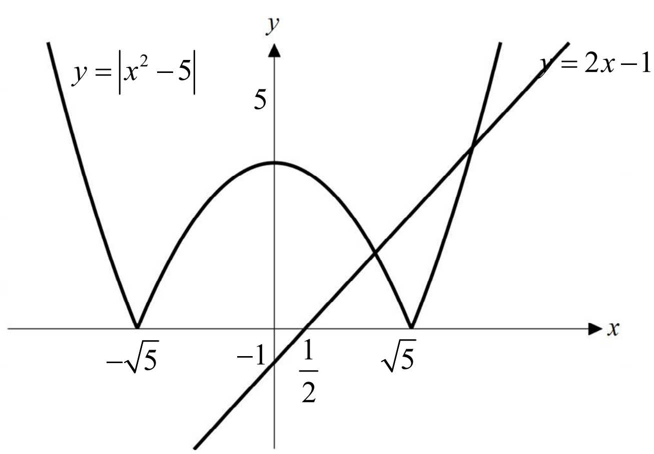
Raffles Institution
H2 Mathematics (9758)
Solution for 2021 A-Level Paper 1

Question 1

No.	Suggested Solution	Remarks for Student
	$\mathrm{f}(1)=5 \Rightarrow \quad a+b+c+d=5 \quad ----(1)$ $\mathrm{f}(-1)=-3 \quad \Rightarrow \quad -a+b-c+d=-3 \quad ----(2)$ $\mathrm{f}'(x)=3ax^2+2bx+c$ $\mathrm{f}'(1)=0 \quad \Rightarrow \quad 3a+2b+c=0 \quad ----(3)$ $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x=6 \quad \Rightarrow \quad \int_0^1 ax^3+bx^2+cx+d\,\mathrm{d}x=6$ $\left[\frac{1}{4}ax^4+\frac{1}{3}bx^3+\frac{1}{2}cx^2+dx\right]_0^1=6$ $\frac{1}{4}a+\frac{1}{3}b+\frac{1}{2}c+d=6 \quad ----(4)$ Using GC to solve for a, b, c and d, we get $a=4, b=-6, c=0, d=7$	

Question 2

No.	Suggested Solution	Remarks for Student
(a)		
(b)	From the above graph, we can see that the 2 graphs intersect at two points. The left-most intersection point can be found using	The question asks for **exact solutions**, so full algebraic working is required

	$$5-x^2=2x-1$$ $$x^2+2x-6=0$$ $$x=\frac{-2\pm\sqrt{4+24}}{2}=-1\pm\sqrt{7}$$ From the graph, $0<x<\sqrt{5}$, so $x=\sqrt{7}-1$ The right-most intersection point can be found using $$x^2-5=2x-1$$ $$x^2-2x-4=0$$ $$x=\frac{2\pm\sqrt{4+16}}{2}=1\pm\sqrt{5}$$ From the graph, $x>\sqrt{5}$, so $x=\sqrt{5}+1$ The solutions are $\sqrt{7}-1$ and $\sqrt{5}+1$.	

Question 3

No.	Suggested Solution	Remarks for Student
(a)	$x^{\frac{1}{2}}+y^{\frac{1}{2}}=3$ Differentiate with respect with x, $$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$ $$\frac{1}{y^{\frac{1}{2}}}\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{x^{\frac{1}{2}}}$$ $$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}=-\left(\frac{y}{x}\right)^{\frac{1}{2}}$$	
(b)	When $x=1$, $1+y^{\frac{1}{2}}=3 \quad \Rightarrow \quad y=4$ Gradient of the curve at $x=1$ is $-\left(\frac{4}{1}\right)^{\frac{1}{2}}=-2$ Equation of normal at $x=1$ is $y-4=\frac{1}{2}(x-1)$ $$y=\frac{1}{2}x+\frac{7}{2}$$	

