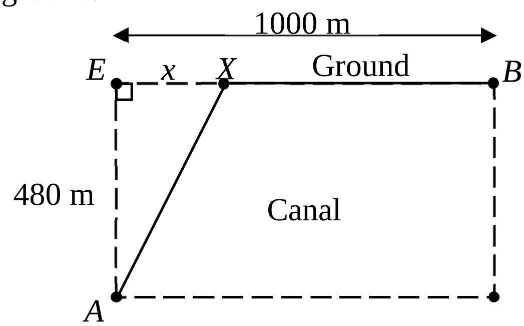
(i)	Let $-\frac{1}{6}t^3 + \frac{1}{2}t^2 + 8t = 0$ From GC, $t = 8.59$ (3 s.f.) is a non-zero solution $\therefore \alpha = 8.5887$ (5sf) $= 8.59$ (3sf)
(ii)	$h = -\frac{1}{6}t^3 + \frac{1}{2}t^2 + 8t$ From GC, at maximum h, $t = 5.1231$ $= 5.12$ (3 s.f.) Time taken to reach its highest point: 5.12 minutes

(iii)	$\tan\theta = \dfrac{h-1.6}{25}$ To find $\dfrac{\mathrm{d}\theta}{\mathrm{d}h}$: $\sec^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}h} = \dfrac{1}{25}$ $\dfrac{\mathrm{d}\theta}{\mathrm{d}h} = \dfrac{1}{25}\cos^2\theta$ When $t=4$, from GC, $h = 29.333$, $\theta = \tan^{-1}\left(\dfrac{29.333-1.6}{25}\right) = 0.83718$ $\left.\dfrac{\mathrm{d}\theta}{\mathrm{d}h}\right\|_{t=4} = \dfrac{1}{25}\cos^2(0.83718) = 0.017932$ $h = -\dfrac{1}{6}t^3 + \dfrac{1}{2}t^2 + 8t$ $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -\dfrac{1}{2}t^2 + t + 8$ $\left.\dfrac{\mathrm{d}h}{\mathrm{d}t}\right\|_{t=4} = -\dfrac{1}{2}(4)^2 + 4 + 8 = 4$ When $t=4, \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{\mathrm{d}\theta}{\mathrm{d}h} \times \dfrac{\mathrm{d}h}{\mathrm{d}t}$ $= 0.017932 \times 4$ $= 0.071728$ $= 0.0717$ rad/min (3.s.f)
(iv)	By Pythagoras' Theorem, $s^2 = x^2 + h^2$. Differentiating w.r.t. t, $2s\dfrac{\mathrm{d}s}{\mathrm{d}t} = 2x\dfrac{\mathrm{d}x}{\mathrm{d}t} + 2h\dfrac{\mathrm{d}h}{\mathrm{d}t}$. $s\dfrac{\mathrm{d}s}{\mathrm{d}t} = x\dfrac{\mathrm{d}x}{\mathrm{d}t} + h\dfrac{\mathrm{d}h}{\mathrm{d}t}$
(v)	At new $t = 5.1231 + 1 = 6.1231$ min, $h = 29.469$ m

	$\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{1}{2}t^2+t+8$ $\left.\dfrac{\mathrm{d}h}{\mathrm{d}t}\right\|_{t=6.1231}=-4.6231$ m/min After walking for 1 min, distance covered is 20 m. $\therefore x=25-20=5$ Since x is decreasing w.r.t. t, $\dfrac{\mathrm{d}x}{\mathrm{d}t}=-20$ m/min $s=\sqrt{5^2+29.469^2}=29.890$ m $\therefore$ When $t=6.1231$, $29.890\dfrac{\mathrm{d}s}{\mathrm{d}t}=5(-20)+29.469(-4.6231)$ $\dfrac{\mathrm{d}s}{\mathrm{d}t}=-7.90$ m / min

9 **NJC Promo 9758/2022/Q11**

Engineers need to lay pipes to connect two factories A and B that are separated by a canal of uniform width 480 m as shown in the diagram. They plan to lay the pipes under the canal in a line from A to X and then under the ground in a line from X to B. The cost of laying the pipes under the canal is five times the cost of laying the pipes under the ground.



The point E is such that AE is perpendicular to the line XB, where EB has a length of 1000 m. X is between E and B such that EX has a length of x m. The cost per metre of laying the pipes under the ground is 180 dollars.

(i) Show that the total cost C, in dollars, of laying the pipes from A to B is given by $C=900\sqrt{230400+x^2}+180(1000-x)$. [2]

(ii) In the context of the question, find the range of values of x such that $\frac{\mathrm{d}C}{\mathrm{d}x} > 0.$ [3]

Let the angle at which the pipes are joined, $\angle AXB$, be θ. For engineering reasons, $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{4}$.

