Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 3 Functions
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 3 Functions
Core Concept Notes

Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
□ Identify the rule of a function □ Find the domain of a function □ Find the range of a function □ Define a function using functions notation □ Identify whether a relation is a function □ Check if a function is one-one <u>Inverse Functions</u> □ Check the condition for the existence of inverse functions <u>Composite Functions</u> □ Check the condition for the existence of composite functions <u>Periodic Functions</u> □ Identify the period of a periodic function	<u>Inverse Functions</u> □ Find the rule of an inverse function □ Find the domain of an inverse function □ Find the range of an inverse function <u>Composite Functions</u> □ Find the rule of a composite function □ Find the domain of a composite function □ Find the range of a composite function using its rule and domain <u>Periodic Functions</u> □ Find the value of a periodic function f(x) given x	<u>Inverse Functions</u> □ Restrict a domain to obtain an inverse function □ Explain the relationship between a function and its inverse <u>Composite Functions</u> □ Find the range of a composite function using the mapping method <u>Periodic Functions</u> □ Sketch the graph of a periodic function

§1 Functions

1.1 Relations

Let X and Y be two sets, where the elements in X and Y may or may not be the same.

A **relation** is a rule that maps some elements of X to some elements of Y.

There are many ways to relate 2 sets of elements, e.g. X and Y.

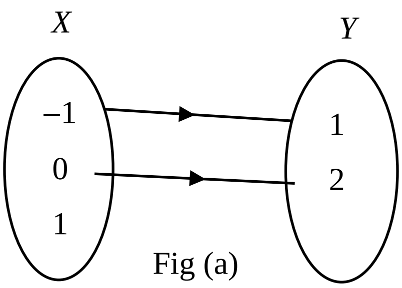


Fig (a)

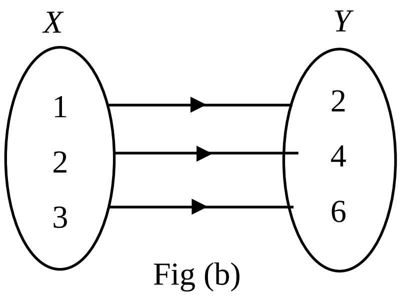


Fig (b)

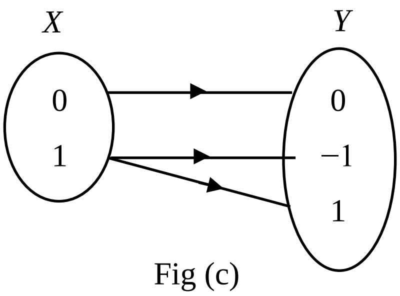


Fig (c)

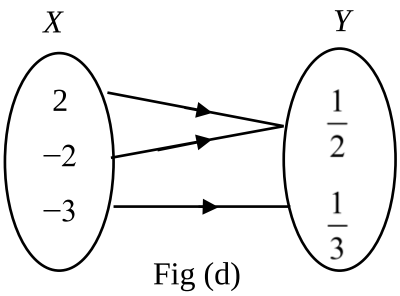


Fig (d)

1.2 Functions, Domain and Range

1. A **function** $\mathrm{f}: X \to Y$ is a relation where f maps **each and every** element $x \in X$ to **exactly one** element $y \in Y$. (In other words, **every** input has **exactly** one output.)
2. **Set** X is known as the **domain** of f, and denoted by $\mathrm{D_f}$. (Set of input)
3. The **element** y is called the **image** of x under f. This is denoted by $\mathrm{f}(x) = y$.
4. The **range** of f (denoted by $\mathrm{R_f}$) is the **set** of all images of $x \in X$. (Set of output)

Q: In Figure (a)-(d), which of the relations are functions? (b) and (d)

1.3 Defining a Function

To define a function, both the rule and the domain of the function must be clearly stated.

Examples:

$$\mathrm{f}: x \mapsto \underbrace{x-1}_{\text{Rule}}, \quad \underbrace{x \geq 0}_{\text{Domain}}$$

$$\mathrm{g}: x \mapsto \underbrace{x^2}_{\text{Rule}}, \quad \underbrace{-1 < x \leq 1}_{\text{Domain}}$$

$$\mathrm{h}: x \mapsto \underbrace{\sin x}_{\text{Rule}}, \quad \underbrace{x \in \mathbb{R}}_{\text{Domain}}$$

Recall:

Notation	Example
(a,b) is equivalent to $\{x \in \mathbb{R} : a < x < b\}$.	$(-10,8)$ is equivalent to $\{x \in \mathbb{R} : -10 < x < 8\}$. $(-\infty,\infty)$ is equivalent to $\{x \in \mathbb{R}\}$.
$(a,b]$ is equivalent to $\{x \in \mathbb{R} : a < x \leq b\}$.	$(-10,8]$ is equivalent to $\{x \in \mathbb{R} : -10 < x \leq 8\}$. $(-\infty,8]$ is equivalent to $\{x \in \mathbb{R} : x \leq 8\}$.
$[a,b)$ is equivalent to $\{x \in \mathbb{R} : a \leq x < b\}$.	$[-10,8)$ is equivalent to $\{x \in \mathbb{R} : -10 \leq x < 8\}$. $[-10,\infty)$ is equivalent to $\{x \in \mathbb{R} : x \geq -10\}$.
$[a,b]$ is equivalent to $\{x \in \mathbb{R} : a \leq x \leq b\}$.	$[-10,8]$ is equivalent to $\{x \in \mathbb{R} : -10 \leq x \leq 8\}$.

Note:

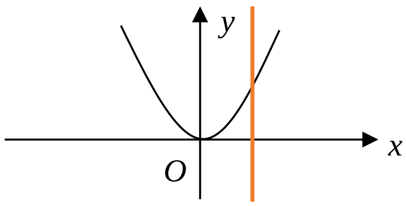
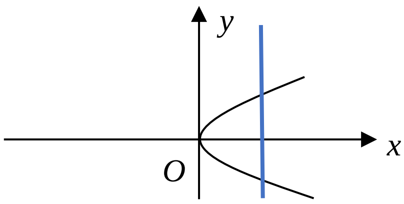
- A function is not completely defined if the domain is not indicated.
- There should be exactly one result (output) originating from each $x \in X$ (input).
- All functions are relations since a function is a special type of relation. However, a relation may not be a function.

1.4 Test for Function (Vertical line test)

(a)	Given a relation $f : X \to Y$ where $X, Y \subseteq \mathbb{R}$, if for **each** $a \in X$, the vertical line $x = a$ cuts the graph of f at **exactly one point**, then f is a function.
(b)	When a relation is **not** a function, we can show this by just giving a counter-example, i.e. find the equation of **a particular** vertical line that cuts the graph more than once or does not cut the graph at all for **some** $x \in X$.

Example 1

Determine which of the following relations are functions:

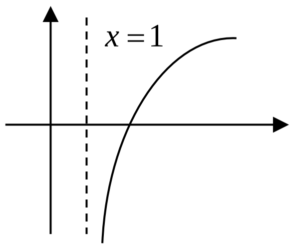
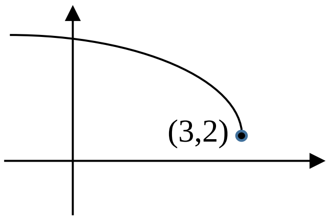
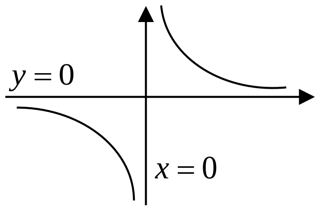
(a) $\mathrm{g}(x) = x^2$ for $x \in \mathbb{R}$  $x = a$ Any vertical line $x = a$, $a \in \mathbb{R}$, cuts the graph of g at exactly one point. $\therefore$ g is a function	**(b)** $\mathrm{h}(x) = \pm\sqrt{x}$ for $x \geq 0$  $x = 1$ The line $x = 1$ cuts the graph of h at 2 points, $(1, -1)$ and $(1, 1)$. $\therefore$ h is not a function.

1.5 Maximal Domain

> The **maximal domain** is the largest possible domain for which a function can be defined.

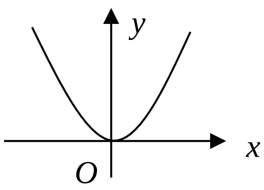
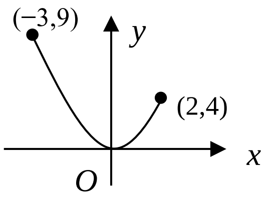
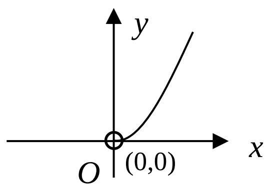
Example 2

State the maximal domain for each of the following functions to exist.

(a) $\mathrm{f}: x \mapsto \ln(x-1)$  $\ln(x-1)$ is defined if $x-1>0 \Rightarrow x>1$ Maximal domain for function f to exist = $(1,\infty)$	**(b)** $\mathrm{g}: x \mapsto 2+\sqrt{3-x}$  $\sqrt{3-x}$ is defined if $3-x \geq 0 \Rightarrow x \leq 3$ Maximal domain for function g to exist = $(-\infty,3]$	**(c)** $\mathrm{h}: x \mapsto \dfrac{1}{x}$  $\dfrac{1}{x}$ is defined if $x \neq 0$ Maximal domain for function h to exist = $\mathbb{R}\setminus\{0\}$

1.6 Restricting the Domain of a Function

Consider the following functions and graphs obtained using GC:

$\mathrm{f}: x \mapsto x^2, \quad x \in \mathbb{R}$ 	$\mathrm{g}: x \mapsto x^2, \quad -3 \leq x \leq 2$ 	$\mathrm{h}: x \mapsto x^2, \quad x>0$ 

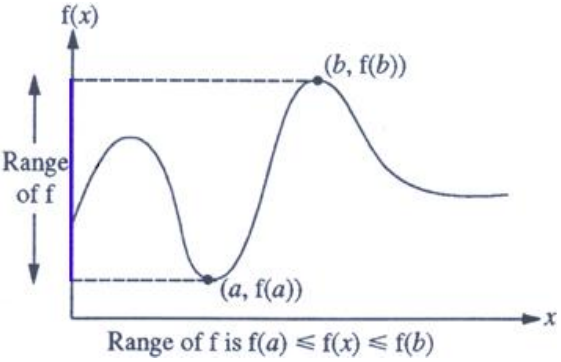
Q: Are functions f, g and h considered the same functions?

Ans:
No, they are not the same function. Though the three functions have the same rule, their domains are not the same.
Two functions are the same when BOTH the rule and domain are the same.