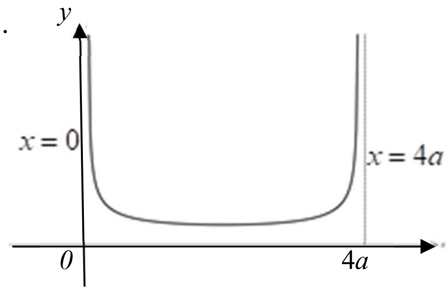
Question 4

No.	Suggested Solution	Remarks for Student
(a)	$z=\dfrac{\left(\cos\left(\frac{1}{16}\pi\right)+\mathrm{i}\sin\left(\frac{1}{16}\pi\right)\right)^2}{\cos\left(\frac{1}{8}\pi\right)-\mathrm{i}\sin\left(\frac{1}{8}\pi\right)}=\dfrac{\left(\mathrm{e}^{\mathrm{i}\frac{1}{16}\pi}\right)^2}{\mathrm{e}^{-\mathrm{i}\frac{1}{8}\pi}}=\dfrac{\mathrm{e}^{\mathrm{i}\frac{1}{8}\pi}}{\mathrm{e}^{-\mathrm{i}\frac{1}{8}\pi}}=\mathrm{e}^{\mathrm{i}\frac{1}{4}\pi}$ $z^2=\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^2=\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}=\mathrm{i}$ **Alternatively,** $z=\dfrac{\left(\cos\left(\frac{1}{16}\pi\right)+\mathrm{i}\sin\left(\frac{1}{16}\pi\right)\right)^2}{\cos\left(\frac{1}{8}\pi\right)-\mathrm{i}\sin\left(\frac{1}{8}\pi\right)}$ $\lvert z\rvert=\dfrac{\left\lvert\cos\left(\frac{1}{16}\pi\right)+\mathrm{i}\sin\left(\frac{1}{16}\pi\right)\right\rvert^2}{\left\lvert\cos\left(\frac{1}{8}\pi\right)-\mathrm{i}\sin\left(\frac{1}{8}\pi\right)\right\rvert}=\dfrac{1^2}{1}=1$ $\arg z=\arg\left(\left(\cos\left(\frac{1}{16}\pi\right)+\mathrm{i}\sin\left(\frac{1}{16}\pi\right)\right)^2\right)-\arg\left(\cos\left(\frac{1}{8}\pi\right)-\mathrm{i}\sin\left(\frac{1}{8}\pi\right)\right)$ $=2\arg\left(\cos\left(\frac{1}{16}\pi\right)+\mathrm{i}\sin\left(\frac{1}{16}\pi\right)\right)-\arg\left(\cos\left(-\frac{1}{8}\pi\right)+\mathrm{i}\sin\left(-\frac{1}{8}\pi\right)\right)$ $=2\left(\frac{1}{16}\pi\right)-\left(-\frac{1}{8}\pi\right)=\frac{1}{4}\pi$ So, z can be written as $z=\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$.	Note that calculator is not allowed in answering this question, thus all steps need to be clearly shown. $\cos\left(\frac{1}{8}\pi\right)-\mathrm{i}\sin\left(\frac{1}{8}\pi\right)$ $=\cos\left(-\frac{1}{8}\pi\right)+\mathrm{i}\sin\left(-\frac{1}{8}\pi\right)$ $=\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}$
(b)(i)	$\left(\cos\theta+\mathrm{i}\sin\theta\right)\left(1+\cos\theta-\mathrm{i}\sin\theta\right)=\mathrm{e}^{\mathrm{i}\theta}\left(1+\mathrm{e}^{-\mathrm{i}\theta}\right)$ $=\mathrm{e}^{\mathrm{i}\theta}+1$ $=\cos\theta+\mathrm{i}\sin\theta+1$	
(b)(ii)	The equation in **(b)(i)** holds for all values of θ, in particular we let $\theta=\dfrac{\pi}{4}$. Then, $z(1+z^*)=1+z$ $\Rightarrow\quad z^4(1+z^*)^4=\left(1+z\right)^4\quad ---(1)$ But, $z^2=\mathrm{i}\quad\Rightarrow\quad z^4=-1$ Then, (1) becomes $-(1+z^*)^4=\left(1+z\right)^4$ $\left(1+z\right)^4+(1+z^*)^4=0$	

