2025 H3 Math Prelim Mark Scheme

	Solution
1	Let P(n) be the statement $\dfrac{x_1+x_2+\ldots+x_n}{n} \geq \left(x_1x_2\ldots x_n\right)^{\frac{1}{n}}$ for $n \in \left\{2,\ 2^2,\ 2^3, \ldots\right\}$. To prove P(2) is true Since $(x_1-x_2)^2 \geq 0$ $x_1{}^2+2x_1x_2+x_2{}^2 \geq 4x_1x_2$ $(x_1+x_2)^2 \geq 4x_1x_2$ $\dfrac{x_1+x_2}{2} \geq \sqrt{x_1x_2}$ Therefore P(2) is true Assume $\mathrm{P}(2^m)$ is true for some $m \in \mathbb{N}$, i.e $\dfrac{x_1+x_2+\ldots+x_{2^m}}{2^m} \geq \left(x_1x_2\ldots x_{2^m}\right)^{\frac{1}{2^m}}$ -----------(1) To prove $\mathrm{P}(2^{m+1})$ is true, i.e. $\dfrac{x_1+x_2+\ldots+x_{2^{m+1}}}{2^{m+1}} \geq \left(x_1x_2\ldots x_{2^{m+1}}\right)^{\frac{1}{2^{m+1}}}$ -------(2) LHS $= \dfrac{x_1+x_2+\ldots+x_{2^{m+1}}}{2^{m+1}}$ $= \dfrac{1}{2}\left[\dfrac{x_1+x_2+\ldots+x_{2^m}}{2^m}+\dfrac{x_{2^m+1}+x_{2^m+2}+\ldots+x_{2^{m+1}}}{2^m}\right]$ $\geq \sqrt{\left(\dfrac{x_1+x_2+\ldots+x_{2^m}}{2^m}\right)\times\left(\dfrac{x_{2^m+1}+x_{2^m+2}+\ldots+x_{2^{m+1}}}{2^m}\right)}$ ---(3) $\geq \sqrt{\left(x_1x_2\ldots x_{2^m}\right)^{\frac{1}{2^m}}\times\left(x_{2^m+1}x_{2^m+2}\ldots x_{2^{m+1}}\right)^{\frac{1}{2^m}}}$ ------------(4) $= \left(x_1x_2\ldots x_{2^m}x_{2^m+1}x_{2^m+2}\ldots x_{2^{m+1}}\right)^{\frac{1}{2^{m+1}}}$ Hence $\mathrm{P}(2^{m+1})$ is true, by principle of mathematical induction, P(n) is true for all $n \in \left\{2,\ 2^2,\ 2^3, \ldots\right\}$.

	Solution
2(a)	$\sum_{i=1}^{n} \lvert S_i \rvert$ $= \sum_{i=1}^{n}$ no. of permutations where the ith object is in the ith position $= \sum_{i=1}^{n} (n-1)!$ $= n(n-1)!$ $= n!$ $\sum_{i<j} \lvert S_i \cap S_j \rvert$ $= \sum_{i<j}$ no. of permutations where the ith & jth object are in their original positions $= \sum_{i<j} (n-2)!$ $= \binom{n}{2}(n-2)!$ $= \frac{n!}{2!(n-2)!}(n-2)!$ $= \frac{n!}{2!}$
2(b)	$D_n = \lvert \overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_n} \rvert$ $= n! - \lvert S_1 \cup S_2 \cup \ldots \cup S_n \rvert$ $= n! - \sum_{i=1}^{n} \lvert S_i \rvert + \sum_{i<j} \lvert S_i \cap S_j \rvert - \sum_{i<j<k} \lvert S_i \cap S_j \cap S_k \rvert + \ldots + (-1)^n \lvert S_1 \cap \ldots \cap S_n \rvert$ $= n! - n! + \frac{n!}{2} - \binom{n}{3}(n-3)! + \ldots + (-1)^n 1$ $= n! - n! + \frac{n!}{2} - \frac{n!}{3!(n-3)!}(n-3)! + \ldots + (-1)^n$ $= n! - n! + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n \frac{n!}{n!}$ $= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$

