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DISCRETE RANDOM VARIABLES AND BINOMIAL DISTRIBUTION

SYLLABUS

- Concept of discrete random variables, probability distributions, expectations and variances.
- Concept of the binomial distribution $B(n, p)$ as an example of a discrete probability distribution.
- Use of $B(n, p)$ as a probability model, including conditions under which the binomial distribution is a suitable model.
- Use of the mean and variance of the binomial distribution.

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LECTURE 1

Lesson Outline

- Introduction to discrete random variables: definition, probability distribution function, cumulative distribution function and expectation.
-

1 DEFINITIONS

A **random variable** is a function that associates every outcome of an experiment with a unique numerical value.

For example, consider the experiment of tossing a fair die. The number that shows up on the die roll can be denoted by a random variable X , where the value of X may correspond to the six outcomes of the die roll: 1, 2, ..., 6.

A **discrete random variable** is a random variable which may only take on a countable number of distinct numerical values such as 0, 1, 2, The random variable X in the example above is a discrete random variable.

A **continuous random variable** is a random variable which can take on an uncountable number of numerical values (i.e. a range of values). For example, if the random variable Y denotes any real number between 0 and 1, then Y is a continuous random variable.

Example 1 (Concept of Random Variables)

X denotes a random variable. Write down the possible values of X and conclude whether it is a discrete or continuous random variable.

Random Variable X	Possible Values of X	Discrete or Continuous
The number of students sitting next to a chosen student.		
The number of pens in a student's pencil case.		
The number of Instagram followers that a student currently has.		
The amount of time a student spent on homework yesterday.		
The height of a student.		

Notes

- We use upper case alphabet letters to represent random variables. We use lower case alphabet letters to represent the values these random variables can take.
- The expression $P(X = x)$ denotes the probability of the random variable X taking on the value x .

Example 2

Let the random variable X denote ‘the number of fours obtained when two dice are thrown’. Find

- (i) $P(X = 0)$ (ii) $P(X = 1)$ (iii) $P(X = 2)$

Solution

(i) $P(X = 0) =$

(ii) $P(X = 1) =$

(iii) $P(X = 2) =$ ■

2 PROBABILITY DISTRIBUTION FUNCTION

Let X be a discrete random variable.

The **probability distribution function (pdf)** (or **probability function**) of X is defined by $P(X = x)$, where x denotes all the possible values that the random variable X can take.

The pdf of X may be presented in a table form or in a formula that lists all the probabilities of the possible values of X . In this case, we may consider the table as the **probability distribution** of X .

For example, let the random variable X denote the number of heads obtained when tossing a fair coin once. Then the probability distribution of X can be represented by either of the following:

This probability distribution has a name - the Bernoulli distribution.

x	0	1
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{2}$

or
$$P(X = x) = \begin{cases} \frac{1}{2}, & x = 0 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$$

Properties of the probability distribution $p(x)$ of a discrete random variable X

1. For all values of x , $0 \leq p(x) \leq 1$.

2. $\sum_{\text{all } x} p(x) = 1$.

That is, if the variable X can take on values $x_1, x_2, \dots, x_n$ such that $P(X = x_1) = p_1$, $P(X = x_2) = p_2$ and so on, then

$$p_1 + p_2 + \dots + p_n = 1.$$

Example 3

Two fair tetrahedral dice, each with faces labeled 1, 2, 3 and 4, are thrown. If X is the random variable denoting ‘the sum of the outcomes on the bottom faces of the two tetrahedral dice’, find the probability distribution of X .

Solution

	1	2	3	4
1				
2				
3				
4				

$$P(X = \quad) = P(X = \quad) =$$

$$P(X = \quad) = P(X = \quad) =$$

$$P(X = \quad) = P(X = \quad) =$$

$$P(X = \quad) =$$

■

Example 4

The probability function of a discrete random variable Y is given by $P(Y = y) = cy^2$, for $y = 0, 1, 2, 3, 4$. Find the value of the constant c .

Solution

$$\begin{aligned}\sum_{y=0}^4 cy^2 &= 1 \\ c(0^2 + 1^2 + 2^2 + 3^2 + 4^2) &= 1 \\ 30c &= 1 \Rightarrow c = \frac{1}{30}\end{aligned}$$

3 CUMULATIVE DISTRIBUTION FUNCTION (NOT IN SYLLABUS)

For a discrete random variable X , the **cumulative probabilities** are obtained by summing all the probabilities up to a particular value.

If X is a discrete random variable with probability function $P(X = x)$ for $x = x_1, x_2, \dots, x_n$, then the **cumulative distribution function** of X is given by $F(t)$, where

$$F(t) = P(X \leq t) = \sum_{x \leq t} P(X = x), \text{ where } t \in \mathbb{R}$$

Note

- $P(X > a) = 1 - P(X \leq a)$.

4 EXPECTATION

The **expectation** of X (or the **mean/expected value** of X), written as $E(X)$, is defined as

$$E(X) = \sum_{\text{all } x} x P(X = x).$$

Notes

1. $E(X)$ is a constant.
2. $E(X)$ represents the mean outcome if the same experiment is repeated under the same conditions a large number of times.

Example 5 (Frequency Table)

The number of books borrowed by 120 children in a month are shown below.

Number of books, x	1	2	3	4	5	6
Frequency (number of children), $f(x)$	15	22	23	19	23	18
Probability that a child has borrowed x books, $P(X = x)$	$\frac{15}{120}$	$\frac{22}{120}$	$\frac{23}{120}$	$\frac{19}{120}$	$\frac{23}{120}$	$\frac{18}{120}$

Find the mean number of books borrowed by a child in a month.

Solution

Method 1: 'O' Level method

Mean number of books

$$= \frac{\text{total number of books borrowed by the children}}{\text{total number of children}}$$

$$= \frac{\sum_{\text{all } x} x f(x)}{\sum_{\text{all } x} f(x)}$$

$$= \frac{1(15) + 2(22) + 3(23) + 4(19) + 5(23) + 6(18)}{15 + 22 + 23 + 19 + 23 + 18}$$

$$= 3.56 \text{ (3 s.f.)}$$

Method 2: Using expectation formula

Mean number of books

$$= E(X)$$

$$= \sum_{\text{all } x} x P(X = x)$$

$$= 1\left(\frac{15}{120}\right) + 2\left(\frac{22}{120}\right) + 3\left(\frac{23}{120}\right) + 4\left(\frac{19}{120}\right) + 5\left(\frac{23}{120}\right) + 6\left(\frac{18}{120}\right)$$

$$= 3.56 \text{ (3 s.f.)}$$

$$P(X = x) = \frac{f(x)}{\sum_{\text{all } x} f(x)}$$

So, the mean number of books borrowed is 3.56. ■

Connecting what we learnt in secondary school and the expectation formula,

$$\frac{\sum x f(x)}{\sum f(x)} = \sum x \left(\frac{f(x)}{\sum f(x)} \right) = \sum x P(X = x) = E(X).$$

Example 6

A discrete random variable X only takes on integer values 1, 2 and 3 with its probability distribution function $P(X = x) = \frac{k}{x!}$, where k is a constant.