In fact, in this case, since $\mathrm{D_g}$ and $\mathrm{D_h}$ are subsets of $\mathrm{D_f}$, we say that the functions g and h are **restrictions of f**.

1.7 <u>Determine the range of a function</u>

The range of a function, f, can be determined by sketching the graph of $y = \mathrm{f}(x)$ to find the set of values y can take.

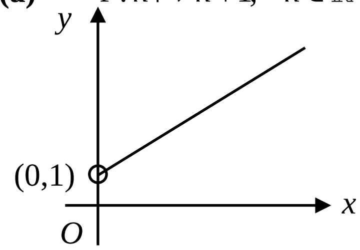
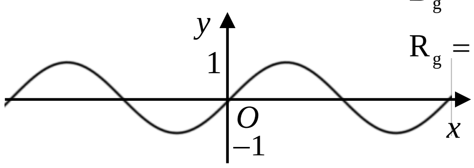
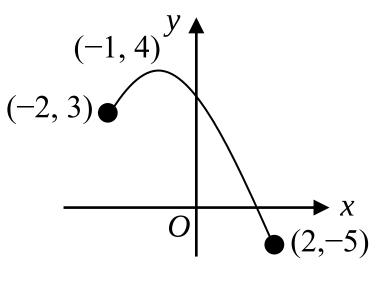
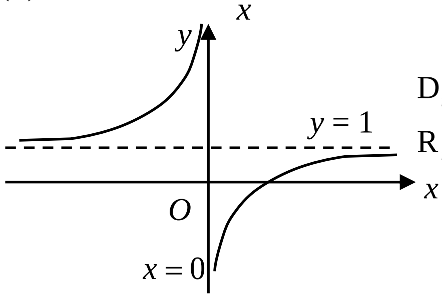


Example 3

Determine the range of each of the following functions.

(a) $\mathrm{f}: x \mapsto x+1, \quad x \in \mathbb{R}^+$ **(b)** $\mathrm{g}: x \mapsto \sin x, \quad x \in \mathbb{R}$

(c) $\mathrm{q}: x \mapsto (3+x)(1-x), \quad -2 \le x \le 2$ **(d)** $\mathrm{r}: x \mapsto 1-\dfrac{1}{x}, \quad x \in \mathbb{R}\setminus\{0\}$

(a) $\mathrm{f}: x \mapsto x+1, \quad x \in \mathbb{R}^+$  $\mathrm{D_f} = (0,\infty)$ $\mathrm{R_f} = (1,\infty)$	**(b)** $\mathrm{g}: x \mapsto \sin x, \quad x \in \mathbb{R}$  $\mathrm{D_g} = (-\infty,\infty)$ $\mathrm{R_g} = [-1,1]$
(c) $\mathrm{q}: x \mapsto (3+x)(1-x), \quad -2 \le x \le 2$  $\mathrm{D_q} = [-2,2]$ $\mathrm{R_q} = [-5,4]$	**(d)** $\mathrm{r}: x \mapsto 1-\dfrac{1}{x}, \quad x \in \mathbb{R}\setminus\{0\}$  $\mathrm{D}_r = \mathbb{R}\setminus\{0\}$ $\mathrm{R}_r = \mathbb{R}\setminus\{1\}$

Note:

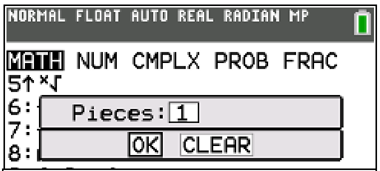
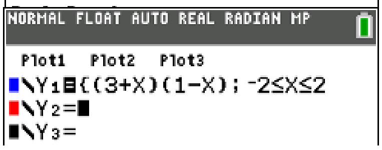
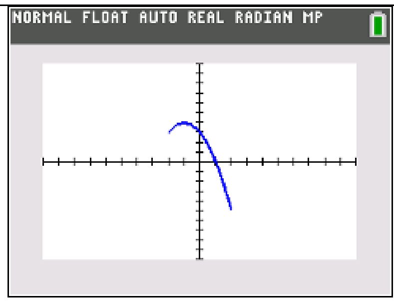
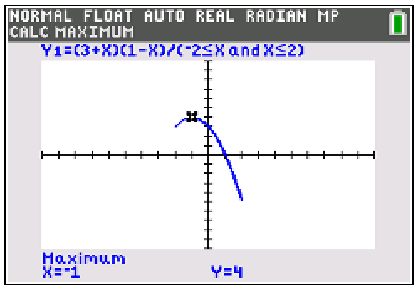
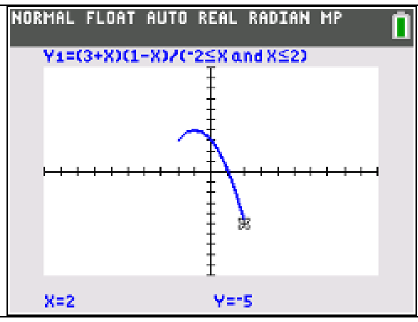
(1) The **range** of a function depends on <u>both</u> the **rule** and the **domain**. Hence, it is important to only consider the part of the graph corresponding to the required domain.
(2) **Range** must be stated in **set notation**.
(3) Indicate **end-points** with ‘open’ (if exclusive) or ‘closed’(if inclusive) circle to remind yourselves whether to include the values when writing down the range.

Q: Can we substitute the endpoints of the domain into the function to find range?

Ans: Generally, no. See Example 3(c). However, if the function is increasing / decreasing on that domain, then we can just substitute the endpoints of the domain into the function to find range (see Example 3(a)).

<u>Using GC to sketch the graph and find the range of a function with the given domain</u>

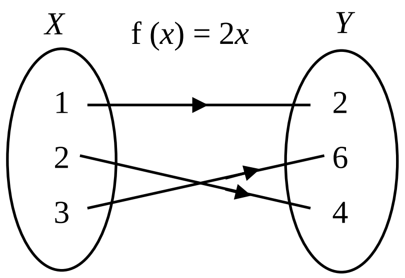
Consider Example 3(c) $q: x \mapsto (3+x)(1-x)$, where $-2 \le x \le 2$. Sketch the graph and find the range of q.

INSTRUCTIONS TO USE KEYS	SCREEN ON GC
Step 1: Enter and plot the function $y=(3+x)(1-x)$ for the restricted domain [–2, 2] using piecewise function. Go to [Y=] and select [MATH], option B: piecewise. In this case, select 1 piece. Inequality symbols "≤" and "≥" can be entered by pressing [2nd][MATH], then under **TEST,** select the required symbol.	
Find the maximum *y*-value for the restricted function as follows: **Step 2:** Press [2nd] and then [TRACE] to bring up the [CALC] option. Select **4:maximum** Press [▶] and [◀] to locate **left** bound (or enter the number directly) and press [ENTER]. Similarly, press [▶] and [◀] to locate **right** bound (or enter the number directly) and press [ENTER].	
Step 4: Observe from the graph that the minimum *y*-value is located at the end-point where *x* = 2. Find its value by pressing [2nd] [TRACE], select **1:value** *x* = 2. What is the corresponding *y*-value? Thus, the range of q over the domain [–2, 2] is [–5, 4].	

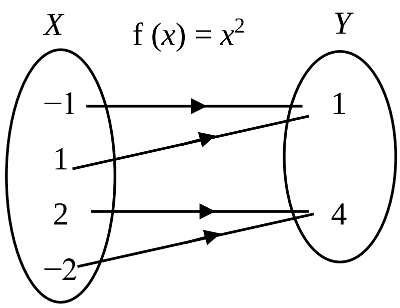
§2 <u>One-One Functions</u>

2.1 <u>Definition</u>

A function f : $\boldsymbol{X} \to \boldsymbol{Y}$ is said to be <u>one-one</u> if each element, $y \in \mathrm{R_f}$ corresponds to exactly one element, $x \in \mathrm{D_f}$, i.e. no 2 distinct elements of $\boldsymbol{X}$ have the same image under f.
In other words, a one-one function is a function BOTH ways:
- Every input has exactly one output, and
- Every output has exactly one input



A one-one function



NOT a one-one function

2.2 <u>To check whether a function is one-one</u>

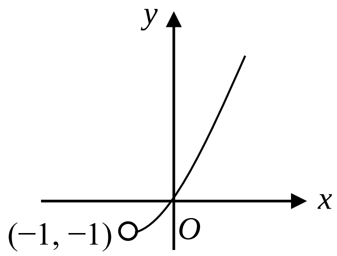
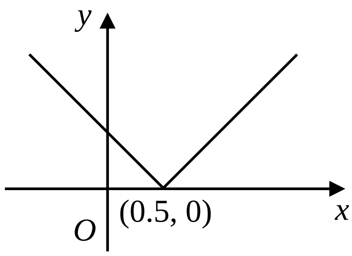
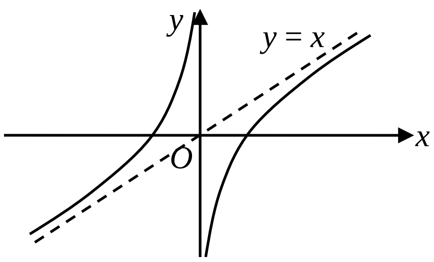
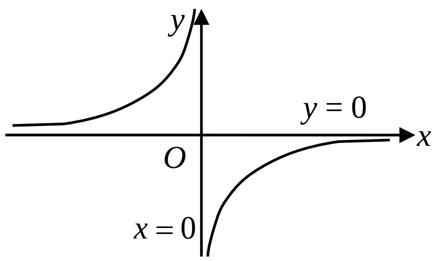
<u>Method 1: Horizontal Line Test</u> (**Need to draw the graph**)

(a) A function f is one-one if **any** horizontal line $y = b$ ($b \in \mathbb{R}$) cuts the graph of f **at most once**, i.e. there is either only one point of intersection, or none at all.

(b) When a function is **<u>not</u>** one-one, we can show this by just giving a counter-example, i.e. find the equation of **one particular** horizontal line that cuts the graph more than once.

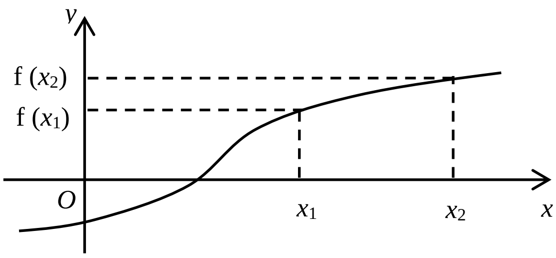
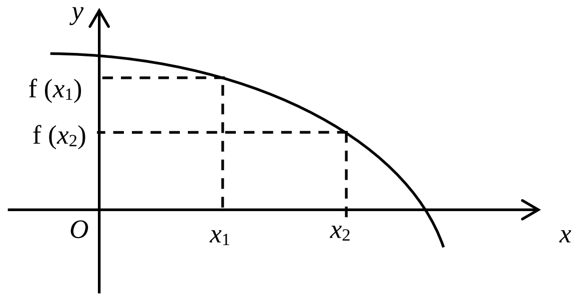
Example 4

Determine whether the following functions are one-one. If the function is not one-one, find the maximal domain such that the function is one-one.