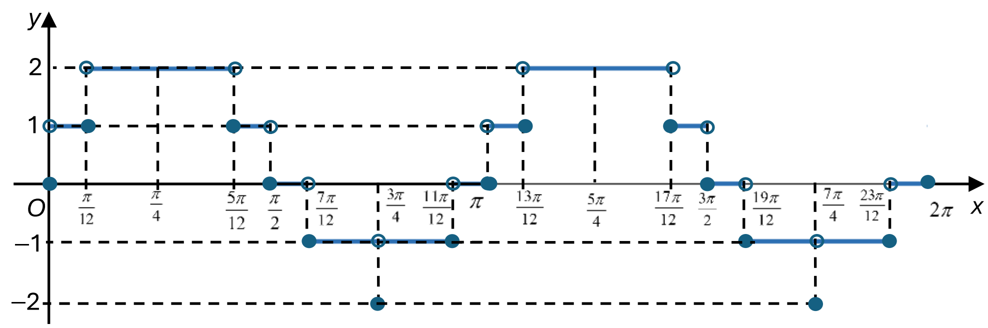
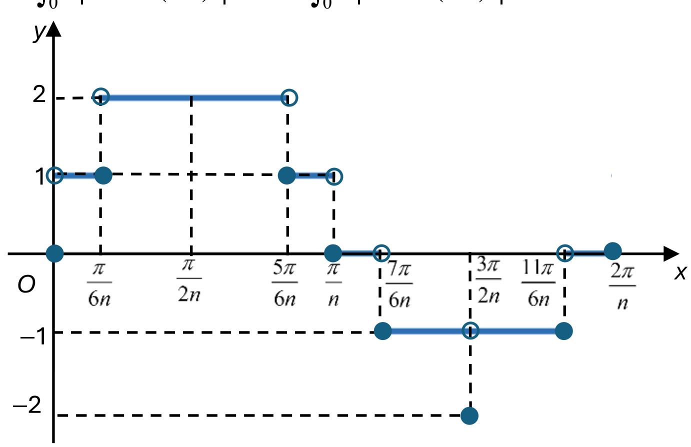
	Solution
2(c)	$\begin{aligned} \lim_{n\to\infty}\frac{D_n}{n!} &= \lim_{n\to\infty}\left(1-\frac{1}{1!}+\frac{1}{2!}-\ldots+(-1)^n\frac{1}{n!}\right) \\ &= 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots \\ &= \mathrm{e}^{-1} \end{aligned}$

	Solution
3(a)	$\begin{aligned} & 1+\frac{1}{(1+k\mathrm{i})^2} \\ &= \frac{(1+k\mathrm{i})^2+1}{(1+k\mathrm{i})^2} \\ &= \frac{(1+k\mathrm{i})^2-\mathrm{i}^2}{(1+k\mathrm{i})^2} \\ &= \frac{[1+(k-1)\mathrm{i}][1+(k+1)\mathrm{i}]}{(1+k\mathrm{i})^2} \\ \mathrm{f}(n) &= 2\left[1+\frac{1}{(1+\mathrm{i})^2}\right]\left[1+\frac{1}{(1+2\mathrm{i})^2}\right]\ldots\left[1+\frac{1}{(1+(n-1)\mathrm{i})^2}\right] \\ &= 2\times\frac{(1)(\cancel{1+2\mathrm{i}})}{(1+\mathrm{i})^{\cancel{2}}}\times\frac{(\cancel{1+\mathrm{i}})(\cancel{1+3\mathrm{i}})}{\cancel{(1+2\mathrm{i})^2}}\ldots \\ & \frac{[1+(\cancel{n-3})\mathrm{i}][1+(\cancel{n-1})\mathrm{i}]}{(1+\cancel{(n-2)\mathrm{i}})^2}\,\frac{[1+(\cancel{n-2})\mathrm{i}][\cancel{1+n\mathrm{i}}]}{\cancel{(1+(n-1)\mathrm{i})^2}}\,\frac{[\cancel{1+(n-1)\mathrm{i}}][1+(n+1)\mathrm{i}]}{(1+n\mathrm{i})^{\cancel{2}}} \\ &= \frac{2}{1+\mathrm{i}}\left(\frac{1+(n+1)\mathrm{i}}{1+n\mathrm{i}}\right) \\ &= \frac{2+2(n+1)\mathrm{i}}{(1-n)+(n+1)\mathrm{i}} \\ &= \frac{[2+2(n+1)\mathrm{i}][(1-n)-(n+1)\mathrm{i}]}{(1-n)^2+(n+1)^2} \\ &= \frac{(n^2+n+2)}{n^2+1}+\frac{(-n^2-n)}{n^2+1}\mathrm{i} \end{aligned}$