Show that $k = \frac{3}{5}$ and find the expectation of X .

Solution

■

Example 7

In a gambling game, a player wins \$5 if all heads or all tails come up when 3 coins are tossed. He loses \$3 if either one or two heads show. Let the random variable X be the player's winnings from one bet.

- (a) Find the probability distribution of X .
 (b) On the basis of $E(X)$, is the game 'fair' (the player wins or loses nothing in the long run)?

Solution

(a) $P(X = 5) =$

$$P(X = -3) =$$

x		
$P(X = x)$		

(b) $E(X) = \sum x P(X = x) =$

Not fair. In the long run, the player is expected to lose money. ■

LECTURE 2

Lesson Outline

- Understand the properties of expectation and variance.
 - Miscellaneous examples on discrete random variables.
-

4.1 Expectation of a Function of a Discrete Random Variable

The definition of the expectation of a discrete random variable X can be extended to any function of X , such as X^2 , $(X+2)^3$, etc.

In general, if $g(X)$ is any function of the discrete random variable X , then,

$$E(g(X)) = \sum_{\text{all } x} (g(x) P(X = x)).$$

For example, let the random variable X be the temperature of a random cup of water in degrees Celsius. Suppose that the random variable Y is the temperature in degrees Fahrenheit.

Given that $Y = \frac{9}{5}X + 32$, we have $E(Y) = \sum_{\text{all } x} \left(\left(\frac{9}{5}x + 32 \right) P(X = x) \right).$

4.2 Properties of Expectation

If a and b are real constants, and X and Y denote discrete random variables, then

- (a) $E(a) = a$
- (b) $E(aX) = a E(X)$
- (c) $E(aX \pm b) = aE(X) \pm b$
- (d) $E(aX \pm bY) = aE(X) \pm bE(Y)$

The proofs of the above results are similar. A partial proof of (c) is shown below.

Proof of (c)

$$\begin{aligned} E(aX + b) &= \sum_{\text{all } x} ((ax + b) P(X = x)) \\ &= \sum_{\text{all } x} (ax P(X = x) + b P(X = x)) \\ &= a \sum_{\text{all } x} (x P(X = x)) + b \sum_{\text{all } x} P(X = x) \\ &= aE(X) + b(1) \\ &= aE(X) + b \end{aligned}$$

In general, if $X_1, X_2, X_3, \dots, X_n$ are n independent observations of X , then

1. $E(X_1) = E(X_2) = \dots = E(X_n) = E(X)$
2. $E(X_1 \pm X_2 \pm \dots \pm X_n) = E(X_1) \pm E(X_2) \pm \dots \pm E(X_n)$
In particular, we have $E(X_1 + X_2 + \dots + X_n) = nE(X)$
3. $E(a_1X_1 \pm a_2X_2 \pm \dots \pm a_nX_n) = (a_1 \pm a_2 \pm \dots \pm a_n)E(X)$,
where $a_1, a_2, \dots, a_n$ are constants.

Example 8

The random variable X has probability distribution given by the following table:

x	1	2	3
$P(X = x)$	0.1	0.6	0.3

Calculate

- (a) $E(3)$ (b) $E(X)$ (c) $E(5X)$
 (d) $E(5X + 3)$ (e) $E(X^2)$ (f) $E(4X^2 - 3)$

Solution

- (a) $E(3) =$
 (b) $E(X) = \sum x P(x = X) = 1 \times 0.1 + 2 \times 0.6 + 3 \times 0.3 = 2.2$
 (c) $E(5X) =$
 (d) $E(5X + 3) =$
 (e) $E(X^2) = \sum x^2 P(X = x) =$
 (f) $E(4X^2 - 3) =$

Note that
 $E(X^2) \neq (E(X))^2$

Example 9

For a game, three dice are thrown. If '1' or '6' appears you will be paid \$1, but if neither '1' nor '6' appears, you will pay \$5. How much would you expect to lose in nine of these games?

Solution

Let the random variable X denote the amount earned in one game.

x	-5	1
$P(X = x)$	$\left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\left(\frac{4}{6}\right) = \frac{8}{27}$	$1 - \frac{8}{27} = \frac{19}{27}$

$$E(X) = \sum x P(X=x) = -5 \times \frac{8}{27} + 1 \times \frac{19}{27} = -\frac{7}{9}$$

$$E(X_1 + X_2 + \dots + X_9) =$$

Since the expectation is negative, I am expected to lose \$7 in 9 games. ■

5 VARIANCE

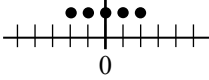
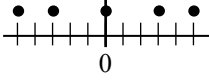
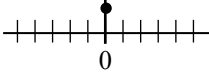
Variance is a measure of the spread of the data. Specifically, it is the expectation of the squared deviations of a random variable from its mean. This gives an idea of the variability of the measurements.

Consider the discrete random variable X and let $E(X) = \mu$.

The **variance** of X , written $\text{Var}(X)$, of a random variable X is given by

$$\text{Var}(X) = E((X - \mu)^2) = \sum_{\text{all } x} (x - \mu)^2 P(X = x).$$

Consider the following 3 data sets:

Data Set	Mean/ Median	Variance
Data Set 1: $\{-2, -1, 0, 1, 2\}$ 	0	$\frac{(-2-0)^2 + (-1-0)^2 + (0-0)^2 + (1-0)^2 + (2-0)^2}{5} = 2$ Moderate spread of data around the mean.
Data Set 2: $\{-5, -3, 0, 3, 5\}$ 	0	$\frac{(-5-0)^2 + (-3-0)^2 + (0-0)^2 + (3-0)^2 + (5-0)^2}{5} = 13.6$ Variance is significantly higher than that of Data Set 1, indicating a much larger spread of data around the mean.
Data Set 3: $1 \{0, 0, 0, 0, 0\}$ 	0	$\frac{(0-0)^2 + (0-0)^2 + (0-0)^2 + (0-0)^2 + (0-0)^2}{5} = 0$ Variance = 0, indicating no spread at all; all data points are exactly the same as the mean.