Question 5

No.	Suggested Solution	Remarks for Student
(a)	$\dfrac{x}{(x+2)(x+3)(x+4)}$ $=\dfrac{A}{x+2}+\dfrac{B}{x+3}+\dfrac{C}{x+4}$ $=\dfrac{A(x+3)(x+4)+B(x+2)(x+4)+C(x+2)(x+3)}{(x+2)(x+3)(x+4)}$ Comparing numerators, $A(x+3)(x+4)+B(x+2)(x+4)+C(x+2)(x+3)=x$ When $x=-2$, $A(1)(2)=-2 \quad \Rightarrow A=-1$ When $x=-3$, $B(-1)(1)=-3 \quad \Rightarrow B=3$ When $x=-4$, $C(-2)(-1)=-4 \quad \Rightarrow C=-2$ So, $\dfrac{x}{(x+2)(x+3)(x+4)}=\dfrac{-1}{x+2}+\dfrac{3}{x+3}+\dfrac{-2}{x+4}$	
(b)	$\displaystyle\sum_{r=1}^{n}\frac{r}{(r+2)(r+3)(r+4)}=\sum_{r=1}^{n}\left(\frac{-1}{r+2}+\frac{3}{r+3}+\frac{-2}{r+4}\right)$ $=-\dfrac{1}{3}+\dfrac{3}{4}-\cancel{\dfrac{2}{5}}$ $-\dfrac{1}{4}+\cancel{\dfrac{3}{5}}-\cancel{\dfrac{2}{6}}$ $-\cancel{\dfrac{1}{5}}+\cancel{\dfrac{3}{6}}-\cancel{\dfrac{2}{7}}$ $-\cancel{\dfrac{1}{6}}+\cancel{\dfrac{3}{7}}-\cancel{\dfrac{2}{8}}$ $+$ $\vdots$ $+$ $-\cancel{\dfrac{1}{n}}+\cancel{\dfrac{3}{n+1}}-\cancel{\dfrac{2}{n+2}}$ $-\cancel{\dfrac{1}{n+1}}+\cancel{\dfrac{3}{n+2}}-\dfrac{2}{n+3}$ $-\cancel{\dfrac{1}{n+2}}+\dfrac{3}{n+3}-\dfrac{2}{n+4}$ $=-\dfrac{1}{3}+\dfrac{3}{4}-\dfrac{1}{4}-\dfrac{2}{n+3}+\dfrac{3}{n+3}-\dfrac{2}{n+4}$ $=\dfrac{1}{6}+\dfrac{1}{n+3}-\dfrac{2}{n+4}$	
(c)	$\displaystyle\sum_{r=1}^{\infty}\frac{r}{(r+2)(r+3)(r+4)}=\frac{1}{6}$	Since the question says "state", working is not required.

Question 6

No.	**Suggested Solution**	**Remarks for Student**
(a)	Note that C is not defined when $4ax - x^2 = 0$, i.e. $x = 0$ or $4a$.  $x = 0$ $x = 4a$	
(b)	Smallest possible value of y occurs when $4ax - x^2$ is the largest. The largest value of $4ax - x^2$ occurs at $x = 2a$. Thus, the smallest possible value of y is $\dfrac{1}{\sqrt{4a(2a)-(2a)^2}} = \dfrac{1}{2a}$.	
(c)	$y = \dfrac{1}{\sqrt{4ax - x^2}} = \dfrac{1}{\sqrt{4a^2 - (x-2a)^2}}$ Thus, $y = \dfrac{1}{\sqrt{4ax - x^2}} = \dfrac{1}{\sqrt{4a^2 - (x-2a)^2}} \xrightarrow{\text{replace } x \text{ by } x+2a} y = \dfrac{1}{\sqrt{4a^2 - x^2}}$ The transformation is a transaltion in the negative x-direction by $2a$ units.	