	Solution
3(b)	$\tan\left[\sum_{k=0}^{n}\arg\left(1+\frac{1}{(1+k\mathrm{i})^2}\right)\right]=-\frac{36}{37}$ $\Rightarrow \tan\left[\arg\left(\mathrm{f}(n)\right)\right]=-\frac{36}{37}$ $\Rightarrow \frac{-n^2-n}{n^2+n+2}=-\frac{36}{37}$ $\Rightarrow n=8$

	Solution
4(a)	Let $a_1=\frac{x}{\sqrt{x+y}}, a_2=\frac{y}{\sqrt{y+z}}, a_3=\frac{z}{\sqrt{z+w}}, a_4=\frac{w}{\sqrt{w+x}}$ and $b_1=\sqrt{x+y}, b_2=\sqrt{y+z}, b_3=\sqrt{z+w}, b_4=\sqrt{w+x}$ Applying Cauchy Schwarz inequality, $\left(a_1^{\ 2}+a_2^{\ 2}+a_3^{\ 2}+a_4^{\ 2}\right)\left(b_1^{\ 2}+b_2^{\ 2}+b_3^{\ 2}+b_4^{\ 2}\right)\geq\left(a_1b_1+a_2b_2+a_3b_3+a_4b_4\right)^2$ Hence $\left(\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+w}+\frac{w^2}{w+x}\right)\left(2w+2x+2y+2z\right)\geq\left(w+x+y+z\right)^2$ --(1) $\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+w}+\frac{w^2}{w+x}\geq\frac{w+x+y+z}{2}$
4(b)	Using Arithmetic-Geometric inequality, $\frac{w+x+y+z}{4}\geq\left(wxyz\right)^{\frac{1}{4}}\Rightarrow\frac{w+x+y+z}{2}\geq 2\left(wxyz\right)^{\frac{1}{4}}$ Hence $\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+w}+\frac{w^2}{w+x}\geq 2\left(wxyz\right)^{\frac{1}{4}}$ -- (2)
4(c)	$wxyz=81$ From (2), $\frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+w}+\frac{w^2}{w+x}\geq 6$

	Solution
	However, since for Arithmetic-Geometric Progression, equality only holds if $w=x=y=z=3$. However since $wx=27, yz=3$, the equality condition is not met. $\therefore \frac{x^2}{x+y}+\frac{y^2}{y+z}+\frac{z^2}{z+w}+\frac{w^2}{w+x}>6$

	Solution
5(a)	 G1 Correct $y=2$ or $y=-1$ with endpoints G1 Correct $y=1$ with endpoints G1 Correct $y=0$ with endpoints G1 Correct points $\left(\frac{3\pi}{4}, -2\right)$ and $\left(\frac{7\pi}{4}, -2\right)$ indicated.
5(b)	For $2\sin(nx)$, period $=\frac{2\pi}{n}$. From graph, $\therefore \int_0^{2\pi}\lceil 2\sin(nx)\rceil \mathrm{d}x = n\int_0^{\frac{2\pi}{n}}\lceil 2\sin(nx)\rceil \mathrm{d}x$ 