Even though all three data sets have the same mean and median, their variances differ significantly:

- **Data Set 1** has a moderate spread.
- **Data Set 2** has a wider spread.
- **Data Set 3** has no spread.

This difference in variance helps distinguish the data sets based on how dispersed the data points are.

$$\begin{aligned}
 \text{Since } \text{Var}(X) &= E\left((X - \mu)^2\right) \\
 &= E\left(X^2 - 2\mu X + \mu^2\right) \\
 &= E\left(X^2\right) - 2\mu E(X) + \mu^2 \\
 &= E\left(X^2\right) - \mu^2 \\
 &= E\left(X^2\right) - (E(X))^2
 \end{aligned}$$

Equivalently, we have

$$\text{Var}(X) = E\left(X^2\right) - (E(X))^2.$$

While variance is mathematically important and is used in various calculations, standard deviation is more practical and intuitive because it is expressed in the same units as the original data, making it more practical for real-world applications.

The positive square root of $\text{Var}(X)$ is called the **standard deviation** of X and is denoted as σ . Therefore $\text{Var}(X)$ can also be denoted as σ^2 .

The standard deviation of X , σ also measures the dispersion of the data points. Like variance, a higher standard deviation means more spread-out data.

5.1 Properties of Variance

If a and b are real constants, and X and Y denote independent discrete random variables, then

- (a) $\text{Var}(a) = 0$
- (b) $\text{Var}(aX) = a^2 \text{Var}(X)$
- (c) $\text{Var}(aX \pm b) = a^2 \text{Var}(X)$
- (d) $\text{Var}(aX \pm bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$

Proof

$$(a) \quad \text{Var}(a) = E(a^2) - (E(a))^2 = a^2 - (a)^2 = 0$$

$$\begin{aligned}
 (b) \quad \text{Var}(aX) &= E\left((aX)^2\right) - (E(aX))^2 \\
 &= E\left(a^2 X^2\right) - (a E(X))^2 \\
 &= a^2 E\left(X^2\right) - a^2 (E(X))^2 \\
 &= a^2 \left(E\left(X^2\right) - (E(X))^2\right) \\
 &= a^2 \text{Var}(X)
 \end{aligned}$$

$$(c) \quad \text{Var}(aX \pm b) = a^2 \text{Var}(X) + \text{Var}(b) = a^2 \text{Var}(X) \quad \blacksquare$$

In general, if $X_1, X_2, X_3, \dots, X_n$ are n independent observations of X , then

1. $\text{Var}(X_1) = \text{Var}(X_2) = \dots = \text{Var}(X_n) = \text{Var}(X),$
2. $\text{Var}(X_1 \pm X_2 \pm \dots \pm X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n)$
 $= n\text{Var}(X)$ and
3. $\text{Var}(a_1X_1 \pm a_2X_2 \pm \dots \pm a_nX_n)$
 $= a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + \dots + a_n^2\text{Var}(X_n)$
 $= (a_1^2 + a_2^2 + \dots + a_n^2)\text{Var}(X),$
 where $a_1, a_2, \dots, a_n$ are constants.

Example 10

Two balls are drawn without replacement, from a box containing 3 red balls and 4 white balls. The balls are drawn at random. If X is the discrete random variable ‘the number of red balls drawn’, find

- (a) $E(X)$
- (b) $\text{Var}(X)$
- (c) the standard deviation of X
- (d) $\text{Var}(2X)$
- (e) $\text{Var}(3X - 2)$.

If X_1 and X_2 are two independent observations of X , find

- (f) $\text{Var}(X_1 + X_2)$ and Note that $\text{Var}(X_1 + X_2) \neq \text{Var}(2X)$
- (g) $\text{Var}(2X_1 - 3X_2 + 5)$.

Solution

Let the random variable X denote the number of red balls drawn.

x	0	1	2
$P(X = x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

$$P(X = 0) = P(WW) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

$$P(X = 1) = P(WR \text{ or } RW) = \frac{4}{7} \times \frac{3}{6} \times 2 = \frac{4}{7}$$

$$P(X = 2) = P(RR) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$

$$(a) \quad E(X) = 0 \times \frac{2}{7} + 1 \times \frac{4}{7} + 2 \times \frac{1}{7} = \frac{6}{7}$$

$$(b) \quad E(X^2) = 0 \times \frac{2}{7} + 1 \times \frac{4}{7} + 2^2 \times \frac{1}{7} = \frac{8}{7}$$

$$\text{Var}(X) =$$

$$(c) \quad \text{the standard deviation of } X = \sqrt{\text{Var}(X)} =$$

$$(d) \quad \text{Var}(2X) =$$

$$(e) \quad \text{Var}(3X - 2) =$$

$$(f) \quad \text{Var}(X_1 + X_2) =$$

$$(g) \quad \text{Var}(2X_1 - 3X_2 + 5) =$$

■

Example 11

At a funfair stall, a player pays 5 cents in order to toss three coins successively. The stall holder pays the player 10 cents if the first coin toss is a head, 5 cents if the second coin toss is a head and 2 cents if the third coin toss is a head. Determine the mean and variance of the player's net gain from tossing the three coins.

*Solution**Method 1*

Let the random variable X denote

x		
$P(X = x)$		

$$E(X) = \quad, E(X^2) = \quad, \text{Var}(X) =$$

Method 2

Let the random variable X denote the player's net gain from 3 tosses.

x	-5	-3	0	2	5	7	10	12
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

(E.g. To obtain -3 cents, the player should earn only 2 cents while paying the 5 cents: $P(X = -3) = P(\text{1st T, 2nd T, 3rd H}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.)

$$E(X) = \frac{1}{8}(-5 - 3 + 0 + 2 + 5 + 7 + 10 + 12)$$

$$= 3.5 \text{ cents}$$

$$E(X^2) = \frac{1}{8}(25 + 9 + 0 + 4 + 25 + 49 + 100 + 144)$$

$$= 44.5$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= 44.5 - 3.5^2$$

$$= 33.25 \text{ cents}^2$$

■

6 MISCELLANEOUS EXAMPLES

Example 12

A bowl contains 9 chips of which 4 are marked \$0.50, 3 are marked \$0.20 and 2 are marked \$0.10. A player who is blindfolded picks two chips at random (without replacement). The player will receive a reward equal to the sum of the 2 chips drawn. If it costs \$0.65 to play the game, is it worthwhile for the player to play the game? (You may assume that it will be worthwhile for the player to play the game if he expects to have a net gain in the long run.)