(a) $\mathrm{f}: x \mapsto x^2 + 2x, \quad x > -1$  Any line $y = a \, (a \in \mathbb{R})$ cuts the graph of f at most once. Therefore f is 1-1.	**(b)** $\mathrm{g}: x \mapsto \lvert 2x - 1 \rvert, \quad x \in \mathbb{R}$  The line $y = 1$ cuts the graph of g at 2 points. Therefore g is not 1-1. For function g to be 1-1, we restrict the domain to be $[0.5, \infty)$ or $(-\infty, 0.5]$.
(c) $\mathrm{h}: x \mapsto x - \dfrac{1}{x}, \quad x \in \mathbb{R} \setminus \{0\}$  The line $y = 1$ cuts the graph of h at 2 points. Therefore h is not 1-1. For function h to be 1-1, we restrict the domain to be $(0, \infty)$ or $(-\infty, 0)$.	**(d)** $\mathrm{k}: x \mapsto \dfrac{1}{x}, \quad x \in \mathbb{R} \setminus \{0\}$  Any line $y = b \, (b \in \mathbb{R})$ cuts the graph of k at most once. Therefore k is 1-1.

Method 2: **(Need to prove function is strictly increasing/decreasing) – This will be covered in Differentiation.**

All **strictly** increasing and **strictly** decreasing functions are one-one.

Strictly Increasing Function	Strictly Decreasing Function
A function f with domain X is said to be **strictly increasing** if and only if for all $x_1,\ x_2 \in X,\quad x_1 < x_2 \quad\Rightarrow\quad \mathrm{f}\left(x_1\right) < \mathrm{f}\left(x_2\right)$ This means that as the value of x increases, the value of $\mathrm{f}\left(x\right)$ also increases. 	A function f with domain X is said to be **strictly decreasing** if and only if for all $x_1,\ x_2 \in X,\quad x_1 < x_2 \quad\Rightarrow\quad \mathrm{f}\left(x_1\right) > \mathrm{f}\left(x_2\right)$ This means that as the value of x increases, the value of $\mathrm{f}\left(x\right)$ decreases. 

Some common strictly increasing/decreasing functions:

Strictly increasing function	**Strictly decreasing function**
$\mathrm{f}: x \to \sqrt{x},\quad x \geq 0$	$\mathrm{p}: x \to -\sqrt{x},\quad x \geq 0$
$\mathrm{h}: x \to \mathrm{e}^{x},\quad x \in \mathbb{R}$	$\mathrm{q}: x \to \dfrac{1}{x},\quad x > 0$
$\mathrm{m}: x \to \ln x,\quad x \in \mathbb{R}^{+}$	$\mathrm{r}: x \to \mathrm{e}^{-x},\quad x \in \mathbb{R}$

Method 3: Algebraic Approach

(a) A function f is one-one if $\mathrm{f}(x_1) = \mathrm{f}(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in \mathrm{D_f}$.

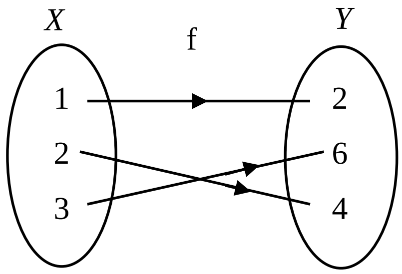
(b) A function is **not** one-one if we can find $x_1, x_2 \in \mathrm{D_f}$ such that $\mathrm{f}(x_1) = \mathrm{f}(x_2)$ where $x_1 \neq x_2$.

Example 4(b): Since $\mathrm{g}\left(0\right) = \mathrm{g}\left(1\right) = 1$, g is not one-one.

§3 Inverse Functions

Consider the following functions f and g. We can always find an inverse relation f^{-1} and g^{-1} which **maps** each **image back** to its initial value.

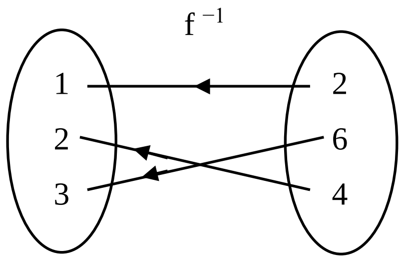
If f^{-1} and g^{-1} are functions, then we say the inverse functions of f and g exist.



$\mathrm{f}(x) = 2x$

$y = 2x$

Note: f is a one-one function

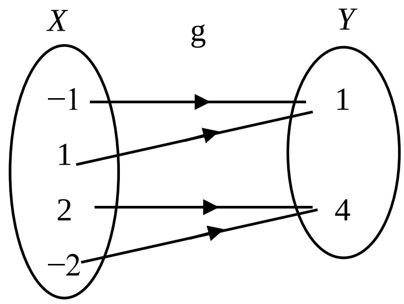


f^{-1} is a function

$y = 2x$

$x = \dfrac{y}{2}$

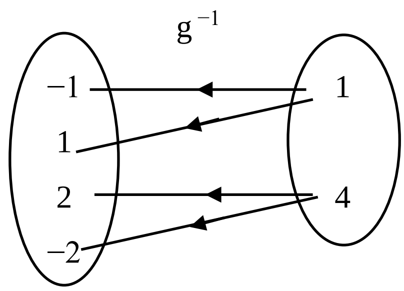
$\mathrm{f}^{-1}(x) = \dfrac{x}{2}$



$g(x) = x^2$

$y = x^2$

Note: g is not a one-one function



g^{-1} is **not** a function

Investigation:

Q: Compare functions f and g and deduce the criteria for the inverse function to exist.

$$\boxed{\mathrm{f}^{-1} \text{ exists} \Leftrightarrow \mathrm{f} \text{ is one-one}}$$

IMPORTANT:

- $\mathrm{f}^{-1}(x) \neq \left[\mathrm{f}(x)\right]^{-1}$ $\left[\text{e.g. } \sin^{-1} x \neq [\sin x]^{-1}\right]$

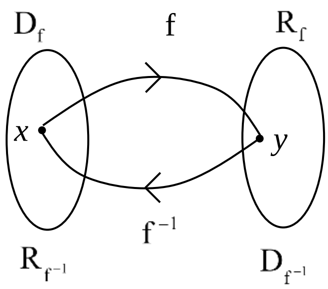
Inverse function

Reciprocal function of f

3.1 Existence of Inverse function

$$\boxed{f^{-1} \text{ exists } \Leftrightarrow f \text{ is one-one}}$$

Q: What is the relationship between the domain and range of f and f^{-1}?



If the inverse function exists,

$$\boxed{\begin{array}{c} D_f = R_{f^{-1}} \\ R_f = D_{f^{-1}}. \end{array}}$$

Example 5

The function f is defined as $\mathrm{f}: x \mapsto 2x-2,\quad x>-2$.

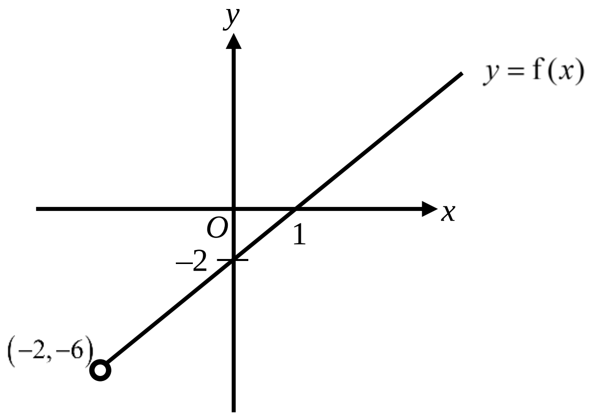
(i) Give a reason why f^{-1} exists.

(ii) Find $\mathrm{f}^{-1}(x)$.

(iii) Write down f^{-1} in a similar form.

Solution:

(i) Method 1:



Since any line $y=a$, where $a\in\mathbb{R}$, cuts the graph of f at most once, f is one-one. Hence, f^{-1} exists.

Method 2:
Since the gradient of the graph is 2 > 0 for all values of *x* within the domain, the graph of f is **strictly increasing** for $x>-2$, f is one-one. Hence f^{-1} exists.

(ii) Let $y=\mathrm{f}(x)$.

$$y=2x-2$$

$$x=\frac{y+2}{2}$$

$$\therefore \mathrm{f}^{-1}(x)=\frac{x+2}{2}$$

(iii) $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(-6,\infty)$

$$\therefore \mathrm{f}^{-1}: x\mapsto \frac{x+2}{2}, x>-6$$

Step 1: Let $y=\mathrm{f}(x)$.
Step 2: Make *x* the subject.
Note: If there are 2 possible expressions, choose the one that corresponds to the set of values that *x* can take, i.e. the domain of f.
Step 3: Replace *y* by *x* to get the expression for $\mathrm{f}^{-1}(x)$.
Step 4: Define f^{-1} in a similar form by specifying the rule and the domain.

***Note the difference between $\mathrm{f}^{-1}(x)$ and f^{-1}.**

$\mathrm{f}^{-1}(x)$ is the expression, i.e. $\mathrm{f}^{-1}(x)=\dfrac{x+2}{2}$, while

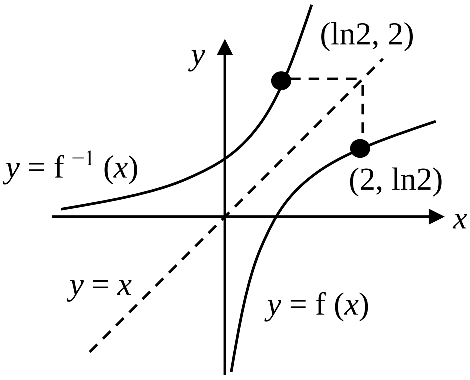
f^{-1} is the definition where both the rule and domain must be clearly defined,
i.e. $\mathrm{f}^{-1}: x\mapsto \dfrac{x+2}{2}, x>-6$.

3.2 Relationship between the graphs of f and f^{-1}

Investigation:

Using a GC, sketch the graphs of f and f^{-1} for which $f : x \mapsto \ln x,\ x \in \mathbb{R}^+$ and state the domain and range of f and f^{-1}.