(iii) State an equation relating θ and x. Hence find the range of values of x when $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{4}$. [3]

(iv) By considering part **(ii)** and part **(iii)**, show that $\frac{\mathrm{d}C}{\mathrm{d}\theta} > 0$ for $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{4}$. [3]

(v) Hence, deduce the value of θ which gives the minimum total cost. [1]

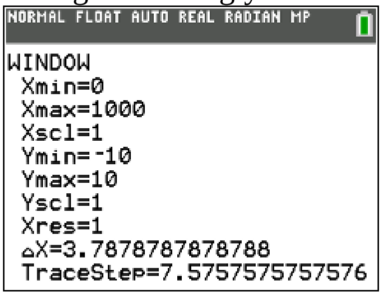
[(ii) $98.0 < x < 1000$ (iii) $x = -480\cot\theta$; $\frac{480}{\sqrt{3}} \le x \le 480$ (v) $\frac{2\pi}{3}$]

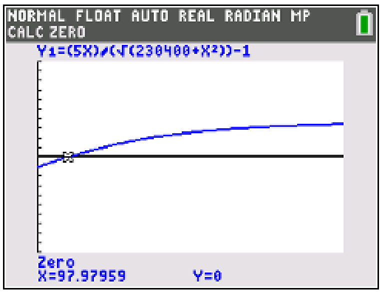
Solutions

(i)	$C = 5(180)(AX) + 180(XB)$ $= 900\sqrt{480^2 + x^2} + 180(1000 - x)$ $= 900\sqrt{230400 + x^2} + 180(1000 - x)$
(ii)	$C = 900\sqrt{230400 + x^2} + 180(1000 - x)$ $\frac{\mathrm{d}C}{\mathrm{d}x} = 180\left(\frac{10x}{2\sqrt{230400 + x^2}} - 1\right)$ $= 180\left(\frac{5x}{\sqrt{230400 + x^2}} - 1\right)$ $\frac{\mathrm{d}C}{\mathrm{d}x} > 0$ $180\left(\frac{5x}{\sqrt{230400 + x^2}} - 1\right) > 0$ $\frac{5x}{\sqrt{230400 + x^2}} - 1 > 0$ **<u>Graphical Method</u>**

We sketch the graph of $y=\dfrac{5x}{\sqrt{230400+x^2}}-1$.

In the context of the question, $0<x<1000$, thus we adjust the WINDOW settings accordingly.





Since $x<1000$ due to the context, we have $98.0<x<1000$.

<u>Algebraic Method</u>

$$\frac{5x}{\sqrt{230400+x^2}}>1$$

$$\begin{aligned}5x&>\sqrt{230400+x^2} \quad \text{since } \sqrt{230400+x^2}>0\\ 25x^2&>230400+x^2 \quad \text{since both sides positive}\\ 24x^2&>230400\\ x^2&>9600\end{aligned}$$

$$\therefore x>\sqrt{9600} \quad \left(\text{or } 40\sqrt{6}\right) \quad \text{since } x>0$$

Since $x<1000$ due to the context, we have $\sqrt{9600}<x<1000$.

(iii)

$$\tan\angle AXE=\frac{480}{x}$$

Since $\angle AXE+\theta=\pi$,

$$\tan(\pi-\theta)=\frac{480}{x}$$

$$x=\frac{480}{-\tan\theta}=-480\cot\theta$$

<u>Algebraic Method</u>

$$\frac{2\pi}{3}\le\theta\le\frac{3\pi}{4}$$

$\tan\dfrac{2\pi}{3}\le\tan\theta\le\tan\dfrac{3\pi}{4}$ since the tangent function is increasing for $\dfrac{2\pi}{3}\le\theta\le\dfrac{3\pi}{4}$

$$-\sqrt{3}\le\tan\theta\le-1$$

$$1\le-\tan\theta\le\sqrt{3}$$

$$\frac{1}{\sqrt{3}}\le\frac{1}{-\tan\theta}\le1$$

$$\frac{480}{\sqrt{3}}\le\frac{480}{-\tan\theta}\le480$$

	Hence, $\frac{480}{\sqrt{3}} \le x \le 480$. **<u>Graphical Method</u>** We can use the GC to sketch the graph of $x = -480\cot\theta$ and observe the range of x.
(iv)	$\frac{\mathrm{d}C}{\mathrm{d}\theta} = \frac{\mathrm{d}C}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}\theta}$ To explain $\frac{\mathrm{d}x}{\mathrm{d}\theta} > 0$: From part (iii), $x = -480\cot\theta$. $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -480\left[-\mathrm{cosec}^2(\theta)\right] = 480\mathrm{cosec}^2\theta > 0$ To explain $\frac{\mathrm{d}C}{\mathrm{d}x} > 0$: From part **(ii)**, we know that when $x > \sqrt{9600}$, $\frac{\mathrm{d}C}{\mathrm{d}x} > 0$. From part **(iii)**, for $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{4}$, we have $\frac{480}{\sqrt{3}} \le x \le 480$. Therefore, for $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{4}$, since $\left[\frac{480}{\sqrt{3}}, 480\right] \subseteq \left(\sqrt{9600}, 1000\right)$, then we have $\frac{\mathrm{d}C}{\mathrm{d}x} > 0$. Hence for $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{4}$, since $\frac{\mathrm{d}C}{\mathrm{d}x} > 0$ and $\frac{\mathrm{d}x}{\mathrm{d}\theta} > 0$, we have $\frac{\mathrm{d}C}{\mathrm{d}\theta} = \frac{\mathrm{d}C}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}\theta} > 0$.
(v)	The value of θ that gives the minimum total cost is $\frac{2\pi}{3}$.