Solution

Let the random variable X denote the amount of money the player earns from drawing 2 chips.

$$P(X = 0.2) = P(2 \text{ \$0.10 chips}) = \left(\frac{2}{9}\right)\left(\frac{1}{8}\right) = \frac{1}{36}$$

$$P(X = 0.3) = P(1 \text{ \$0.10 chip, 1 \$0.20 chip}) = 2\left(\frac{2}{9}\right)\left(\frac{3}{8}\right) = \frac{6}{36}$$

$$P(X = 0.4) = P(2 \text{ \$0.20 chips}) = \left(\frac{3}{9}\right)\left(\frac{2}{8}\right) = \frac{3}{36}$$

$$P(X = 0.6) = P(1 \text{ \$0.10 chip, 1 \$0.50 chip}) = 2\left(\frac{2}{9}\right)\left(\frac{4}{8}\right) = \frac{8}{36}$$

$$P(X = 0.7) = P(1 \text{ \$0.20 chip, 1 \$0.50 chip}) = 2\left(\frac{3}{9}\right)\left(\frac{4}{8}\right) = \frac{12}{36}$$

$$P(X = 1) = P(2 \text{ \$0.50 chips}) = \left(\frac{4}{9}\right)\left(\frac{3}{8}\right) = \frac{6}{36}$$

x	0.2	0.3	0.4	0.6	0.7	1
$P(X = x)$	$\frac{1}{36}$	$\frac{6}{36}$	$\frac{3}{36}$	$\frac{8}{36}$	$\frac{12}{36}$	$\frac{6}{36}$

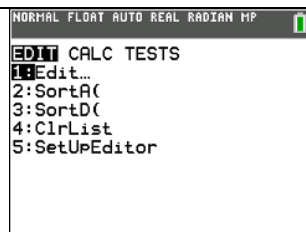
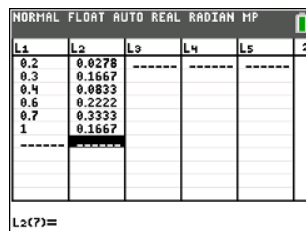
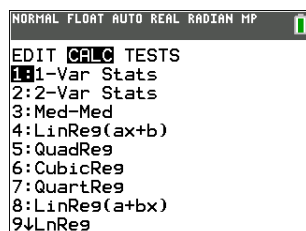
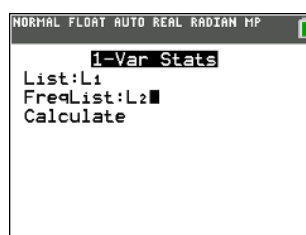
$$\begin{aligned} E(X) &= \sum_{\text{all } x} xP(X = x) = \frac{1}{36}(0.2 + 1.8 + 1.2 + 4.8 + 8.4 + 6) \\ &= \frac{28}{9} \approx \$0.62 \end{aligned}$$

$$\$0.62 - \$0.65 = -\$0.03$$

Therefore, it will not be worthwhile for the player to play the game. ■

The value of $E(X)$ can be verified using GC.

- 1) Press [STAT] followed by [ENTER] to select **1: Edit**.
- 2) Position the cursor at the first row of L1 and enter the data 0.2, 0.3, 0.4, 0.6, 0.7 and 1 in the column for L1.
Then position the cursor at the second column (L2) and enter the probabilities $\frac{1}{36}, \frac{6}{36}, \frac{3}{36}, \frac{8}{36}, \frac{12}{36}$ and $\frac{6}{36}$.
- 3) Press [STAT] followed by '→' for the CALC menu.
Press [ENTER] to select **1: 1-Var Stats**.
- 4) Press [2nd][1] and [2nd][2] for L1, L2 respectively to get the results.

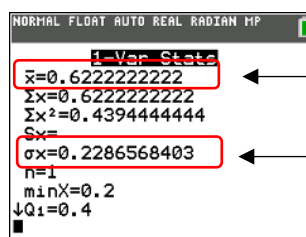





From G.C., $E(X) = \bar{x} = 0.62$

Note:

We can use GC to find $\text{Var}(X)$:

$$\text{Var}(X) = \sigma_x^2 = (0.22866)^2 = 0.0523 \text{ (3 s.f.)}$$



Example 13

An unbiased disc has a single dot marked on one side and two dots marked on the other side. The disc and an unbiased six-sided die are thrown and the random variable X denotes the sum of the numbers of dots showing on the disc and on the top of the die.

(i) Tabulate the probability distribution of X .

(ii) Find $E(X)$ and show that $\text{Var}(X) = \frac{19}{6}$.

(iii) Show that $P(X \geq 4 | X \leq 7) = \frac{8}{11}$.

Two independent observations of X_1 and X_2 are taken of X .

(iv) Find $\text{Var}(X_1 - X_2)$.

(v) Find $P(X_1 - X_2 \geq 5)$ and hence write down $P(|X_1 - X_2| \geq 5)$.

Solution

(i)	x	2	3	4	5	6	7	8
	$P(X = x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

$$(ii) \quad E(X) = (2+8)\left(\frac{1}{12}\right) + (3+\dots+7)\left(\frac{2}{12}\right) = 5$$

$$E(X^2) = (2^2+8^2)\left(\frac{1}{12}\right) + (3^2+\dots+7^2)\left(\frac{2}{12}\right) = \frac{169}{6}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{169}{6} - 5^2 = \frac{19}{6} \quad (\text{shown})$$

$$(iii) \quad P(X \geq 4 | X \leq 7) = \frac{P(4 \leq X \leq 7)}{P(X \leq 7)}$$

$$= \frac{P(X=4) + \dots + P(X=7)}{P(X \leq 7)} = \frac{4\left(\frac{2}{12}\right)}{\frac{11}{12}} = \frac{8}{11} \quad (\text{shown})$$

$$(iv) \quad \text{Var}(X_1 - X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \frac{19}{3}$$

$$(v) \quad P(X_1 - X_2 \geq 5)$$

$$= P(X_1 = 7, X_2 = 2) + P(X_1 = 8, X_2 = 2) + P(X_1 = 8, X_2 = 3)$$

$$= \left(\frac{2}{12}\right)\left(\frac{1}{12}\right) + \left(\frac{1}{12}\right)\left(\frac{1}{12}\right) + \left(\frac{1}{12}\right)\left(\frac{2}{12}\right) = \frac{5}{144}$$

$$\therefore P(|X_1 - X_2| \geq 5) = P(X_2 - X_1 \geq 5 \text{ or } X_1 - X_2 \geq 5)$$

$$= 2P(X_1 - X_2 \geq 5) = \frac{5}{72} \quad \blacksquare$$

LECTURE 3

Lesson Outline

- Understand the properties of the binomial distribution, and how to calculate probabilities, expectation and variance of a binomial distribution.
 - Usage of the graphic calculator to solve questions where the binomial distribution is a suitable model with known parameters, as well as finding the mode of a binomial variable.
-

7 BINOMIAL DISTRIBUTION

Let us consider the rolling of a fair die, and suppose that rolling a '6' is a successful roll.

$$P(\text{success}) = P(\text{rolling a '6'}) = \frac{1}{6}$$

$$P(\text{failure}) = P(\text{not rolling a '6'}) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Let the random variable X denote the number of successful rolls in 4 repeated rolls of the die.