[Note: Refer to page 15 for the GC keystrokes on how to sketch inverse functions]



$D_f = (0, \infty)$; $R_f = (-\infty, \infty)$
$D_{f^{-1}} = (-\infty, \infty)$; $R_{f^{-1}} = (0, \infty)$

Q: What is the geometrical relationship between the graphs of f and f^{-1}?

If the inverse function exists, The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

GC Keystrokes to Draw Inverse Function

Example: Sketch the graphs of f and f^{-1} for which $\mathrm{f}: x \mapsto \ln x,\ x \in \mathbb{R}^+$

<table>
<tr><td>

Steps on GC

1. Press $\boxed{\text{Y=}}$ to display equation editor and enter the expression into Y1.
2. Enter Y2 = X

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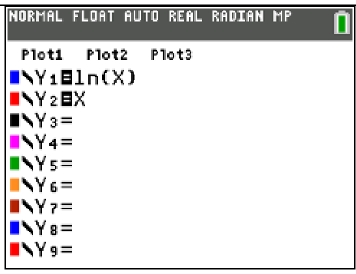

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3. You may use **zoom:ZSquare** to obtain a more proportionate graph.

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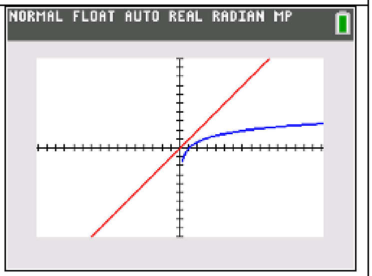

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4. To draw the inverse function on the same diagram, press $\boxed{\text{2nd}}\boxed{\text{PRGM}}$ to display the DRAW menu and select $\boxed{8}$ for **DrawInv.**

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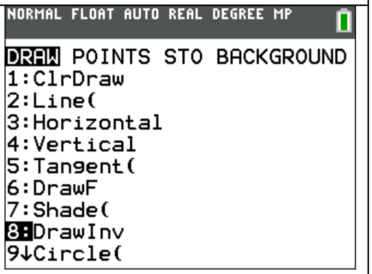

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5. Press $\boxed{\text{ALPHA}}\boxed{\text{TRACE}}$ to display the YVAR menu followed by $\boxed{\text{ENTER}}$ to select Y1.

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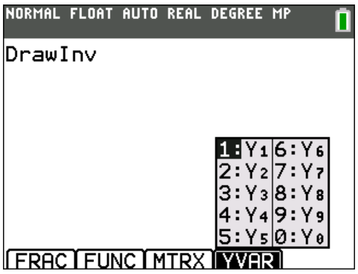

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6. Press $\boxed{\text{ENTER}}$ to display the graph of the inverse function.

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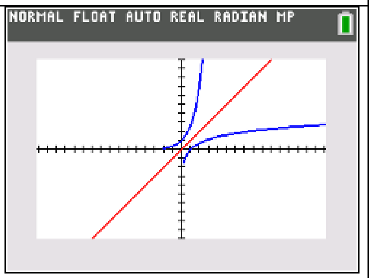

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Example 6

The functions f and g are defined by

$\mathrm{f}: x \mapsto 4-x^2, \quad x \in \mathbb{R},$

$\mathrm{g}: x \mapsto 4-x^2, \quad x \le 0.$

(i) Explain why the inverse function of g exists but the inverse function of f does not exist.

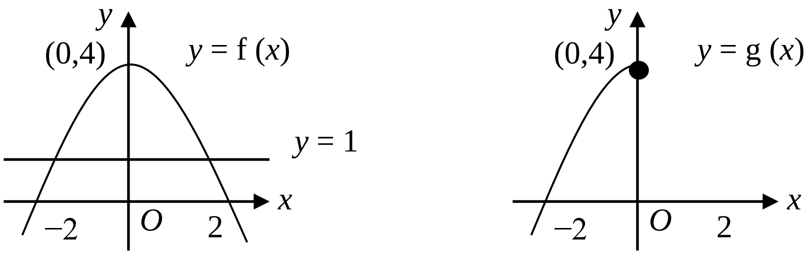
(ii) Find, in a similar form, g^{-1}.

(iii) Sketch the graphs of the functions g and g^{-1} on the same diagram.

(iv) Write down the equation of the line in which the graph $y=\mathrm{g}(x)$ must be reflected in order to obtain the graph of $y=\mathrm{g}^{-1}(x)$.Hence find the exact solution of the equation $\mathrm{g}(x)=\mathrm{g}^{-1}(x)$.

Solution:

(i)



Since any line $y=a$, where $a \in \mathbb{R}$, cuts the graph of g at most once, g is one-one. Hence, g has an inverse.

Since the line $y = 1$ cuts the graph of f at 2 points, f is not one-one. Hence, f does not have an inverse.

OR

$\mathrm{f}(1)=\mathrm{f}(-1)=3$, f is not one-one, hence f does not have an inverse.

Step 1: Let $y=\mathrm{f}(x)$.

Step 2: Make x the subject.

Note: If there are 2 possible expressions, choose the one that corresponds to the set of values that x can take, i.e. the domain of f.

Step 3: Replace y by x to get the expression for $\mathrm{f}^{-1}(x)$.

Step 4: Define f^{-1} in a similar form by specifying the rule and the domain.

(ii) Let $y=\mathrm{g}(x)=4-x^2$

$\Rightarrow x^2=4-y$

$\Rightarrow x=\pm\sqrt{4-y}$

$\Rightarrow x=\sqrt{4-y}$ or $-\sqrt{4-y}$

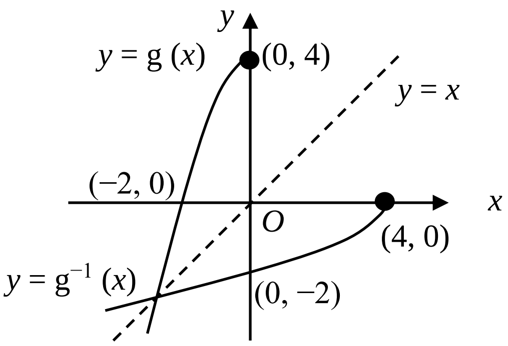
Since $x \le 0$, $\therefore x=-\sqrt{4-y}$

$\therefore \mathrm{g}^{-1}(x)=-\sqrt{4-x}$

$D_{\mathrm{g}^{-1}}=R_{\mathrm{g}}=(-\infty,4]$

Hence, $\mathrm{g}^{-1}: x \mapsto -\sqrt{4-x}, \quad x \le 4$

iii) Required Graph:



(iv) The graph of g^{-1} can be obtained by reflecting the graph of g in the line $y = x$.

$$g(x) = g^{-1}(x) \Rightarrow g(x) = x$$
$$\therefore 4 - x^2 = x$$
$$x^2 + x - 4 = 0$$
$$x = \frac{-1 \pm \sqrt{1 - 4(1)(-4)}}{2}$$
$$= \frac{-1 \pm \sqrt{17}}{2}$$
$$= \frac{-1 + \sqrt{17}}{2} \text{ or } \frac{-1 - \sqrt{17}}{2}$$

Since $x \le 0$, $\therefore x = \dfrac{-1 - \sqrt{17}}{2}$

Note:

For this example, since the graph of $y = x$, $y = g(x)$ and $y = g^{-1}(x)$ cut at a common point, solving $g(x) = g^{-1}(x)$ is equivalent to solving $g(x) = x$ (or $g^{-1}(x) = x$).

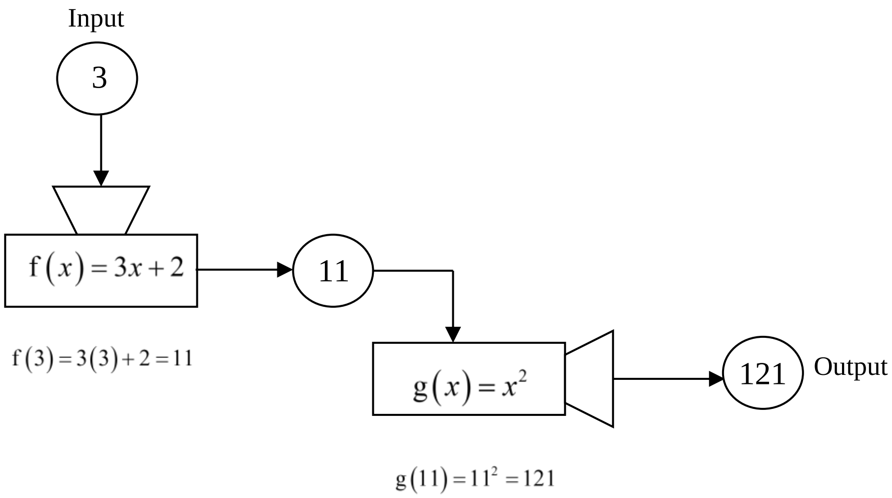
Solving $g(x) = g^{-1}(x)$ directly, which is $4 - x^2 = -\sqrt{4 - x}$, is usually more difficult and is not recommended.

§4 Composite Functions

In this section, we shall investigate the existence of the composition of two functions.

The composition of **f followed by g** is denoted by gf.

For example, with many processes, more than one machine operation is required to produce an output. Suppose an output is the result of one function being applied after another, e.g. $\mathrm{f}(x) = 3x + 2$ followed by $\mathrm{g}(x) = x^2$.



A new function h is formed

i.e. $\mathrm{h}(x) = \mathrm{gf}(x) = \mathrm{g}(3x+2) = (3x+2)^2$ and $\mathrm{h}(3) = \mathrm{g}(\mathrm{f}(3)) = \mathrm{g}(11) = 121$.

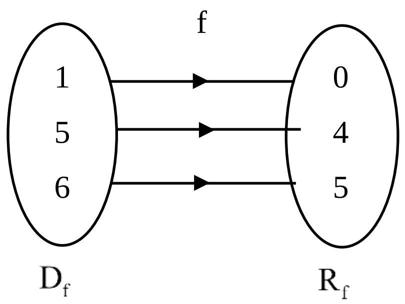
h is said to be the composition of f followed by g, or simply gf.

