MOE H3 Math Mathematical Proofs and Reasoning

Problem Set 1

1. Determine whether each of the following statements is true or false? Give a direct proof if it is true, and give a counter-example if it is false.
   (a) The set of prime numbers is closed under addition.
   (b) The set of positive rational numbers is closed under division.

2. Let a, b and c be nonzero integers. Use the definition of divisibility and write down a direct proof for each of the following statements. (Indicate every step clearly.)
   (a) If a divides b, then ac divides bc.
   (b) If a divides b and b divides a, then $a = \pm b$.

3. Show that 3 divides $n(n+1)(2n+1)$ for any integer n.

4. Prove that for all integers a, if the remainder is NOT 2 when a is divided by 4, then $4 \mid a^3 + 23a$.

5. For any integer $n > 1$, let the standard factored form of n be given by
$$n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}.$$
Prove that n is a perfect square if and only if $k_1, k_2, \dots, k_r$ are all even integers.

6. Prove the following bi-conditional statement:
For all integers a and b, $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.

7. Is the following statement true?
If $\dfrac{n(n+1)}{2}$ is odd, then so is $\dfrac{(n+1)(n+2)}{2}$.

8. Let a, j, r, s be fixed integers. Prove: if j divides $r + s$, then
$$\exists\, b \in \mathbb{Z}\ (a = bj + r) \quad \Longleftrightarrow \quad \exists\, c \in \mathbb{Z}\ (a = cj - s).$$

9. Prove that, if $3 \nmid a$, then $3 \mid a^2 + 5$.

10. Prove the following statement:
If m and n are any two integers with the same parity, then $4 \mid m^2 - n^2$.

Hints

1. *Hint for Question 3.* Consider cases: $n = 3k + r$, $r = 0, 1, 2$.
2. *Hint for Question 4.* Consider cases: $a = 4k + r$, $r = 0, 1, 3$.
3. *Hint for Question 5.* For the "only if" part, may use the uniqueness of prime factorization.
4. *Hint for Question 6.* For the "only if" part, consider prove by contrapositive.
5. *Hint for Question 9.* Consider cases for a.