	Solution
	$\int_0^{2\pi} \lceil 2\sin(nx) \rceil \mathrm{d}x$ $= n\int_0^{\frac{2\pi}{n}} \lceil 2\sin(nx) \rceil \mathrm{d}x$ $= n\left\{\left[\left(\frac{\pi}{6n}\right)+2\left(\frac{7\pi}{6n}-\frac{5\pi}{6n}\right)+\left(\frac{\pi}{6n}\right)\right]-\left(\frac{11\pi}{6n}-\frac{7\pi}{6n}\right)\right\}$ $= n\left\{\frac{\pi}{n}\right\}$ $= \pi \qquad (k=1)$
5(c)	From graph in (a), $\int_0^{2\pi} \lceil n\sin(nx) \rceil \mathrm{d}x = \left(\frac{\pi}{n}\right)n$ $= \pi$

	Solution
6(a)(i)	Given $2\mathrm{f}(x) - \mathrm{f}\left(\frac{1}{x}\right) = 3x$ ———— (1) Replace x with $\frac{1}{x}$ in equation (1) $2\mathrm{f}\left(\frac{1}{x}\right) - \mathrm{f}(x) = \frac{3}{x}$ ———— (2)
6(a)(ii)	2 ×(1) + (2): $3\mathrm{f}(x) = 6x + \frac{3}{x}$ $\mathrm{f}(x) = 2x + \frac{1}{x}, \quad x \neq 0$
6(b)	$\mathrm{g}(x) + \mathrm{g}(-x) - \mathrm{g}\left(\frac{1}{x}\right) = x$ ———— (1) Replace x with $-x$: $\mathrm{g}(-x) + \mathrm{g}(x) - \mathrm{g}\left(-\frac{1}{x}\right) = -x$ ———— (2) (1) − (2): $\mathrm{g}\left(-\frac{1}{x}\right) - \mathrm{g}\left(\frac{1}{x}\right) = 2x$

	Solution
	Replace x with $\frac{1}{x}$: $\mathrm{g}(-x)-\mathrm{g}(x)=\frac{2}{x}$ ———————— (3) Taking (3) − (1): $2\mathrm{g}(x)-\mathrm{g}\left(\frac{1}{x}\right)=x-\frac{2}{x}$ ———————— (4) Replace x with $\frac{1}{x}$: $2\mathrm{g}\left(\frac{1}{x}\right)-\mathrm{g}(x)=\frac{1}{x}-2x$ ———————— (5) Taking 2×(4) + (5): $3\mathrm{g}(x)=-\frac{3}{x}$ $\mathrm{g}(x)=-\frac{1}{x}$

	Solution
7(a)	$x^5+1=(x+1)(x^4-x^3+x^2-x+1)$ Assume that m has an odd divisor $2k+1>1$. Let $m=(2k+1)r$ for some positive integer r. Then $2^m+1=2^{(2k+1)r}+1$ $=\left(2^r\right)^{2k+1}+1$ $=\left(2^r+1\right)\left[2^{2kr}-2^{(2k-1)r}+\ldots+2^{2r}-2^r+1\right]$ This contradicts the primality of 2^m+1. Hence m cannot have any odd divisor. So m must be a power of 2, i.e. $m=2^k$, where $k\in\mathbb{Z}$.
7(b)	If p divides $2^{2^n}+1$, then $2^{2^n}+1\equiv 0 \pmod{p}$ $2^{2^n}\equiv -1 \pmod{p}$ Squaring both sides, $2^{2^{n+1}}\equiv 1 \pmod{p}$ Let d be the smallest positive integer such that $2^d\equiv 1 \pmod{p}$.