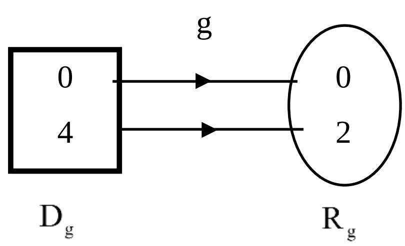
4.1 <u>Existence of Composite Functions</u>

<u>Investigation 1</u>

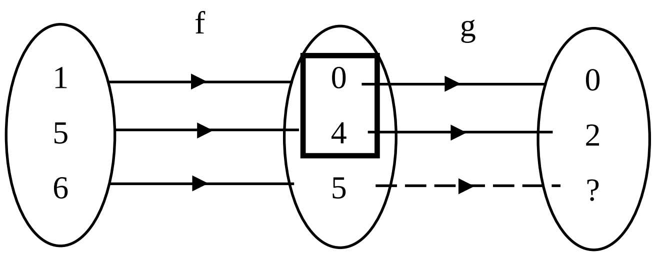
Consider the functions $\mathrm{f}: x \mapsto x-1, \quad x \in \{1, 5, 6\}$

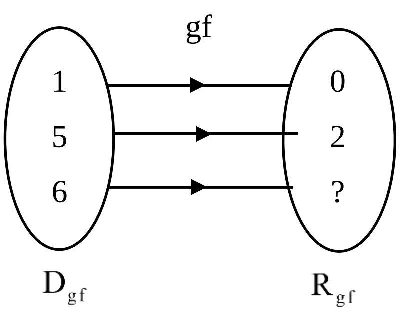
$\mathrm{g}: x \mapsto \sqrt{x}, \quad x \in \{0, 4\}$





Note that $\mathrm{R_f} = \{0, 4, 5\}$ is a NOT a subset of $\mathrm{D_g} = \{0, 4\}$, i.e. $\mathrm{R_f} \not\subseteq \mathrm{D_g}$





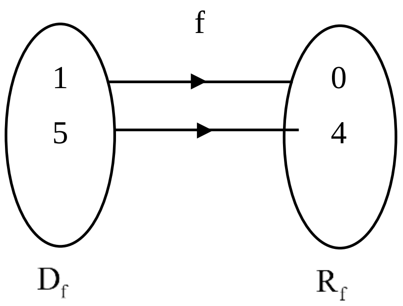
Since there exists an element in its domain with no image, gf is <u>not</u> a function, i.e. gf has an input that does not have an output.

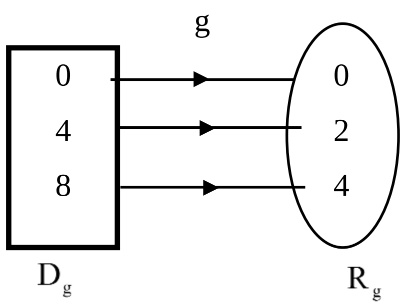
Conclusion: Since $\mathrm{R_f} \not\subseteq \mathrm{D_g}$, gf does not exist.

Investigation 2

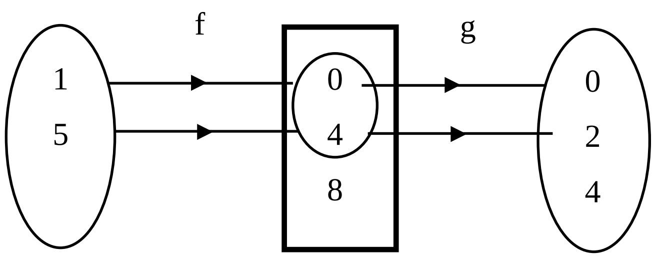
Consider the functions $\mathrm{f}: x \mapsto x-1, \quad x \in \{1, 5\}$

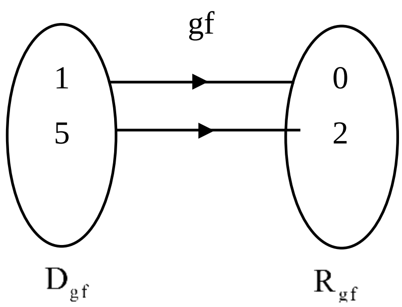
$$\mathrm{g}: x \mapsto \frac{x}{2}, \quad x \in \{0, 4, 8\}$$





Note that $\mathrm{R_f} = \{0, 4\}$ is a subset of $\mathrm{D_g} = \{0, 4, 8\}$, i.e. $\mathrm{R_f} \subseteq \mathrm{D_g}$





Since every element in its domain has exactly 1 image, gf is a function.

$$\mathrm{gf}: x \mapsto \frac{x-1}{2}, \quad x \in \{1, 5\}$$

Conclusion: Since $\mathrm{R_f} \subseteq \mathrm{D_g}$, gf exists.

Q: What can be said about $\mathrm{D_{gf}}$ and $\mathrm{D_f}$?

$$\boxed{\mathrm{D_{gf}} = \mathrm{D_f}}$$

What can be said about the $\mathrm{R_{gf}}$ and $\mathrm{R_g}$?

$$\boxed{\mathrm{R_{gf}} \subseteq \mathrm{R_g}}$$

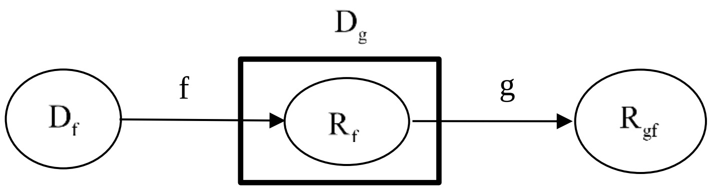
Quick Summary on Composite Functions:

(a) Composite function g of f (gf for all $x \in \mathrm{D_f}$) exists $\Leftrightarrow \mathrm{R_f} \subseteq \mathrm{D_g}$.

(b) gf denotes the composition of f followed by g.

(c) Domain of composite function, $\mathrm{D_{gf}} = \mathrm{D_f}$,

(d) Range of composite function, $\mathrm{R_{gf}} \subseteq \mathrm{R_g}$.



Note:

(i) In general, $\mathrm{fg} \neq \mathrm{gf}$

(ii) $\mathrm{f}^n(x) = \underbrace{\mathrm{f}\left(\mathrm{f}\left(\mathrm{f}\left(\ldots\mathrm{f}\left(\mathrm{f}\right.\right.\right.\right.}_{\text{function f applied } n \text{ times}} \left.\left.(x)\right)\right)$

E.g. $\mathrm{f}^2(x) = \mathrm{ff}(x)$ and $\mathrm{f}^3(x) = \mathrm{fff}(x)$

(iii) $\left[\mathrm{f}(x)\right]^n = \underbrace{\left[\mathrm{f}(x)\right] \times \left[\mathrm{f}(x)\right] \times \cdots \times \left[\mathrm{f}(x)\right]}_{\text{function f multiplied } n \text{ times}}$

$\therefore \mathrm{f}^n(x) \neq \left[\mathrm{f}(x)\right]^n$

(iv) $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$; but they have **different** domains.

(a) $\mathrm{D_{ff^{-1}}} = \mathrm{D_{f^{-1}}} = \mathrm{R_f}$, while

(b) $\mathrm{D_{f^{-1}f}} = \mathrm{D_f}$

Example 7

Function f and g are defined by $\mathrm{f}: x \mapsto -x^2, \quad x \in \mathbb{R}$,

$\mathrm{g}: x \mapsto \mathrm{e}^x, \quad x \in \mathbb{R}$.

Show that gf exists. Find its rule, domain and range.

Solution:

$\mathrm{R_f} = (-\infty, 0] \qquad \mathrm{D_g} = (-\infty, \infty)$

Since $\mathrm{R_f} \subseteq \mathrm{D_g}$, $\therefore$ gf exists.

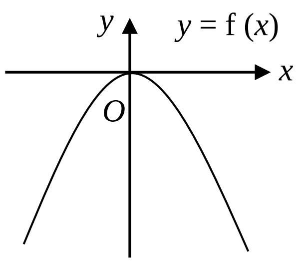
$\mathrm{gf}(x) = \mathrm{g}(-x^2) = \mathrm{e}^{-x^2}$

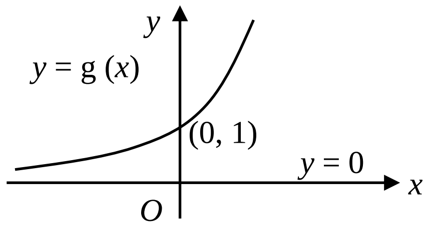
$\mathrm{D_{gf}} = \mathrm{D_f} = \mathbb{R}$

$\therefore \mathrm{gf}: x \mapsto \mathrm{e}^{-x^2}, \quad x \in \mathbb{R}$

Range of gf:

Method 1: Use the graphs of both *y* = f(*x*) and *y* = g(*x*)



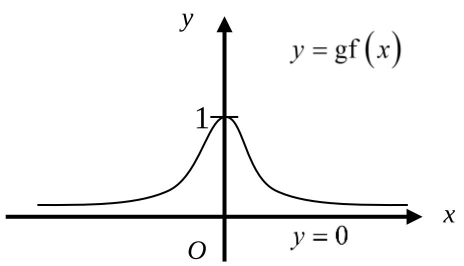


$\mathrm{D_f} = (-\infty, \infty) \xrightarrow{\mathrm{f}} \mathrm{R_f} = (-\infty, 0] \xrightarrow{\mathrm{g}} \mathrm{R_{gf}} = (0, 1]$

Using range of f as the restricted domain of g

$\therefore \mathrm{R_{gf}} = (0, 1]$

Method 2: Using the graph of *y* = gf(*x*).



$\therefore \mathrm{R_{gf}} = (0, 1]$

Note: This method is not always suitable.

Note:

Q: When will $\mathrm{R_{gf}} = \mathrm{R_g}$?