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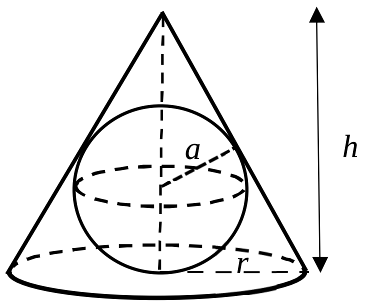
(a)



A decorative artifact, as shown in the diagram above, is in the shape of a cone with radius r and height h with a sphere of fixed radius a inscribed in it.

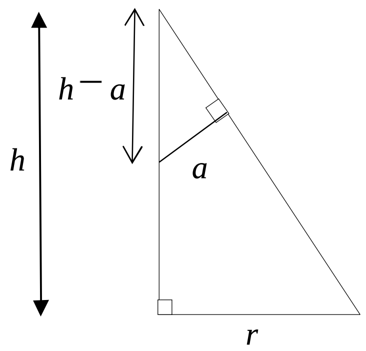
(i) Show that the volume of the cone, V, is given by $V=\dfrac{\pi a^2h^2}{3(h-2a)}$. [3]

(ii) Use differentiation to find, in terms of a, the minimum value of V. Leave your answer in exact form. [5]

(b) In a triangle PQR, $PQ=3$ cm and $PR=2$ cm. If angle QPR is increasing at a constant rate of 0.1 radians per second, find the rate of increase of the length QR at the instant when angle QPR is $\dfrac{\pi}{3}$ radians. [4]

[(aii) $\dfrac{8\pi a^3}{3}$ (b) 0.196 cm/s]

Solutions

(ai)	 By similar triangles, $\dfrac{h-a}{\sqrt{r^2+h^2}}=\dfrac{a}{r}$ or $\dfrac{h}{\sqrt{(h-a)^2-a^2}}=\dfrac{r}{a}$

$$(h-a)(r)=a\sqrt{r^2+h^2}$$

$$\Rightarrow (h-a)^2(r)^2=a^2(r^2+h^2)$$

$$\mathrm{r}^2(h^2-2ah+a^2-a^2)=a^2h^2$$

$$\mathrm{r}^2=\frac{a^2h^2}{h^2-2ah}=\frac{a^2h}{h-2a}$$

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{a^2h}{h-2a}\right)h=\frac{\pi a^2h^2}{3(h-2a)}$$

(aii)

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{\pi a^2}{3}\left[\frac{(h-2a)(2h)-h^2}{(h-2a)^2}\right]$$

$$=\frac{\pi a^2}{3}\left[\frac{h^2-4ah}{(h-2a)^2}\right]=\frac{\pi a^2 h}{3}\left[\frac{h-4a}{(h-2a)^2}\right]$$

When V is minimum, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$

$h=0$ (rej since h>0) or $h=4a$

Recall $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{\pi a^2 h}{3}\left[\frac{h-4a}{(h-2a)^2}\right]$ which is in factorised form where h>0 and $(h-2a)^2$ >0. Applying 1st derivative test,

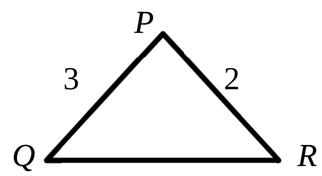
h	$(4a)^-$	$4a$	$(4a)^+$
$h-4a$	<0	0	>0
$\frac{\mathrm{d}V}{\mathrm{d}h}$	<0	0	>0

$\therefore V$ is minimum at $h=4a$

Minimum $V=\frac{\pi a^2(4a)^2}{3(4a-2a)}=\frac{16\pi a^4}{3(2a)}=\frac{8\pi a^3}{3}$

(b)

Let $QR=x$ cm and $\angle QPR=\theta$ radians.



Using cosine rule,

$$x^2=3^2+2^2-2(3)(2)\cos\theta$$

$$x^2=13-12\cos\theta$$

Differentiating wrt t,

$$2x\frac{\mathrm{d}x}{\mathrm{d}t}=12\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} \quad \text{or} \quad 2x\frac{\mathrm{d}x}{\mathrm{d}\theta}=12\sin\theta$$

When $\theta=\frac{\pi}{3}$, $x=\sqrt{7}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ = 0.1,

$$2\sqrt{7}\frac{\mathrm{d}x}{\mathrm{d}t}=12\left(\frac{\sqrt{3}}{2}\right)(0.1) \quad \text{or} \quad \frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{0.3\sqrt{3}}{\sqrt{7}}\approx 0.196 \text{ cm/s}$$