	Solution
	So d divides 2^{n+1}, but does not divide 2^n (since $2^{2^n} \equiv -1 \bmod p$). So $d = 2^{n+1}$. By Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod p$ Since d divides $p - 1$, so $p - 1 = k2^{n+1}$ for some integer k. Hence $p = k2^{n+1} + 1$, where $k \in \mathbb{Z}$.
7(c)	Let p and q be prime factors of $F_n = 2^{2^n} + 1$. By **(ii)**, $p = r2^{n+1} + 1$ and $q = s2^{n+1} + 1$ for some $r, s \in \mathbb{Z}$. Since $(r2^{n+1} + 1)(s2^{n+1} + 1)$ $= (rs2^{n+1} + r + s)2^{n+1} + 1$ is also of the form $k2^{n+1} + 1$. Hence any factor of $F_n = 2^{2^n} + 1$, whether prime or composite, must be of the form $k2^{n+1} + 1$, where $k \in \mathbb{Z}$.
7(d)	$641 = 5^4 + 2^4$ $\Rightarrow 5^4 + 2^4 \equiv 0 \pmod{641}$ $\Rightarrow 5^4\, 2^{28} + 2^{32} = 2^{28}(5^4 + 2^4) \equiv 0 \pmod{641}$ $641 = 5(2^7) + 1$ $\Rightarrow 5(2^7) + 1 \equiv 0 \pmod{641}$ $\Rightarrow 5^4\, 2^{28} - 1$ $= [5^2\, 2^{14} + 1]\, [5^2\, 2^{14} - 1]$ $= [5^2\, 2^{14} + 1]\, [5(2^7) + 1]\, [5(2^7) - 1] \equiv 0 \bmod 641$ Subtracting, we get $2^{32} + 1 \equiv 0 \bmod 641$ That is, 641 divides $F_5 = 2^{32} + 1$.

	Solution
8(a)	$D_q(x^n)=\dfrac{(qx)^n-x^n}{(q-1)x}$ $=\dfrac{(q^n-1)x^n}{(q-1)x}$ $=\dfrac{(q-1)(q^{n-1}+q^{n-2}+\ldots+1)}{q-1}x^{n-1}$ $=(q^{n-1}+q^{n-2}+\ldots+1)x^{n-1}$ $\lim\limits_{q\to1}D_q(x^n)=\lim\limits_{q\to1}\left(q^{n-1}+q^{n-2}+\ldots+1\right)x^{n-1}$ $=nx^{n-1}$ $=\mathrm{f}'(x)$
8(b)	$\mathrm{e}=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ $\ln\mathrm{e}=\ln\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ since $\mathrm{f}(x)=\ln x$ is continuous, $1=\lim\limits_{n\to\infty}\ln\left(1+\dfrac{1}{n}\right)^n$ Let $h=\dfrac{1}{n}$, we have $1=\lim\limits_{h\to0}\ln\left(1+h\right)^{\frac{1}{h}}$ $1=\lim\limits_{h\to0}\dfrac{\ln\left(1+h\right)}{h}$
8(c)	$\lim\limits_{q\to1}D_q(\ln x)=\lim\limits_{q\to1}\dfrac{\ln(qx)-\ln x}{(q-1)x}$ $=\lim\limits_{q\to1}\dfrac{\ln\frac{qx}{x}}{q-1}\dfrac{1}{x}$ $=\dfrac{1}{x}\lim\limits_{q\to1}\dfrac{\ln q}{q-1}$ Let $h=q-1$, we have

	Solution
	$\lim_{q\to 1} D_q(\ln x) = \frac{1}{x}\lim_{q\to 1}\frac{\ln q}{q-1}$ $= \frac{1}{x}\lim_{h\to 0}\frac{\ln(1+h)}{h}$ $= \frac{1}{x}$ $= \mathrm{f}'(x)$
8(d)(i)	$D_q\left[\mathrm{f}(x)\,\mathrm{g}(x)\right] = \frac{\mathrm{f}(qx)\,\mathrm{g}(qx) - \mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}$ $= \frac{\mathrm{f}(qx)\,\mathrm{g}(qx) - \mathrm{f}(qx)\,\mathrm{g}(x) + \mathrm{f}(qx)\,\mathrm{g}(x) - \mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}$ $= \frac{\mathrm{f}(qx)\,\mathrm{g}(qx) - \mathrm{f}(qx)\,\mathrm{g}(x)}{(q-1)x} + \frac{\mathrm{f}(qx)\,\mathrm{g}(x) - \mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}$ $= \mathrm{f}(qx)\frac{\mathrm{g}(qx) - \mathrm{g}(x)}{(q-1)x} + \mathrm{g}(x)\frac{\mathrm{f}(qx) - \mathrm{f}(x)}{(q-1)x}$ $= \mathrm{f}(qx)D_q\,\mathrm{g}(x) + \mathrm{g}(x)D_q\,\mathrm{f}(x)$ $D_q\left[\mathrm{f}(x)\,\mathrm{g}(x)\right] = \frac{\mathrm{f}(qx)\,\mathrm{g}(qx) - \mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}$ $= \frac{\mathrm{f}(qx)\,\mathrm{g}(qx) - \mathrm{f}(x)\,\mathrm{g}(qx) + \mathrm{f}(x)\,\mathrm{g}(qx) - \mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}$ $= \frac{\mathrm{f}(qx)\,\mathrm{g}(qx) - \mathrm{f}(x)\,\mathrm{g}(qx)}{(q-1)x} + \frac{\mathrm{f}(x)\,\mathrm{g}(qx) - \mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}$ $= \mathrm{g}(qx)\frac{\mathrm{f}(qx) - \mathrm{f}(x)}{(q-1)x} + \mathrm{f}(x)\frac{\mathrm{g}(qx) - \mathrm{g}(x)}{(q-1)x}$ $= \mathrm{f}(x)D_q\,\mathrm{g}(x) + \mathrm{g}(qx)D_q\,\mathrm{f}(x)$
8(d)(ii)	The product rule for the first principle shows symmetry while the product rule of q-derivative is asymmetric.