What is the probability distribution of X ?

x	0	1	2	3	4
$P(X = x)$	$P(X = 0) =$	$P(X = 1) =$	$P(X = 2) =$	$P(X = 3) =$	$P(X = 4) =$

The above experiment is known as a binomial experiment, where there are 4 identical and independent trials of rolling a die, and for each trial, the probability of success (rolling a '6') is constant at $\frac{1}{6}$ (this also means that

the probability of failure (not rolling a '6') is $1 - \frac{1}{6} = \frac{5}{6}$).

A random variable denoting the number of success, r , out of a total of n trials, is modelled by a binomial distribution if:

1. The experiment consists of **n independent repeated** trials (see Example 24).
2. The result of each trial is only **one of two possible outcomes**, labeled “success” or “failure”.
3. The **probability of “success”** of each trial **remains constant** and is denoted by p .

Example 14

A fixed number, n , of cars is observed and the number of those cars that are red is denoted by R .

State, in context, two assumptions needed for R to be well modelled by a binomial distribution. [N09/II/11]

Solution

Assumptions:

1. The colour of each car must be _____ of the colour of any other car.
2. The probability that any one car is red remains _____ throughout the sample.

7.1 Definition of a Binomial Distribution

Let the probability that an experiment results in a successful outcome be p and the probability that the outcome is a failure be q , where $q = 1 - p$. Let X be the random variable that denotes ‘the number of successful outcomes in n independent trials’.

The probability of obtaining a particular sequence of r successes and $n - r$ failures is given as $p^r q^{n-r}$ by applying the multiplication principle to the independent trials.

Since there are $\binom{n}{r}$ such sequences with r successes and $n - r$ failures, the probability distribution of X is given by

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r} \text{ for } r = 0, 1, 2, \dots, n \text{ (in MF27)}$$

Notes

1. $\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$
2. The values $P(X=x)$ for $x=0, 1, 2, \dots, n$ can be obtained by considering the terms in the binomial expansion of $(p+q)^n$, noting that $p+q=1$:

$$(p+q)^n = p^n + {}^nC_1 p^{n-1} q + {}^nC_2 p^{n-2} q^2 + \dots + {}^nC_{n-1} p q^{n-1} + q^n$$

$$\therefore 1 = P(X=n) + P(X=n-1) + \dots + P(X=1) + P(X=0)$$

If X follows a **binomial distribution**, then we write

$$X \sim B(n, p),$$

where n is the number of independent trials and p is the probability of a successful outcome.

n and p are called the **parameters** of the binomial distribution.

The **mean** (expectation or expected value) and **variance** of X are given by

$$E(X) = np, \text{ Var}(X) = npq, \text{ where } q = (1-p). \text{ (in MF27)}$$

7.2 Examples of Binomial Distribution with Known Parameters

Example 15 (Use of GC to calculate probabilities)

Given that the random variable X is such that $X \sim B(8, 0.3)$, find

- (a) $P(X=5)$,
- (b) $P(X \leq 4)$,
- (c) $P(X > 2)$,
- (d) $P(2 \leq X < 5.5)$.

Solution

- (a) To find $P(X=5)$:

1) Press **[2ND] [DISTR]** and select **A: binompdf(**.

2) Enter the values of n , p and x .

NORMAL	FLOAT	AUTO	REAL	RADIAN	MP	NORMAL	FLOAT	AUTO	REAL	RADIAN	MP	NORMAL	FLOAT	AUTO	REAL	RADIAN	MP
DISTR						binompdf						binompdf(8,0.3,5)					
8: χ^2 cdf(						trials:8											
9: Ppdf(						p:0.3											
0: Pcdf(						x value:5											
AB binompdf(						Paste											
B: binomcdf(																	
C: poissonpdf(																	
D: poissoncdf(																	
E: geometpdf(																	
F: geometcdf(																	

(b) To find $P(X \leq 4)$:

- 1) Press **[2ND] [DISTR]** and select **B: binomcdf**.
- 2) Enter the values of n , p and x .

NORMAL FLOAT AUTO REAL RADIAN MP	NORMAL FLOAT AUTO REAL RADIAN MP	NORMAL FLOAT AUTO REAL RADIAN MP
DISTR DRAW 8: χ^2 cdf(9: F pdf(0: F cdf(A: binompdf(B: binomcdf (C: poissonpdf(D: poissoncdf(E: geometpdf(F: geometcdf(	binomcdf trials: 8 p: 0.3 x value: 4 Paste	binomcdf(8,0.3,4)94203235

(c) $P(X > 2) =$

(d) $P(2 \leq X < 5.5) =$

■

Example 16

According to the Health Promotion Board, 28% of Primary 1 students in Singapore are short-sighted. A sample of 20 Primary 1 students are randomly selected for a survey.

Explain why a binomial distribution is appropriate to model the number students in the sample who are short-sighted.

Find the probability that the number of students in the sample who are short-sighted is

- (i) exactly three;
- (ii) less than three;
- (iii) more than 2;
- (iv) at least 2 but less than 7.

Ryan is one of the 20 students in the sample, and he is short-sighted.

- (v) Find the probability that exactly two of the remaining 19 students are short-sighted.

Solution

A binomial distribution is appropriate because

- there are only two possible outcomes: either a **student is short-sighted** or he is **not**.
- a randomly chosen student being short-sighted is **independent** of any other student being short-sighted, since the whole cohort of Primary 1 students in Singapore is large.
- the **probability** of a student being short-sighted is **constant** at 0.28.

Let the random variable X denote the number of students out of 20 who are short-sighted.

Then $X \sim B(20, 0.28)$.

(i) |

(ii) |

(iii)

(iv)

Let the random variable Y denote the number of students out of 19 who are short-sighted.

Then $Y \sim B(19, 0.28)$

(v) $P(Y = 2) =$ ■

Example 17

If the random variable X is such that $X \sim B(10, p)$ where $p < 0.5$ and

$\text{Var}(X) = \frac{15}{8}$, find

(a) p , (b) $E(X)$, (c) $P(X = 2)$.

