Hwa Chong Institution 华侨中学
Mathematics (SMTP)

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MA302.5.X2 – Simultaneous Equations with Matrices

Matrices can also be used to solve several unknowns quickly in simultaneous equations.

X1. For a pair of simultaneous equations involving two unknowns, it can be rewritten into a matrix equation. For example, given equations $2x+y=5$ and $4x+3y=21$, by writing them into a matrix form, we have

$$\begin{pmatrix} 2x+y \\ 4x+3y \end{pmatrix} = \begin{pmatrix} 5 \\ 21 \end{pmatrix}.$$

However, the matrix on the left hand side can be broken apart into a product of two:

$$\begin{pmatrix} 2x+y \\ 4x+3y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Hence we can rewrite the matrix equation as

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 21 \end{pmatrix}.$$

Eventually, the problem simplifies into solving for the column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$.

S1. **Rewrite** the following simultaneous equations into matrix equations.

(a) $2x-3y=5$ and $4y+6x=12$

$$\begin{pmatrix} 2x-3y \\ 4y+6x \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 2x-3y \\ 6x+4y \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 & -3 \\ 6 & 4 \end{pmatrix}}_{\text{coefficient matrice}}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 5 \\ 12 \end{pmatrix}}_{\text{constant matrice}}$$

(b) $4x - \frac{1}{y} = 5$ and $3xy + 2 = 7y$

$$\begin{pmatrix} 4x - \frac{1}{y} \\ 3xy + 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7y \end{pmatrix}$$

$$\begin{pmatrix} 4x - \frac{1}{y} \\ 3x + \frac{2}{y} \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ \frac{1}{y} \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

(c) $2 - x^2 + 3y = 0$ and $4x^2 + 6y - 9 = 0$

$$\begin{pmatrix} 2 - x^2 + 3y \\ 4x^2 + 6y - 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x^2 - 3y \\ 4x^2 + 6y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x^2 \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

S2. Let $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{B}$, where $\mathbf{A}$ is a two 2×2 matrix. Make $\begin{pmatrix} x \\ y \end{pmatrix}$ the subject and explain the significance of the determinant of **A**, i.e. $\det(\mathbf{A})$.

$$A \begin{pmatrix} x \\ y \end{pmatrix} = B$$

$$A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} B$$

If determinant $= 0$, there are no solution to x and y.

We have thus arrive at the following:

Given a pair of simultaneous equations with unknowns x and y, the solution can be obtained by

Step 1: Converting into the matrix equation $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{B}$.

Step 2: Solve for $\begin{pmatrix} x \\ y \end{pmatrix}$.

Of course, the method fails when $\det(\mathbf{A}) = 0$. In this case, the two equations may have no or infinite number of solutions, depending on the matrix B.

S3. Use a matrix method to solve the following pairs of simultaneous equations.

(a) $6x^2 - 11y = -28$ and $5y - 3x^2 = 13$

$$\begin{pmatrix} 6x^2 - 11y \\ -3x^2 + 5y \end{pmatrix} = \begin{pmatrix} -28 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -11 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} x^2 \\ y \end{pmatrix} = \begin{pmatrix} -28 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ y \end{pmatrix} = \begin{pmatrix} 6 & -11 \\ -3 & 5 \end{pmatrix}^{-1}\begin{pmatrix} -28 \\ 13 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 5 & 11 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} -28 \\ 13 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$x^2 = -1$

x = no solution.

y = no solution.

(b) $4\sqrt{x} - 5\sqrt{y} = 13$ and $10\sqrt{y} - 8\sqrt{x} = 18$

$$\begin{pmatrix} 4 & -5 \\ -8 & 10 \end{pmatrix}\begin{pmatrix} \sqrt{x} \\ \sqrt{y} \end{pmatrix} = \begin{pmatrix} 13 \\ 18 \end{pmatrix}$$

det $= 4(10) - (-5)(-8)$

$= 0$

Since determinant $= 0$, there is no solution as

(c) $2x^2+3y^2=11$ and $x^2-y^2=3$

$$\begin{pmatrix} 2x^2+3y^2 \\ x^2-y^2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.2 & 0.6 \\ 0.2 & -0.4 \end{pmatrix}\begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$\therefore x = \pm 2$

$y = \pm 1$

$(x, y) = (2,1), (2,-1), (-2,1), (-2,-1)$.

S4. Find the inverse of $\begin{pmatrix} 7 & -3 \\ 3 & 2 \end{pmatrix}$ and solve the simultaneous equations $7x-3y=-29$ and $12x+8y=16$.

Inverse of $\begin{pmatrix} 7 & -3 \\ 3 & 2 \end{pmatrix} = \frac{1}{23}\begin{pmatrix} 2 & 3 \\ -3 & 7 \end{pmatrix}$

$$= \begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ -\frac{3}{23} & \frac{7}{23} \end{pmatrix}$$

$$\begin{pmatrix} 7x-3y \\ 12x+8y \end{pmatrix} = \begin{pmatrix} -29 \\ 16 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -3 \\ 12 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -29 \\ 16 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -29 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{23} & \frac{3}{23} \\ -\frac{3}{23} & \frac{7}{23} \end{pmatrix}\begin{pmatrix} -29 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$\therefore x = -2$

$y = 5$