A: $\mathrm{R_{gf}} = \mathrm{R_g}$ when $\mathrm{R_f} = \mathrm{D_g}$

Example 8 [2014(9740)/I/1]

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{1}{1-x}, \quad x \in \mathbb{R}, x \neq 1, x \neq 0.$$

(i) Show that $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$. [4]

(ii) Find $\mathrm{f}^3(x)$ in simplified form. [1]

Solution:

(i) Note that $D_{\mathrm{f}} = \mathbb{R} \setminus \{0,1\}$ and $R_{\mathrm{f}} = \mathbb{R} \setminus \{0,1\}$.

$$\mathrm{f}(x) = \frac{1}{1-x}$$

$$\mathrm{f}^2(x) = \mathrm{f}\left[\mathrm{f}(x)\right]$$

$$= \frac{1}{1-\left(\frac{1}{1-x}\right)}$$

$$= \frac{1}{\left(\frac{-x}{1-x}\right)}$$

$$= \frac{1-x}{-x}$$

$$= \frac{x-1}{x}, x \in \mathbb{R}, x \neq 1, x \neq 0$$

Let $y = \frac{1}{1-x}$

$$y - xy = 1$$

$$xy = y - 1$$

$$x = \frac{y-1}{y}$$

$$\therefore \mathrm{f}^{-1}(x) = \frac{x-1}{x}, x \in \mathbb{R}, x \neq 1, x \neq 0$$

$\therefore \mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$ (shown)

(ii) [Using concept of $\mathrm{f}\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$]

Since $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$,

$$\mathrm{f}^3(x) = \mathrm{f}^2\mathrm{f}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x, \ x \in \mathbb{R}, x \neq 1, x \neq 0$$

OR

$$\mathrm{f}^3(x) = \mathrm{f}\mathrm{f}^2(x) = \mathrm{f}\mathrm{f}^{-1}(x) = x, \ x \in \mathbb{R}, x \neq 1, x \neq 0$$

<u>Alternative (Longer method)</u>

$$\mathrm{f}^3(x) = \mathrm{f}^2\left[\mathrm{f}(x)\right]$$

$$= \frac{\left(\frac{1}{1-x}\right) - 1}{\left(\frac{1}{1-x}\right)}$$

$$= \frac{\frac{x}{1-x}}{\left(\frac{1}{1-x}\right)}$$

$$= x, \quad x \in \mathbb{R}, x \neq 1, x \neq 0$$

or

$$\mathrm{f}^3(x) = \mathrm{f}\left[\mathrm{f}^2(x)\right]$$

$$= \frac{1}{1-\left(\frac{x-1}{x}\right)}$$

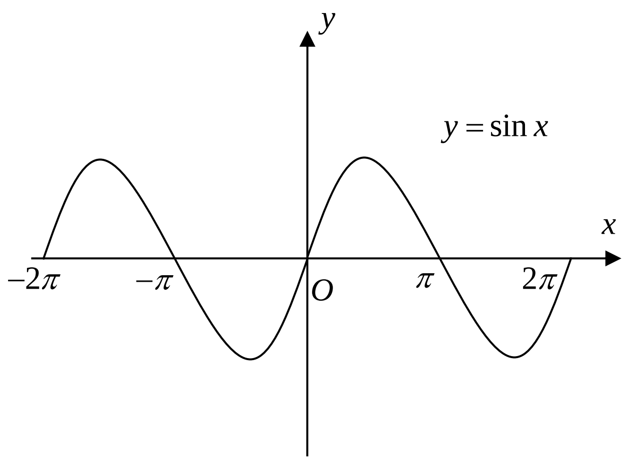
$$= \frac{1}{\left(\frac{1}{x}\right)}$$

$$= x \quad , x \in \mathbb{R}, x \neq 1, x \neq 0$$

§5 Periodic Functions

A function that repeats its value in regular intervals or periods is said to be a periodic function.

Investigation:



Given that $\mathrm{f}(x) = \sin x, x \in \mathbb{R}$. What do you observe about the graph of sin *x*?

Do you think f is a periodic function?

Answer:
Notice that the graph of sin *x* repeats itself at regular intervals.

Yes. f is a periodic function since sin *x* repeats itself every 2π radians i.e. $\sin x = \sin(x + 2\pi)$.

Definition

A function f is periodic if there is a positive constant *a* such that

$$\mathrm{f}(x) = \mathrm{f}(x+a) \text{ for all } x \in \mathrm{D_f}.$$

The smallest such value of a is the **period** of f.
You may visualise *a* as the "width" of the basic pattern of the function.

Other examples of periodic graphs include the trigonometric graphs of $y = \cos x$, which repeats itself every 2π radians, and $y = \tan x$ which repeats itself every of π radians. 2π and π radians are known as the *periods* of $\cos x$ and $\tan x$ respectively.

Q: Can we say that the period for sin *x* and cos *x* is 4π or 6π radians instead (since it also repeats itself every 4π or 6π radians)?
Ans: No, because 4π or 6π radians is not the smallest value.

Example 9

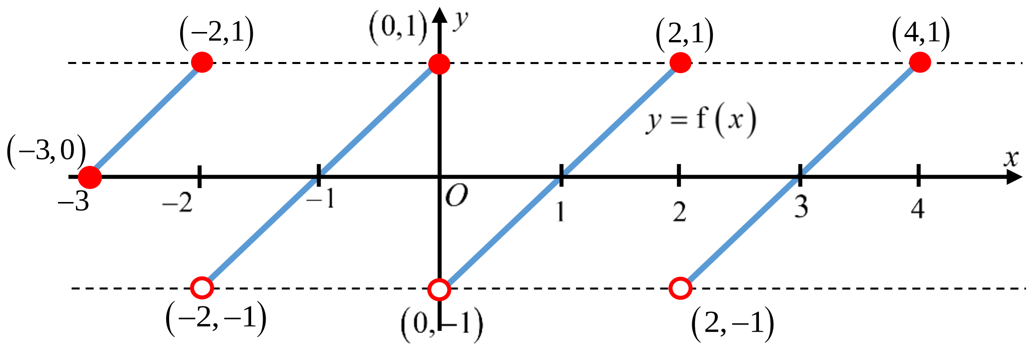
If $\mathrm{f}(x)=x-1$ for $0<x\le 2$ and $\mathrm{f}(x+2)=\mathrm{f}(x)$ for all real values of *x*, find the values of $\mathrm{f}(4)$, $\mathrm{f}(-2)$ and $\mathrm{f}(-3)$. Hence sketch the graph of $\mathrm{f}(x)$ for $-3\le x\le 4$.

Solution:

$\mathrm{f}(4)=\mathrm{f}(2)=1$

$\mathrm{f}(-2)=\mathrm{f}(2)=1$

$\mathrm{f}(-3)=\mathrm{f}(1)=0$



Note: The function is periodic with a period of 2. So we draw $\mathrm{f}(x)=x-1$ for $0<x\le 2$. This pattern then repeats at every two units.

Example 10 [N2009/I/4(i), (ii)]

It is given that

$$\mathrm{f}(x)=\begin{cases}7-x^2 & \text{for } 0<x\le 2,\\ 2x-1 & \text{for } 2<x\le 4,\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+4)$ for all real values of *x*.

(i) Evaluate $\mathrm{f}(27)+\mathrm{f}(45)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-7\le x\le 10$. [3]

Solution:

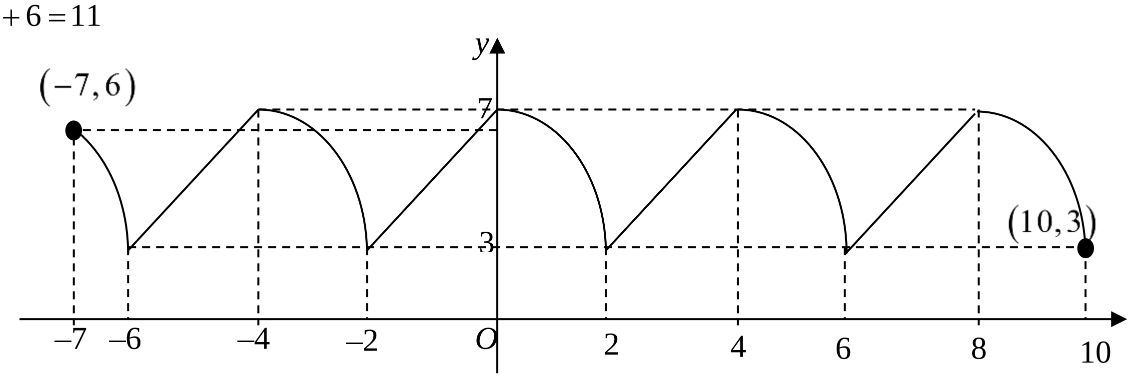
(i)

$\mathrm{f}(27)+\mathrm{f}(45)$

$=\mathrm{f}(3)+\mathrm{f}(1)$

$=5+6=11$

(ii)

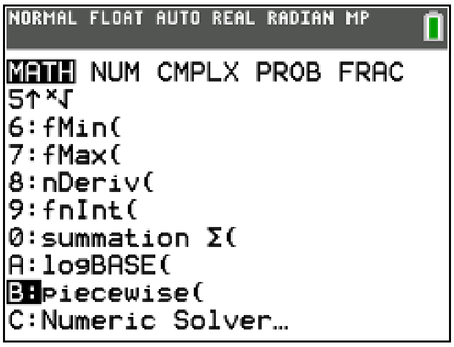


GC Keystrokes to Draw Piecewise Function

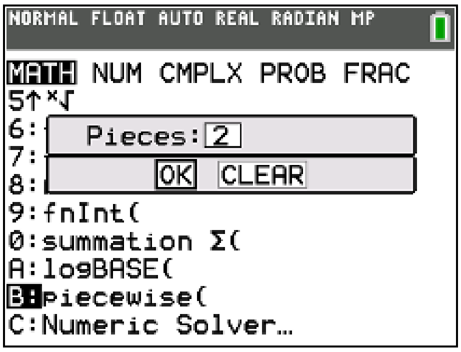
Example: To graph $\mathrm{f}(x)=\begin{cases} 7-x^2 & \text{for } 0<x\le 2, \\ 2x-1 & \text{for } 2<x\le 4. \end{cases}$

Steps on GC

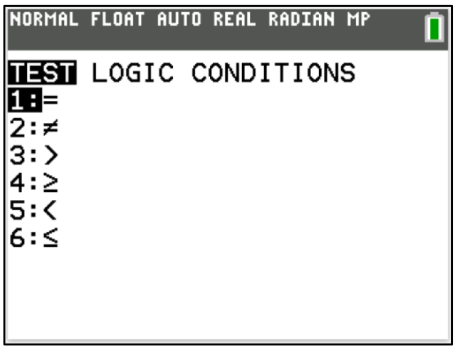
1. Go to [Y=] and select [MATH], option B: piecewise

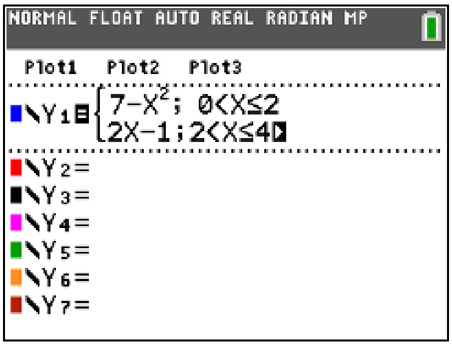


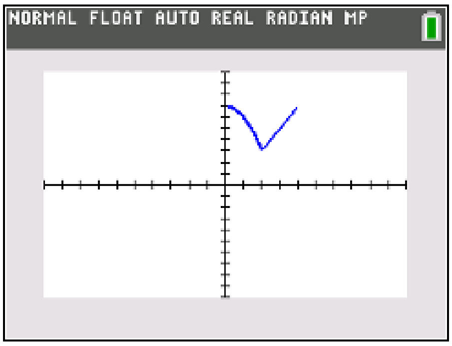
2. Select the required number of pieces for the piecewise function.