Solution

(a) $\text{Var}(X) = \frac{15}{8} \Rightarrow$

$$16p^2 - 16p + 3 = 0$$

$$p = 0.25 \text{ or } p = 0.75 \text{ (rejected)}$$

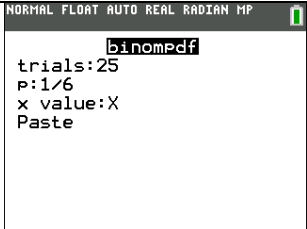
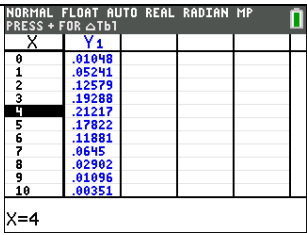
(b) $E(X) = 10 \times 0.25 = 2.5$

(c) $P(X = 2) = 0.28156 \approx 0.282$ (3 s.f.) ■

Example 18 (Modal value of a binomial variable)

Given that $X \sim B\left(25, \frac{1}{6}\right)$, find the **mode** of this distribution, that is, the value of k such that $P(X = k)$ is largest.

Solution

1) Press [Y=] . Move the cursor to select Y1 . Press [2ND] [DISTR] , select A: binompdf . Key in 25, 1/6, X . Select [Paste] .	
2) Press [2ND] [TABLE] . From the table, we can see that $X = 4$ gives the highest probability.	

From the G.C.,

$$P(X = 3) = 0.19288$$

$$P(X = 4) = 0.21217$$

$$P(X = 5) = 0.17822$$

For your shown working, you need to present at least the values of $P(X = k - 1)$, $P(X = k)$ and $P(X = k + 1)$, where $P(X = k)$ is the largest.

Therefore the mode of the distribution is 4. ■

Example 19

Find the mode of X where $X \sim B(24, 0.4)$.

Solution

From the G.C.,

$$P(X = 8) = 0.13598$$

$$P(X = 9) = 0.16116$$

$$P(X = 10) = 0.16116$$

$$P(X = 11) = 0.13674$$

The modes of the distribution are 9 and 10. ■

Example 20

Every morning, Alex takes either Bus 74 or 198 to ACJC. The probability that he takes Bus 74 is 0.8 on any day. If he takes Bus 74, there is a probability of 0.05 that he will be late for school and the probability that he will be late is 0.15 if he chooses to take Bus 198. If Alex is late 3 times or more, he will have to see the Discipline Master for punishment.

- (a) Find the probability that in a period of 5 school days, Alex takes Bus 198 at least once.
- (b) Show that the probability of Alex being late on any school day is 0.07.
- (c) Find the probability that Alex will not be punished over a period of 10 consecutive days.
- (d) Find the probability that Alex will not be punished over a period of 10 consecutive days, given that he was late on the first day.

Solution

- (a) Let the random variable X denote the number of days Alex takes Bus 198 to ACJC, out of 5 days.

Then $X \sim B(5, 0.2)$.

$$P(X \geq 1) = 1 - P(X = 0) = 0.672 \text{ (3 s.f.)}$$

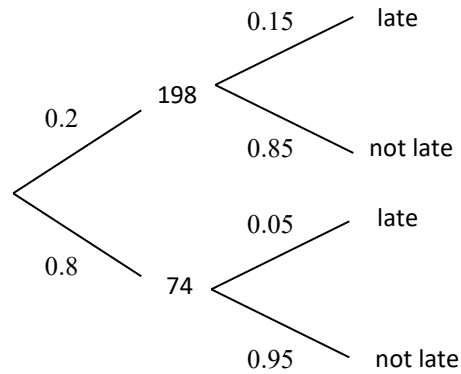
- (b) $P(\text{Alex is late on any day})$
 $= (0.2)(0.15) + (0.8)(0.05)$
 $= 0.07$

- (c) Let the random variable Y denote the number of days that Alex is late over ten days.

Then $Y \sim B(10, 0.07)$.

$$P(\text{Alex is not punished})$$

$$= P(Y \leq 2) = 0.972 \text{ (3 s.f.)}$$



- (d) Let the random variable W denote the number of days that Alex is late over **nine** days.

Then $W \sim B(9, 0.07)$.

Method 1

$$P(\text{Alex not punished} \mid \text{Alex late on first day})$$

$$= P(\text{Alex late} < 2 \text{ times on remaining 9 days})$$

$$= P(W \leq 1) = 0.873$$

Method 2

$$P(\text{Alex not punished} \mid \text{Alex late on first day})$$

$$= \frac{P(\text{Alex unpunished} \cap \text{Alex late on first day})}{P(\text{Alex late on first day})}$$

$$= \frac{0.07 \times P(W \leq 1)}{0.07} = 0.873$$

LECTURE 4

Lesson Outline

- Understand approaches to binomial distribution questions with unknown parameters.
- Explore the effects of varying the parameters of a binomial distribution on its graph.

7.3 Examples of Binomial Distribution with Unknown Parameters

Example 21 (Find unknown n)

A box contains a large number of red and blue marbles such that the probability of picking a red marble is p . Marbles are picked at random from the box.

- (a) Let n be the minimum number of marbles that must be picked so that the probability that there is at least one red marble among them is greater than 0.995.
- (i) Find the relationship between n and p .
- (ii) Given that $p = \frac{1}{20}$, find n .
- (b) Find the maximum number of marbles that must be picked so that the probability that there is at most 10 red marbles among them is greater than 0.995.

Why is there a need for large number of red and blue marbles to be contained in the box?

Solution

- (a) Let the random variable X denote the number of red marbles out of n picks.

Then $X \sim B(n, p)$.

- (i) $P(X \geq 1) > 0.995$
 $\Rightarrow 1 - P(X = 0) > 0.995$
 $\Rightarrow P(X = 0) < 0.005$
 $\Rightarrow (1 - p)^n < 0.005$
- (ii) $(1 - p)^n < 0.005$ where $p = 0.05$

$$\begin{aligned} &\Rightarrow (0.95)^n < 0.005 \\ &\Rightarrow n \lg 0.95 < \lg 0.005 \\ &\Rightarrow n > \frac{\lg 0.005}{\lg 0.95} \\ &\Rightarrow n > 103.29 \\ &\therefore n = 104 \end{aligned}$$

- (b) Let the random variable Y denote the number of red marbles out of n picks.

$$\text{Then } Y \sim B\left(n, \frac{1}{20}\right).$$

We want to find maximum n such that $P(Y \leq 10) > 0.995$.