Question 7

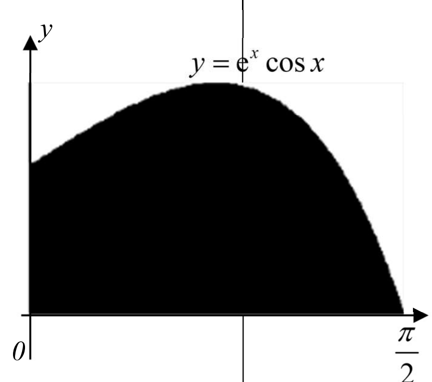
No.	Suggested Solution	Remarks for Student
(a)	$y=\mathrm{e}^{\sin^{-1}x} \quad \Rightarrow \quad \ln y=\sin^{-1}x$ Differentiate with respect to x, $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$ $\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}=y$ $\left(1-x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=y^2$ Differentiate with respect to x, $\left(1-x^2\right)2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}=y$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=y+x\frac{\mathrm{d}y}{\mathrm{d}x} \quad ----(1)$ **Alternatively,** $y=\mathrm{e}^{\sin^{-1}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}\mathrm{e}^{\sin^{-1}x}$ $\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}=y$ Differentiating wrt x $\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\frac{\mathrm{d}y}{\mathrm{d}x}+\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y}{\mathrm{d}x}$ $(-x)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}$ $(-x)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=y$ $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=y+x\frac{\mathrm{d}y}{\mathrm{d}x}$	
(b)	Differentiate (1) with respect to x, $\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ When $x=0$, $y=1,\ \frac{\mathrm{d}y}{\mathrm{d}x}=1,\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=1,\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=2.$ So, $\mathrm{e}^{\sin^{-1}x}=1+x+\frac{1}{2!}x^2+\frac{2}{3!}x^3+\cdots=1+x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$	

Question 8

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)</td><td>The direction vectors of the plane are $\begin{pmatrix}2\\-3\\1\end{pmatrix}$ and $\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}1\\-3\\-1\end{pmatrix}=\begin{pmatrix}2\\5\\1\end{pmatrix}$
A normal vector to the plane $=\begin{pmatrix}2\\-3\\1\end{pmatrix}\times\begin{pmatrix}2\\5\\1\end{pmatrix}=\begin{pmatrix}-8\\0\\16\end{pmatrix}=-8\begin{pmatrix}1\\0\\-2\end{pmatrix}$
Equation of the plane $\mathbf{r}.\begin{pmatrix}1\\0\\-2\end{pmatrix}=\begin{pmatrix}3\\2\\0\end{pmatrix}.\begin{pmatrix}1\\0\\-2\end{pmatrix}=3$
The Cartesian equation of the plane is $x-2z=3$.</td><td></td></tr>
<tr><td>(b)</td><td>Two lines are perpendicular if their direction vectors are perpendicular.
$\begin{pmatrix}2\\-3\\1\end{pmatrix}.\begin{pmatrix}1\\2\\4\end{pmatrix}=2-6+4=0$
Thus, l_1 is perpendicular to l_2.</td><td></td></tr>
<tr><td>(c)(i)</td><td>$\mathbf{r}_1-\mathbf{r}_2=\begin{pmatrix}3\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}-\left[\begin{pmatrix}-4\\1\\-3\end{pmatrix}+\mu\begin{pmatrix}1\\2\\4\end{pmatrix}\right]=\begin{pmatrix}7\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}-\mu\begin{pmatrix}1\\2\\4\end{pmatrix}$
Since $\mathbf{r}_1-\mathbf{r}_2$ is perpendicular to l_1,
$$(\mathbf{r}_1-\mathbf{r}_2).\begin{pmatrix}2\\-3\\1\end{pmatrix}=0$$
$$\left[\begin{pmatrix}7\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}-\mu\begin{pmatrix}1\\2\\4\end{pmatrix}\right].\begin{pmatrix}2\\-3\\1\end{pmatrix}=0$$
$$14+14\lambda=0$$
$$\lambda=-1$$
Likewise for the line l_2, $(\mathbf{r}_1-\mathbf{r}_2).\begin{pmatrix}1\\2\\4\end{pmatrix}=0$
$$\left[\begin{pmatrix}7\\1\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix}-\mu\begin{pmatrix}1\\2\\4\end{pmatrix}\right].\begin{pmatrix}1\\2\\4\end{pmatrix}=0$$
$$21-21\mu=0$$
$$\mu=1$$</td><td></td></tr>
</table>