3. Key in the required function. For inequalities, select [2nd][MATH], and find them under **TEST**.







SUMMARY

1. A **function** $\mathrm{f}: X \to Y$ is a relation where f maps each element $x \in X$ to **exactly one** element $y \in Y$. (In other words, every input has exactly one output.)

2. Finding range of a function (Know how to use GC)
   (a) Sketch the graph according to its **domain**
   (b) Take note of asymptotes and turning points

3. **ONE-ONE FUNCTIONS**
   (a) **Horizontal Line test (MUST SKETCH GRAPH)**
   - The function f is one-one if **any** horizontal line $y = b$ ($b \in \mathbb{R}$) cuts the graph of f **at most once.**
   - If not one-one: find a counter-example (usually a line that cuts the graph twice). Remember to sketch this line onto the graph.

   (b) All **strictly increasing and decreasing functions** are **one-one**.
   - Use differentiation to prove increasing $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ or decreasing $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$.
   - Algebraic Approach
     - A function f is one-one if $\mathrm{f}(x_1) = \mathrm{f}(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in \mathrm{D_f}$.

4. **INVERSE FUNCTIONS**
- The inverse of a function f is denoted by f^{-1}. f^{-1} exists $\Leftrightarrow$ f is one-one.
- $\mathrm{D}_{\mathrm{f}^{-1}} = \boxed{\mathrm{R_f}}$ and $\mathrm{R}_{\mathrm{f}^{-1}} = \boxed{\mathrm{D_f}}$
- The graph of f^{-1} can be obtained by $\boxed{\text{reflecting}}$ the graph of f about the line $\boxed{y = x}$
- If in the process of finding inverse, $\pm$ occurs, use the domain of f to determine which expression to choose.

5. **COMPOSITE FUNCTIONS**
- If f and g are two functions such that $\boxed{\mathrm{R_f} \subseteq \mathrm{D_g},}$ then the composite function g of f exists and is defined by $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$ for all $x \in \mathrm{D_f}$.
- Domain of composite function gf, $\mathrm{D_{gf}} = \boxed{\mathrm{D_f}}$
- Range of composite function gf,

  (a) Method 1: Sketch both functions and use $\mathrm{D_f} \xrightarrow{\mathrm{f}} \mathrm{R_f} \xrightarrow{\mathrm{g}} \mathrm{R_{gf}}$

  [**Using $\mathrm{R_f}$ as restricted domain of g]

  (b) Method 2: Find the composite function and sketch the graph, keeping in mind the domain of the first function

6. $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$; but they have **different** $\boxed{\text{domains.}}$

   (a) $\mathrm{D}_{\mathrm{ff}^{-1}} = \mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R_f}$, (b) $\mathrm{D}_{\mathrm{f}^{-1}\mathrm{f}} = \mathrm{D_f}$

H2 Mathematics (9758)
Chapter 3 Functions
Discussion Questions

<u>Level 1</u>

1 Sketch the graphs of each of the following functions. State its domain and give its corresponding range.

(a) $\mathrm{f}: x \mapsto \dfrac{2x-3}{x-1}, x \in \mathbb{R}, x > 1$

(b) $\mathrm{g}: x \mapsto -(x-2)^2 + 4, x \in \mathbb{R}, x \leq 2$

(c) $\mathrm{h}: x \mapsto \ln(x-1), x \in \mathbb{R}, 1 < x < 3$

2 The function f is defined by

$$\mathrm{f}: x \mapsto -(x-2)^2 + 4, \ x \in \mathbb{R}, \ x > 0.$$

(i) Sketch the graph of f.

(ii) State the domain and range of f.

(iii) Explain why f^{-1} does not exist.

3 The function f is defined as $\mathrm{f}: x \mapsto \dfrac{2x-3}{x-1}, x \in \mathbb{R}, x > 1$.

(i) By sketching the graph of $y = \mathrm{f}(x)$, explain why f^{-1} exists.

(ii) Find f^{-1} in a similar form.

4 Functions g and h are defined by

$$\mathrm{g}: x \mapsto x^2 + 1, \ x \in \mathbb{R}, x \geq 0,$$
$$\mathrm{h}: x \mapsto 2x + 3, \ x \in \mathbb{R}, x > 2.$$

(i) Show that gh exists.

(ii) Find gh in a similar form.

(iii) Find the range of gh.

Level 2

5 The function g is defined as $\mathrm{g}: x \mapsto \ln\left(x^2\right), x \in \mathbb{R}, x < 0$.

(i) State the domain and range of g.

(ii) Give a reason why g^{-1} exists.

(iii) Find the rule, domain and range of g^{-1}.

6 **2013/CJC Prelim/II/3 (modified)**

Functions f and g are defined by

$$\mathrm{f}: x \mapsto (x-2)^2-1, \quad x \in \mathbb{R},\ x<2$$
$$\mathrm{g}: x \mapsto \ln(x^2+1), \quad x \in \mathbb{R}$$

Only one of the composite functions fg and gf exists. Give the rule and domain of the composite function that exists, and explain why the other composite does not exist.

7 **2018/ACJC Promo/Q9(part)**

Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{x+3}{4-x}, \quad x \in \mathbb{R},\ x \neq 4,$$
$$\mathrm{g}: x \mapsto \frac{1}{x}, \quad x \in \mathbb{R},\ x < 0.$$

(i) Show that the composite function fg exists. [1]

(ii) Find the range of fg. [2]

(iii) Find an expression for $\mathrm{fg}\left(x\right)$ and hence, or otherwise, find $\left(\mathrm{fg}\right)^{-1}\left(\frac{1}{2}\right)$. [3]

8 It is given that $\mathrm{f}\left(x\right)=\left(x-1\right)^2+2,\ x \in \mathbb{R},\ 0 \le x < 2$ and that $\mathrm{f}\left(x\right)=\mathrm{f}\left(x+2\right)$ for all real values of *x*.

(i) State the period of f.

(ii) Evaluate $\mathrm{f}\left(1\right)$ and $\mathrm{f}\left(-2\right)$.

(iii) Sketch the graph of $y=\mathrm{f}\left(x\right)$ for $-2 < x < 3$.

9 **2010/MJC JC1 MYE/I/5 (Modified)**

The function f is defined by

$$\mathrm{f}: x \mapsto x^2-4x-5, \quad x \ge 2.$$

(i) Show that f^{-1} exists. [1]

(ii) Find f^{-1} in a similar form. [3]

(iii) Write down the equation of the line in which the graph of $y=\mathrm{f}\left(x\right)$ must be reflected to obtain the graph of $y=\mathrm{f}^{-1}\left(x\right)$. [1]

(iv) Sketch the graphs of $y=\mathrm{f}\left(x\right)$ and $y=\mathrm{f}^{-1}\left(x\right)$ on the same diagram. Hence find the exact solution of the equation $\mathrm{f}\left(x\right)=\mathrm{f}^{-1}\left(x\right)$. [4]

Level 3

10 2018/ACJC Promo/Q9(modified)

The function h is defined by

$$\mathrm{h}: x \mapsto \left|\frac{x+3}{4-x}\right|, \quad x \in \mathbb{R},\ x \neq 4.$$

(i) Sketch the graph of h and state its range. [2]

(ii) Explain why the inverse function h^{-1} does not exist. [1]

(iii) The function h^{-1} exists if the domain of h is restricted to $x \leq k$. State the greatest value of k. [1]

(iv) Using the domain in **(iii)**, find $\mathrm{h}^{-1}(x)$ and state the domain of h^{-1}. [4]

11 2013/VJC Prelim/II/3 (Modified)

The function f is defined by

$$\mathrm{f}: x \mapsto x^2 - 2x + 2, \quad 1 < x \leq 3.$$

(i) Sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$ on a single diagram, indicating clearly the domains of the respective functions. [3]

(ii) Without using the graphing calculator, find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

(iii) State the range of values of x satisfying the equation $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{ff}^{-1}(x)$. [1]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{x+a}{x+1}, \quad x \geq 0,$$

where a is a constant and $a > 1$.

(iv) Show that the composite function gf exists and find, in exact form, the range of gf. [4]

12 At a local meteorological station, the daily average temperature recorded by the instrument is in degrees Fahrenheit, $^{\circ}\mathrm{F}$. The meteorological station master wants to record the temperature in degrees Celsius, $^{\circ}\mathrm{C}$ and he uses the following function c to do the conversion:

$$\mathrm{c}: x \mapsto \frac{5}{9}(x - 32), \quad x > -459.67.$$

(i) Given that the average temperature of a particular day is given as $50\,^{\circ}\mathrm{F}$, express the temperature in terms of $^{\circ}\mathrm{C}$. [1]

(ii) Define c^{-1} and explain the significance of this function in the context of the question. [3]

A physicist wants to record the temperature in Kelvin, K and he has the following function k which converts temperature in degrees Celsius, $^{\circ}\mathrm{C}$ to Kelvin, K:

$$\mathrm{k}: x \mapsto x + 273.15, \quad x > -273.15.$$

(iii) The physicist wants to convert the temperature from degrees Fahrenheit, $^{\circ}\mathrm{F}$ to Kelvin, K directly. Define a composite function to meet his requirement [2]

13 2014/DHS Promo (Modified)

The function f is defined by

$$\mathrm{f}(x)=\begin{cases}2x+3 & \text{for } 0<x\le 4,\\ -4x+27 & \text{for } 4<x\le 6,\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+6)$ for all real values of x.