From the G.C.,

When $n = 88$, $P(Y \leq 10) = 0.99554$

When $n = 89$, $P(Y \leq 10) = 0.99514$

When $n = 90$, $P(Y \leq 10) = 0.99471$

Hence, the maximum number of marbles required is 89

NORMAL FLOAT AUTO REAL RADIAN MP				
PRESS + FOR Δ Tb1				
X	Y1			
85	.99659			
86	.99627			
87	.99592			
88	.99554			
89	.99514			
90	.99471			
91	.99425			
92	.99375			
93	.99323			
94	.99267			
95	.99208			

X=89

This will be explained later in Example 24(ii)(b).

There is a need for a large number of marbles to be contained in the box so that _____

Example 22 (Find unknown p)

X is a random variable such that $X \sim B(10, p)$ where $p < 0.25$ and $P(X = 3) = 0.2013$. Find the value of p .

Solution

- 1) Press **[Y=]**. Move the cursor to select **Y1**.
Press **[2ND] [DISTR]**, select **A: binompdf**.
- 2) Key in **10, X, 3**. Select **[Paste]**.
- 3) Move cursor to select **Y2** and input **Y2 = 0.2013**
- 4) Adjust the window settings by pressing **[2ND] [WINDOW]** and key in the values as shown in the screenshot.

NORMAL FLOAT AUTO REAL RADIAN MP				
binompdf				
trials:10				
p:X				
x value:3				
Paste				

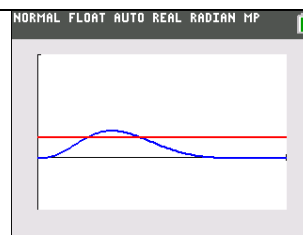
NORMAL FLOAT AUTO REAL RADIAN MP				
Plot1	Plot2	Plot3		
Y1=binompdf(10,X,3)				
Y2=0.2013				
Y3=				
Y4=				
Y5=				
Y6=				
Y7=				
Y8=				
Y9=				

NORMAL FLOAT AUTO REAL RADIAN MP				
WINDOW				
Xmin=0				
Xmax=1				
Xscl=1				
Ymin=-0.5				
Ymax=1				
Yscl=1				
Xres=1				
X=0.0037878787878788				
TraceStep=0.007575757575...				

- 5) Press **[GRAPH]**
- 6) Press **[2ND] [TRACE]** and select option **5: intersect** to find the x values of the two points of intersection.

From the G.C., the two values are $x = 0.200$ or 0.415 (3 s. f.)

- 7) Sketch the graph in your answer, and conclude that $p = 0.200$ (3 s.f.) since $p < 0.25$.



Example 23 (Find unknown r)

In a class of 18 students, the number of students who are late for school on any given day is tracked.

- (i) State, in context, what must be assumed for the number of latecomers in the class on any day to be well modelled by a binomial distribution.
- (ii) Assuming that the probability that each student is late for school on any given day is 0.1, find the least value of the integer r such that the probability that less than r students are late for school on any day is more than 0.9.

Solution

- (i) Assumptions for the number of latecomers in the class per day to be well modelled by a binomial distribution:
- The probability that each student is late for school on any given day is _____.
 - The event that students are late for school is _____ of one another on any given day.
- (ii) Let X be the random variable “number of latecomers on any day in the class of 18 students”.

Then $X \sim B(18, 0.1)$.

Solve $P(X < r) > 0.9$ for r .

$$P(X < r) > 0.9 \Rightarrow P(X \leq r-1) > 0.9.$$

From G.C.,

$$\text{When } r = 3, P(X \leq r-1) = 0.73380.$$

$$\text{When } r = 4, P(X \leq r-1) = 0.90180.$$

$$\text{When } r = 5, P(X \leq r-1) = 0.97181.$$

Hence the least value of r is 4.

Plot1	Plot2	Plot3
Y1=	binomcdf(18,0.1,X-1)	
Y2=		
Y3=		
Y4=		
Y5=		
Y6=		
Y7=		
Y8=		
Y9=		

X	Y1
0	0
1	0.1501
2	0.4503
3	0.7338
4	0.9018
5	0.9718
6	0.9936
7	0.9998
8	0.9999
9	1
10	1

Summary table for finding unknown parameters

Example	Given	In G.C, use
21	$X \sim B\left(n, \frac{1}{20}\right)$ $P(X \leq 10) \geq 0.995$	$Y = \text{binomcdf}\left(X, \frac{1}{20}, 10\right)$
22	$X \sim B(10, p)$, $P(X = 3) = 0.2013$	$Y = \text{binompdf}(10, X, 3)$
23	$X \sim B(18, 0.1)$, $P(X < r) > 0.9$	$Y = \text{binomcdf}(18, 0.1, X - 1)$

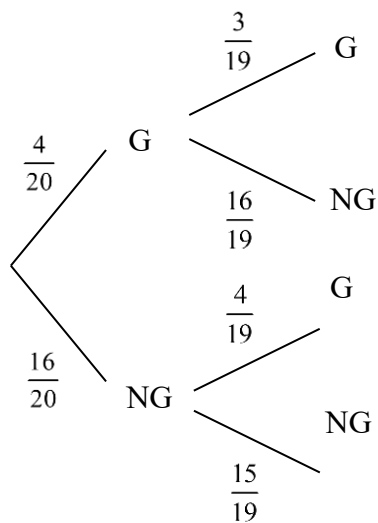
Note

- Normally, we will use the “table” function in our GC to find unknown n or r , since these unknowns are positive integers.
- We will use the “graph” function on our GC to find the unknown p , since this is a value between 0 and 1.

Example 24 (an explanation on “ n independent repeated trials”)

- (i) In a **small plot of land, 20 seeds are sown**, of which only 4 will germinate. If 5 seeds are randomly selected without replacement, find the probability that
- there is at least one seed that germinates.
 - the first randomly chosen seed will germinate,
 - the second randomly chosen seed will germinate.
- Can the number of seeds that will germinate out of the 5 seeds be well modelled by $B(5, 0.2)$?
- (ii) In a **huge plot of land, 1000 seeds are sown**, of which 10% will germinate. If 5 seeds are randomly selected without replacement, find the probability that
- the first randomly chosen seed will germinate,
 - the second randomly chosen seed will germinate.
- Can the number of seeds that will germinate out of the 5 seeds be well modelled by $B(5, 0.1)$?
- Hence find the probability that
- there is at least one seed that germinates.