	Solution
8(e)(i)	$$D_q\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right]=D_q\left[\mathrm{f}(x)\frac{1}{\mathrm{g}(x)}\right]$$ $$=\mathrm{f}(qx)D_q\left[\frac{1}{\mathrm{g}(x)}\right]+\frac{1}{\mathrm{g}(x)}D_q\,\mathrm{f}(x)$$ $$=\mathrm{f}(qx)\frac{\frac{1}{\mathrm{g}(qx)}-\frac{1}{\mathrm{g}(x)}}{(q-1)x}+\frac{1}{\mathrm{g}(x)}D_q\,\mathrm{f}(x)$$ $$=\mathrm{f}(qx)\frac{\mathrm{g}(x)-\mathrm{g}(qx)}{(q-1)x}\frac{1}{\mathrm{g}(qx)\,\mathrm{g}(x)}+\frac{\mathrm{g}(qx)D_q\,\mathrm{f}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{-\mathrm{f}(qx)D_q\,\mathrm{g}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}+\frac{\mathrm{g}(qx)D_q\,\mathrm{f}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{\mathrm{g}(qx)D_q\,\mathrm{f}(x)-\mathrm{f}(qx)D_q\,\mathrm{g}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ Or $$D_q\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right]=D_q\left[\mathrm{f}(x)\frac{1}{\mathrm{g}(x)}\right]$$ $$=\mathrm{f}(x)D_q\left[\frac{1}{\mathrm{g}(x)}\right]+\frac{1}{\mathrm{g}(qx)}D_q\,\mathrm{f}(x)$$ $$=\mathrm{f}(x)\frac{\frac{1}{\mathrm{g}(qx)}-\frac{1}{\mathrm{g}(x)}}{(q-1)x}+\frac{1}{\mathrm{g}(qx)}D_q\,\mathrm{f}(x)$$ $$=\mathrm{f}(x)\frac{\mathrm{g}(x)-\mathrm{g}(qx)}{(q-1)x}\frac{1}{\mathrm{g}(qx)\,\mathrm{g}(x)}+\frac{\mathrm{g}(x)D_q\,\mathrm{f}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{-\mathrm{f}(x)D_q\,\mathrm{g}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}+\frac{\mathrm{g}(x)D_q\,\mathrm{f}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{\mathrm{g}(x)D_q\,\mathrm{f}(x)-\mathrm{f}(x)D_q\,\mathrm{g}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$