	The position vectors are $\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}1\\5\\-1\end{pmatrix}$ and $\begin{pmatrix}-4\\1\\-3\end{pmatrix}+\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-3\\3\\1\end{pmatrix}$ **Alternatively,** $\begin{pmatrix}1\\2\\4\end{pmatrix}\times\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}14\\7\\-7\end{pmatrix}=7\begin{pmatrix}2\\1\\-1\end{pmatrix}$ Since $\mathbf{r}_1-\mathbf{r}_2$ is perpendicular to both l_1 and l_2, then $\mathbf{r}_1-\mathbf{r}_2=k\begin{pmatrix}2\\1\\-1\end{pmatrix}$, for some $k\in\mathbb{R}$ $\begin{pmatrix}7+2\lambda-\mu\\1-3\lambda-2\mu\\3+\lambda-4\mu\end{pmatrix}=k\begin{pmatrix}2\\1\\-1\end{pmatrix}$ From G.C., $\lambda=-1,\ \mu=1$ The position vectors are $\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}1\\5\\-1\end{pmatrix}$ and $\begin{pmatrix}-4\\1\\-3\end{pmatrix}+\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-3\\3\\1\end{pmatrix}$	Note that $\mathbf{b}_1\times\mathbf{b}_2$ is a vector perpendicular to both lines l_1 and l_2, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are direction vectors of l_1 and l_2.
(c)(ii)	Length of common perpendicular is $\left\|\begin{pmatrix}1\\5\\-1\end{pmatrix}-\begin{pmatrix}-3\\3\\1\end{pmatrix}\right\|=\left\|\begin{pmatrix}4\\2\\-2\end{pmatrix}\right\|=\sqrt{24}=2\sqrt{6}$	

Question 9

No.	Suggested Solution	Remarks for Student
(a)	$f'(x) = e^x \cos x - e^x \sin x$ For stationary point, $f'(x) = 0$ $e^x \cos x - e^x \sin x = 0$ $\tan x = 1$ $x = \frac{\pi}{4}$, since $0 \le x \le \frac{\pi}{2}$ The coordinates of the stationary point is $\left(\frac{\pi}{4}, e^{\frac{\pi}{4}} \cos\frac{\pi}{4}\right) = \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}\right)$ $f''(x) = e^x \cos x - e^x \sin x - \left(e^x \sin x + e^x \cos x\right)$ $= -2e^x \sin x$ So, $f''\left(\frac{\pi}{4}\right) = -2e^{\frac{\pi}{4}} \sin\frac{\pi}{4} = -\sqrt{2}e^{\frac{\pi}{4}} < 0$ Thus, stationary point is a maximum point.	
(b)	$\int e^{2x} \cos 2x \, dx = \frac{1}{2}e^{2x} \cos 2x + \int e^{2x} \sin 2x \, dx$ $= \frac{1}{2}e^{2x} \cos 2x + \frac{1}{2}e^{2x} \sin 2x - \int e^{2x} \cos 2x \, dx$ $2\int e^{2x} \cos 2x \, dx = \frac{1}{2}e^{2x} \cos 2x + \frac{1}{2}e^{2x} \sin 2x + C$ $\int e^{2x} \cos 2x \, dx = \frac{1}{4}e^{2x}\left(\cos 2x + \sin 2x\right) + D$	
(c)	Volume $= \pi\int_0^{\frac{\pi}{2}} y^2 \, dx = \pi\int_0^{\frac{\pi}{2}} e^{2x} \cos^2 x \, dx$ $= \frac{1}{2}\pi\int_0^{\frac{\pi}{2}} e^{2x}\left(1 + \cos 2x\right) dx$ $= \frac{1}{2}\pi\int_0^{\frac{\pi}{2}} e^{2x} \, dx + \frac{1}{2}\pi\int_0^{\frac{\pi}{2}} e^{2x} \cos 2x \, dx$ $= \frac{1}{2}\pi\left[\frac{1}{2}e^{2x}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\pi\left(\frac{1}{4}\right)\left[e^{2x}\left(\sin 2x + \cos 2x\right)\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{4}\pi\left(e^{\pi} - 1\right) + \frac{1}{8}\pi\left(-e^{\pi} - 1\right) = \frac{1}{8}\pi\left(e^{\pi} - 3\right)$  	