[DHS/2009/2/7/modified]

*Solution***(i)****(a)** $P(\text{at least one will germinate})$

$$= 1 - P(\text{none will germinate})$$

$$= 1 - \frac{16}{20} \cdot \frac{15}{19} \cdot \frac{14}{18} \cdot \frac{13}{17} \cdot \frac{12}{16} = 1 - \frac{91}{323} = \frac{232}{323}$$

(b) $P(\text{first seed selected will germinate}) = \frac{4}{20} = \frac{1}{5}$ **(c)** $P(\text{second seed selected will germinate}) = \frac{4}{20} \cdot \frac{3}{19} + \frac{16}{20} \cdot \frac{4}{19} = \frac{1}{5}$

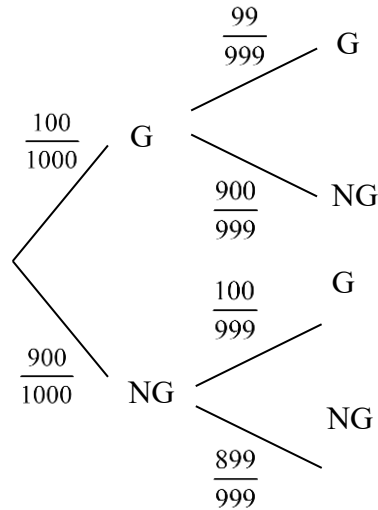
Define the following two events:

 A : first seed germinates B : second seed germinates

$$P(B) = \frac{1}{5} \text{ from (c) and } P(B|A) = \frac{3}{19}$$

Since $P(B|A) \neq P(B)$, $\therefore A$ and B are **not independent**.Hence the number of seeds that germinate out of the 5 seeds selected **cannot** be modelled by $B(5, 0.2)$.

(ii)

(a) $P(\text{first seed selected will germinate}) = 0.1$ (b) $P(\text{second seed selected will germinate})$

$$= \frac{100}{1000} \cdot \frac{99}{999} + \frac{900}{1000} \cdot \frac{100}{999} = \frac{1}{10}$$

Define the following two events:

 A : first seed germinates B : second seed germinates

$$P(B) = \frac{1}{10} \text{ from (b) and } P(B|A) = \frac{99}{999} \approx \frac{1}{10} = P(B)$$

Since $P(B|A) \approx P(B)$, $\therefore$ Events A and B are **approximately independent**. Hence the number of seeds that germinate out of the 5 seeds selected **can** be modelled by $B(5, 0.1)$.

(c) Let the random variable X denote the number of germinating seeds out of 5 seeds.Then $X \sim B(5, 0.1)$

$$P(X \geq 1) = 1 - P(X = 0) = 0.410$$

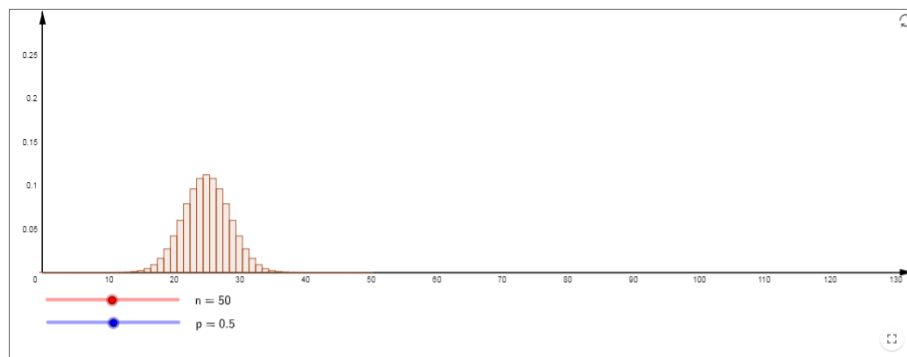
■

7.4 Self Exploration on the Binomial Distribution

What does a binomial distribution ‘look’ like? The probability distribution of a binomial variable can be graphed out as a **histogram**.

The parameters n and p alters the shape of the graph. By using the **GeoGebra applet “Binomial Distribution – Further Exploration”**, explore what happens when you vary the values of n and p .

	Binomial Distribution – Further Exploration
	GeoGebra applet https://ggbm.at/xwftzyv5



Exploration Questions

- For small n , what happens to the graph of the binomial distribution if p is small? If p is large?
- For small p , what do you notice about the graph of the binomial distribution as n becomes larger?

ANNEX A: PRACTICE QUESTIONS

Discrete Random Variable

- 1 For each of the following experiments, tabulate the probability distribution of X and find its expectation and variance.
- (a) The discrete random variable X denotes the number of “sixes” showing when two ordinary fair dice are thrown.
 - (b) Two red balls and two white balls are placed in a bag. Balls are drawn one by one, at random and without replacement. The random variable X is the number of white balls drawn before the first red ball is drawn.
- 2 The discrete random variable X has the probability distribution as shown in the table below, where p is a constant.

x	0	1	2	3
$P(X = x)$	p	$\frac{1}{2}p$	$\frac{1}{4}p$	$\frac{1}{20}p$

- (a) Show that $p = \frac{5}{9}$.
 - (b) Find $E(X)$ and $\text{Var}(X)$, giving your answers as exact fractions.
- 3 A discrete random variable X has the following probability distribution.

x	1	2	n	7
$P(X = x)$	0.4	0.3	p	0.1

- (i) Write down the value of p .
- (ii) Given that $E(X) = 2.5$, find n .
- (iii) Find $\text{Var}(X)$.

Binomial Distributions

- 4 In a lucky draw with many tickets distributed, 10% of the tickets will win prizes. John was given 6 tickets for this draw. Calculate the probability that
- (i) John does not win any prize,
 - (ii) John will win more than 1 prize.
- 5 A random variable X has the distribution $B(12, p)$.
- (a) Given that $p = 0.25$, find
 - (i) $P(X < 5)$,
 - (ii) $P(X \geq 7)$.
 - (b) Given that $P(X = 0) = 0.05$, find the value of p to 3 decimal places.
 - (c) Given that the variance is 1.92, find the possible values of p .

*Answers***1(a)**

x	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$
$E(X) = \frac{1}{3}, \text{Var}(X) = \frac{5}{18}$			

(b)

x	0	1	2
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
$E(X) = \frac{2}{3}, \text{Var}(X) = \frac{5}{9}$			

2 **(b)** $E(X) = \frac{23}{36}, \text{Var}(X) = \frac{875}{1296}$

3 **(i)** 0.2 **(ii)** $n = 4$ **(iii)** 3.45

4 **(i)** 0.531 **(ii)** 0.114

5 **(a)(i)** 0.842 **(ii)** 0.0143 **(b)** 0.221 **(c)** 0.2 or 0.8