	Solution
	Alternatively, $$D_q\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right]=\frac{\dfrac{\mathrm{f}(qx)}{\mathrm{g}(qx)}-\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}}{(q-1)x}$$ $$=\frac{\dfrac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}(qx)}{\mathrm{g}(qx)\,\mathrm{g}(x)}}{(q-1)x}$$ $$=\frac{\dfrac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}(qx)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{\dfrac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(qx)\,\mathrm{g}(qx)+\mathrm{f}(qx)\,\mathrm{g}(qx)-\mathrm{f}(x)\,\mathrm{g}(qx)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{\dfrac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(qx)\,\mathrm{g}(qx)}{(q-1)x}-\dfrac{\mathrm{f}(x)\,\mathrm{g}(qx)-\mathrm{f}(qx)\,\mathrm{g}(qx)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{\dfrac{\mathrm{f}(qx)\left[\mathrm{g}(x)-\mathrm{g}(qx)\right]}{(q-1)x}-\dfrac{\left[\mathrm{f}(x)-\mathrm{f}(qx)\right]\mathrm{g}(qx)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{-\dfrac{\mathrm{f}(qx)\left[\mathrm{g}(qx)-\mathrm{g}(x)\right]}{(q-1)x}+\dfrac{\left[\mathrm{f}(qx)-\mathrm{f}(x)\right]\mathrm{g}(qx)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ $$=\frac{\mathrm{g}(qx)D_q\,\mathrm{f}(x)-\mathrm{f}(qx)D_q\,\mathrm{g}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$$ Or

	Solution
	$D_q\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right]=\frac{\frac{\mathrm{f}(qx)}{\mathrm{g}(qx)}-\frac{\mathrm{f}(x)}{\mathrm{g}(x)}}{(q-1)x}$ $=\frac{\frac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}(qx)}{\mathrm{g}(qx)\,\mathrm{g}(x)}}{(q-1)x}$ $=\frac{\frac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}(qx)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$ $=\frac{\frac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}(x)+\mathrm{f}(x)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}(qx)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$ $=\frac{\frac{\mathrm{f}(qx)\,\mathrm{g}(x)-\mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}-\frac{\mathrm{f}(x)\,\mathrm{g}(qx)-\mathrm{f}(x)\,\mathrm{g}(x)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$ $=\frac{\frac{\mathrm{g}(x)\left[\mathrm{f}(qx)-\mathrm{f}(x)\right]}{(q-1)x}-\frac{\left[\mathrm{g}(x)-\mathrm{g}(qx)\right]\mathrm{f}(x)}{(q-1)x}}{\mathrm{g}(qx)\,\mathrm{g}(x)}$ $=\frac{\mathrm{g}(x)D_q\,\mathrm{f}(x)-\mathrm{f}(x)D_q\,\mathrm{g}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$
8(e)(ii)	As $q\to 1$, $\lim_{q\to 1}\frac{\mathrm{g}(qx)D_q\,\mathrm{f}(x)-\mathrm{f}(qx)D_q\,\mathrm{g}(x)}{\mathrm{g}(qx)\,\mathrm{g}(x)}$ $=\frac{\lim_{q\to 1}\mathrm{g}(qx)D_q\,\mathrm{f}(x)-\mathrm{f}(qx)D_q\,\mathrm{g}(x)}{\lim_{q\to 1}\mathrm{g}(qx)\,\mathrm{g}(x)}$ $=\frac{\lim_{q\to 1}\mathrm{g}(qx)\lim_{q\to 1}D_q\,\mathrm{f}(x)-\lim_{q\to 1}\mathrm{f}(qx)\lim_{q\to 1}D_q\,\mathrm{g}(x)}{\left[\mathrm{g}(x)\right]^2}$ $=\frac{\mathrm{g}(x)\,\mathrm{f}'(x)-\mathrm{f}(x)\,\mathrm{g}'(x)}{\left[\mathrm{g}(x)\right]^2}$ Or

	Solution
	$\lim_{q\to 1}\dfrac{g(x)D_q\,f(x)-f(x)D_q\,g(x)}{g(qx)\,g(x)}$ $=\dfrac{\lim_{q\to 1} g(x)D_q\,f(x)-\lim_{q\to 1} f(x)D_q\,g(x)}{\lim_{q\to 1} g(qx)\,g(x)}$ $=\dfrac{g(x)\lim_{q\to 1} D_q\,f(x)-f(x)\lim_{q\to 1} D_q\,g(x)}{\left[g(x)\right]^2}$ $=\dfrac{g(x)\,f'(x)-f(x)\,g'(x)}{\left[g(x)\right]^2}$ As $q\to 1$, the quotient rule for q-derivative aligns with the quotient rule from the first principle.