Question 10

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)</td><td>

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN$$
$$\int \frac{1}{N}\,\mathrm{d}N = \int k\,\mathrm{d}t$$
$$\ln|N| = kt + C$$
$$\ln N = kt + C, \quad \text{since } N > 0$$
$$N = \mathrm{e}^{kt+C} = A\mathrm{e}^{kt}, \text{ where } A = \mathrm{e}^{c}$$

Given that $N\big|_{t=0} = 100 \quad \Rightarrow \quad A = 100$

$N\big|_{t=2} = 300 \quad \Rightarrow \quad 100\mathrm{e}^{2k} = 300$

$$\mathrm{e}^{2k} = 3$$
$$k = \frac{1}{2}\ln 3$$

When $N = 1000$, $1000 = 100\mathrm{e}^{kt}$

$$t = \frac{1}{k}\ln 10 = \frac{2\ln 10}{\ln 3} \approx 4.19 \text{ mins}$$

</td><td></td></tr>
<tr><td>(b)</td><td>

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN - d$$
$$\int \frac{1}{kN-d}\,\mathrm{d}N = \int 1\,\mathrm{d}t$$
$$\frac{1}{k}\ln|kN-d| = t + C$$
$$\ln|kN-d| = kt + kC$$
$$|kN-d| = \mathrm{e}^{kt+kC}$$
$$kN - d = \pm\mathrm{e}^{kt+kC}$$
$$kN - d = A\mathrm{e}^{kt} \text{ where } A = \pm\mathrm{e}^{kC}$$

When $t = 0$, $N = 100$. So, $100k - d = A$

Thus $kN - d = (100k - d)\mathrm{e}^{kt}$

$$\mathrm{e}^{kt} = \frac{kN-d}{100k-d}$$
$$t = \frac{1}{k}\ln\left(\frac{kN-d}{100k-d}\right)$$
$$t = \frac{2}{\ln 3}\ln\left(\frac{N\ln 3 - 2d}{100\ln 3 - 2d}\right)$$

To find N in terms of t and d,

</td><td>To give the answer for t in terms of N and d, thus need to substitute in the value of k found in (a)</td></tr>
</table>

	$\frac{\ln 3}{2}t = \ln\left(\frac{N\ln 3 - 2d}{100\ln 3 - 2d}\right)$ $\frac{N\ln 3 - 2d}{100\ln 3 - 2d} = e^{\frac{t}{2}\ln 3} = 3^{\frac{t}{2}}$ $N\ln 3 - 2d = 3^{\frac{t}{2}}\left(100\ln 3 - 2d\right)$ $N = \frac{1}{\ln 3}\left[3^{\frac{t}{2}}\left(100\ln 3 - 2d\right) + 2d\right]$ $= \frac{2}{\ln 3}\left[3^{\frac{t}{2}}\left(50\ln 3 - d\right) + d\right]$	
(c) (i)	For the number of bacteria to decrease, $\frac{\mathrm{d}N}{\mathrm{d}t} < 0$ $kN - d < 0$ $d > kN = \left(\frac{1}{2}\ln 3\right)100 = 50\ln 3 \approx 54.9$ The range of d is $d > 54.9$	
(c) (ii)	When $N = 0$, $t = \frac{2}{\ln 3}\ln\left(\frac{-2\left(58\right)}{100\ln 3 - 2(58)}\right) \approx 5.35$ mins	