(i) Find the value of $\mathrm{f}(-17)+\mathrm{f}(17)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-8\le x\le 13$. [3]

14 2016(9740)/I/10(b)

The function g, with domain the set of non-negative integers, is given by

$$\mathrm{g}(n)=\begin{cases}1 & \text{for } n=0,\\ 2+\mathrm{g}\left(\frac{1}{2}n\right) & \text{for } n \text{ even},\\ 1+\mathrm{g}(n-1) & \text{for } n \text{ odd}.\end{cases}$$

(i) Find $\mathrm{g}(4)$, $\mathrm{g}(7)$ and $\mathrm{g}(12)$. [3]

(ii) Does g have an inverse? Justify your answer. [2]

Answer Key
1(a) $\mathrm{D_f}=(1,\infty)$, $\mathrm{R_f}=(-\infty,2)$ **(b)** $\mathrm{D_g}=(-\infty,2]$, $\mathrm{R_g}=(-\infty,4]$ **(c)** $\mathrm{D_h}=(1,3)$, $\mathrm{R_h}=(-\infty,\ln 2)$
2 (ii) $\mathrm{D_f}=(0,\infty)$, $\mathrm{R_f}=(-\infty,4]$
3 (ii) $\mathrm{f}^{-1}:x\mapsto \dfrac{x-3}{x-2},\ x\in\mathbb{R},\ x<2$
4 (ii) $\mathrm{gh}:x\mapsto (2x+3)^2+1,\ x\in\mathbb{R},\ x>2$ **(iii)** $(50,\infty)$
5 (i) $\mathrm{D_g}=(-\infty,0)$, $\mathrm{R_g}=(-\infty,\infty)$ **(iii)** $\mathrm{g}^{-1}:x\mapsto -\sqrt{\mathrm{e}^x},\ x\in\mathbb{R}$, $\mathrm{R_{g^{-1}}}=(-\infty,0)$
6 $\mathrm{gf}:x\mapsto \ln\left[\left((x-2)^2-1\right)^2+1\right],\ x\in\mathbb{R},\ x<2$
7 (ii) $\mathrm{R_{fg}}=\left(-1,\dfrac{3}{4}\right)$ **(iii)** $(\mathrm{fg})^{-1}\left(\dfrac{1}{2}\right)=-\dfrac{3}{2}$
8 (i) 2 **(ii)** $\mathrm{f}(1)=2$, $\mathrm{f}(-2)=3$
9 (ii) $\mathrm{f}^{-1}:x\mapsto 2+\sqrt{x+9},\quad x\geq -9$ **(iii)** $y=x$ **(iv)** $x=\dfrac{5+3\sqrt{5}}{2}$
10 (iii) Greatest $k=-3$ **(iv)** $\mathrm{h}^{-1}(x)=\dfrac{4x+3}{x-1}$, $\mathrm{D_{h^{-1}}}=[0,1)$
11 (ii) $x=2$ **(iii)** $1<x\leq 3$ **(iv)** $\mathrm{R_{gf}}=\left[\dfrac{5+a}{6},\ \dfrac{1+a}{2}\right)$
12 (i) $10\ ^{\circ}\mathrm{C}$ **(ii)** $\mathrm{c}^{-1}:x\mapsto \dfrac{9}{5}x+32,\quad x>-273.15$ **(iii)** $\mathrm{kc}:x\mapsto \dfrac{5}{9}(x-32)+273.15,\quad x>-459.67$
13 (i) 12
14 (b)(i) $\mathrm{g}(4)=6$, $\mathrm{g}(7)=8$, $\mathrm{g}(12)=9$

H2 Mathematics (9758)
Chapter 3 Functions
Extra Practice Questions

1 **2018/NYJC Promo/7 (Modified)**

The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \mathrm{e}^{\sqrt{x}}, \qquad x \in \mathbb{R}, x \geq 0,$$

$$\mathrm{g}: x \mapsto (x-1)^2, \qquad x \in \mathbb{R}, x < 2.$$

(i) Explain why g does not have an inverse. [1]

(ii) Show that fg exists. [2]

(iii) Find the range of fg. [2]

2 **2008(9740)/II/4**

The function f is defined by $\mathrm{f}: x \mapsto (x-4)^2 + 1$ for $x \in \mathbb{R}, x > 4$.

(i) Sketch the graph of $y = \mathrm{f}(x)$. Your sketch should indicate the position of the graph in relation to the origin. [2]

(ii) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [3]

(iii) On the same diagram as in part **(i)**, sketch the graph of $y = \mathrm{f}^{-1}(x)$. [1]

(iv) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$, and hence find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [5]

3 **2009/TPJC Prelim/I/9**

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \ln(x^2), \qquad x \in \mathbb{R}, x \neq 0,$$

$$\mathrm{g}: x \mapsto \mathrm{e}^{|x|} + 1, \qquad x \in \mathbb{R}.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$ and state the range of the function f. [3]

(ii) Show that the composite function fg exists. [2]

(iii) Explain why f^{-1} does not exist. [2]

(iv) The function f has an inverse if its domain is restricted to $x < a$. State the greatest value of a. Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

4 **2018/SAJC Promo/6**

The functions f, g and h are defined by

$$\mathrm{f} : x \mapsto \frac{1}{(1+x)(3-x)},\ x \in \mathbb{R},\ x<1$$

$$\mathrm{g} : x \mapsto \ln(2-x),\ x \in \mathbb{R},\ x<2$$

$$\mathrm{h} : x \mapsto 4\mathrm{e}^{-x},\ x \in \mathbb{R},\ x>\ln 2$$

(i) Find $\mathrm{f}^{-1}(x)$. [3]

(ii) Explain why the composite function hg does not exist. [2]

(iii) Find $\mathrm{gh}(x)$ and find the range of gh. [3]

(iv) Hence, without finding the inverse of gh, find the **exact** value of $(\mathrm{gh})^{-1}\left(\ln\frac{4}{3}\right)$. [3]

5 **2009(9740)/II/3(modified)**

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{ax}{bx-a},\ \text{ for } x \in \mathbb{R},\ x \neq \frac{a}{b},$$

where a and b are non-zero constants, and $a \neq b$.

(i) Find $\mathrm{f}^{-1}(x)$. Hence or otherwise find $\mathrm{f}^{2}(x)$ and state the range of f^{2}. [5]

(ii) State the value of $\mathrm{f}^{2023}(1)$ in terms of a and b. [3]

(iii) The function g is defined by $\mathrm{g} : x \mapsto \frac{1}{x}$ for all real non-zero x. State whether the composite function fg exists, justifying your answer. [2]

(iv) Solve the equation $\mathrm{f}^{-1}(x) = x$. [3]

6 Given $\mathrm{f}(x) = \begin{cases} x^2 & \text{for } 0<x\leq 1, \\ 3-2x & \text{for } 1<x\leq 3, \end{cases}$

and $\mathrm{f}(x+3) = \mathrm{f}(x)$ for all values of x.

Sketch the graph for $-4 \leq x \leq 7$. Evaluate $\mathrm{f}(26)$.

7 2009/NYJC Prelim/I/2

The functions f and g are defined by

$$\mathrm{f}: x \to x^2 - 1, \quad x \in \mathbb{R}$$
$$\mathrm{g}: x \to \sqrt{x+4}, \quad x \in \mathbb{R}^+$$

(i) Show that the composite function fg exists and define fg in similar form. [2]

(ii) If h is a function defined such that h(x) = gfg(x) for all $x > 0$, show that the composite function gfg exists. Hence, find an expression for h(x). [2]

(iii) Solve $\mathrm{h}(x) = \mathrm{g}^{-1}\mathrm{g}(x)$, giving your answer in exact form. [2]

8 2013/IJC Prelim/I/13 (Modified)

It is given that

$$\mathrm{f}(x) = \begin{cases} \dfrac{3}{(2x+3)^2 - 2} & \text{for } -2 \le x < -1, \\ -3 & \text{for } -1 \le x < 0, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+2)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-\dfrac{5}{2} \le x \le \dfrac{5}{2}$. [3]

(ii) Find the exact value of $\mathrm{f}(-3) + \mathrm{f}(4.5)$. [4]

The function h is defined by

$$\mathrm{h}: x \mapsto \frac{3}{(2x+3)^2 - 2} \qquad \text{for } -2 \le x < a.$$

(iii) Write down the greatest value of a such that h^{-1} exists. [1]

Assume that a takes the value found in part **(iii)**.

(iv) Sketch, on a single diagram, the graphs of $y = \mathrm{h}(x)$, $y = \mathrm{h}^{-1}(x)$ and $y = \mathrm{h}^{-1}\mathrm{h}(x)$. [2]

(v) Explain why the x-coordinate of the point of intersection of the curves in part **(iv)** satisfies the equation

$$4x^3 + 12x^2 + 7x - 3 = 0.$$

and find the value of this x-coordinate, correct to 4 significant figures. [3]

9 2016/H2 Specimen Paper/II/1

The function f is defined as follows:

$$\mathrm{f}: x \mapsto 3\cos x - 2\sin x, \qquad x \in \mathbb{R}, -\pi \le x < \pi.$$

(i) Write $\mathrm{f}(x)$ as $R\cos(x+\alpha)$, where R and α are constants to be found. Hence, or otherwise, find the range of f and sketch the curve. [4]

(ii) If the domain of f is further restricted, the function f^{-1} exists. Write down the largest value of b for which f^{-1} exists in the interval $[-\alpha, b]$. Find $\mathrm{f}^{-1}(x)$. [3]

Answer Key

No	**Year**	**JC/CI**	**Answers**
1	**2018**	**NYJC**	(iii) $R_{fg} = [1, \infty)$
2	**2008**	**A Level**	(ii) $f^{-1}(x) = 4 + \sqrt{x-1}$, $D_{f^{-1}} = (1, \infty)$ (iv) $y = x$; $x = \dfrac{9+\sqrt{13}}{2}$
3	**2009**	**TPJC**	(i) $R_f = \mathbb{R}$ (iv) $a = 0$ $f^{-1} : x \to -e^{\frac{x}{2}}, \quad x \in \mathbb{R}$ $D_{f^{-1}} = \mathbb{R}$
4	**2018**	**SAJC**	(i) $f^{-1}(x) = 1 - \sqrt{4 - \dfrac{1}{x}}$ (iii) $gh(x) = \ln(2 - 4e^{-x})$ $R_{gh} = (-\infty, \ln 2)$ (iv) $a = \ln 6$
5	**2009**	**A Level**	(i) $f^{-1} : x \mapsto \dfrac{ax}{bx-a}, x \in \mathbb{R}, x \neq \dfrac{a}{b}$ $f^2 : x \mapsto x, x \in \mathbb{R}, x \neq \dfrac{a}{b}$; $R_{f^2} = \mathbb{R} \setminus \left\{\dfrac{a}{b}\right\}$ (ii) $\dfrac{a}{b-a}$ (iv) $x = 0$ or $x = \dfrac{2a}{b}$
6			–1
7	**2009**	**NYJC**	(i) $fg : x \mapsto x + 3, \quad x \in \mathbb{R}^+$ (ii) $h(x) = \sqrt{x+7}$ (iii) $x = \dfrac{1}{2}(1 + \sqrt{29})$
8	**2013**	**IJC**	(ii) -4.5 (iii) $a = -\dfrac{3}{2}$ (v) $x = -1.781$
9	**2016**	**Specimen**	(i) $\sqrt{13}\cos\left(x + \tan^{-1}\dfrac{2}{3}\right)$; $R_f = \left[-\sqrt{13}, \sqrt{13}\right]$ (ii) $b = \pi - \tan^{-1}\dfrac{2}{3}$ $f^{-1} : x \mapsto \cos^{-1}\dfrac{x}{\sqrt{13}} - \tan^{-1}\dfrac{2}{3}, -\sqrt{13} \leq x \leq \sqrt{13}$