Question 11

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)</td><td>

$x\big|_{\theta=\beta} = a(\beta - \sin\beta),$

$x\big|_{\theta=2\pi-\beta} = a\left(2\pi - \beta - \sin\left(2\pi-\beta\right)\right) = a\left(2\pi - \beta + \sin\beta\right)$

Distance $PS = a\left(2\pi - \beta + \sin\beta\right) - a(\beta - \sin\beta)$

$= a\left(2\pi - 2\beta + 2\sin\beta\right)$

$= 2a\left(\pi - \beta + \sin\beta\right)$

</td><td></td></tr>
<tr><td>(b)</td><td>

Area of shaded region

$= \int_{a(\beta-\sin\beta)}^{a(2\pi-\beta+\sin\beta)} y \, \mathrm{d}x$

$= \int_{\beta}^{2\pi-\beta} y \frac{\mathrm{d}x}{\mathrm{d}\theta} \mathrm{d}\theta$

$= \int_{\beta}^{2\pi-\beta} a(1-\cos\theta)a(1-\cos\theta)\,\mathrm{d}\theta$

$= a^2 \int_{\beta}^{2\pi-\beta} 1 - 2\cos\theta + \cos^2\theta \,\mathrm{d}\theta$

$= a^2 \int_{\beta}^{2\pi-\beta} 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2} \mathrm{d}\theta$

$= a^2 \int_{\beta}^{2\pi-\beta} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta \,\mathrm{d}\theta$

$= a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\beta}^{2\pi-\beta}$

$= a^2 \left[\begin{array}{l} \frac{3}{2}\left(2\pi-\beta-\beta\right) - 2\left(\sin\left(2\pi-\beta\right) - \sin\beta\right) \\ +\frac{1}{4}\left(\sin 2\left(2\pi-\beta\right) - \sin 2\beta\right) \end{array}\right]$

$= a^2 \left[3\left(\pi-\beta\right) + 4\sin\beta - \frac{1}{2}\sin 2\beta\right]$

$= \frac{1}{2}a^2\left(6\pi - 6\beta + 8\sin\beta - \sin 2\beta\right)$

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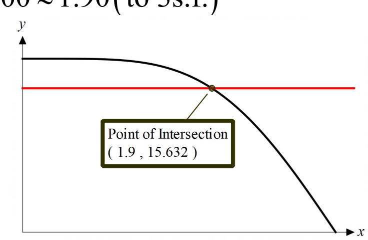
<tr><td>(c)</td><td>

$\frac{1}{2}a^2\left(6\pi - 6\beta + 8\sin\beta - \sin 2\beta\right) = 7.8159a^2$

$6\pi - 6\beta + 8\sin\beta - \sin 2\beta = 15.6318$

Solving using GC, $\beta \approx 1.9000 \approx 1.90\left(\text{to 3s.f.}\right)$



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(d)	$PS = 50$ $2a(\pi - \beta + \sin\beta) = 50$ $a = \dfrac{25}{\pi - 1.9000 + \sin 1.9000} \approx 11.42652$ Least height of the arc = Distance PQ $= y\big\vert_{\theta=\beta}$ $= a(1-\cos\beta)$ $\approx 11.42652(1-\cos 1.90)$ $\approx 15.1\,\text{m}$ Greatest height of the arc = Largest value of y = Largest value of $a(1-\cos\beta)$ $= a(1+1)$, occurs when $\beta = \pi$ $\approx 2(11.42652)$ $\approx 22.9\,\text{m}$	Differentiation is not required for finding the least and greatest height of the arc as it can be seen from the given diagram.