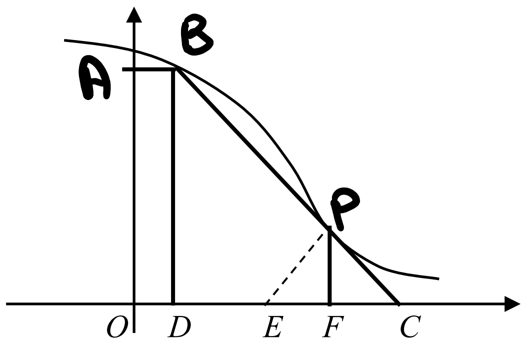
Differentiation: Rate of Change, Maxima & Minima Problems

1. **ACJC Prelim/2020/01/Q11**



The diagram above shows part of a roller coaster track, made up by a curve with parametric equations

$$x = 3t^2 - 10t - 1, \qquad y = 16\tan^{-1} t^3 - 4t + 16, \qquad \text{for } -2 \le t \le 0,$$

with points B and P on the curve.

Rods are used to support the track. AB is a horizontal metal rod. Another straight metal rod, fixed from point B to C, is tangential to the track at point P. In order to further strengthen the support, vertical metal rods BD and PF, of length m metres and $4(5-\pi)$ metres respectively, are fixed.

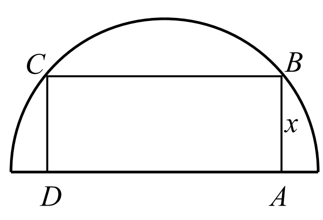
(i) Find the x-coordinate of point P. [2]

Hence show that the equation of line BC is $5x + 4y + 16\pi - 140 = 0$. [3]

(ii) Find m, giving your answer to 3 decimal places. [3]

For regular maintenance, an extendable ladder, fixed at point E, is extended till it meets rod BC. If the ladder and rod BC meet at point P, then the length of the extended ladder, l, is at its shortest length. Find the exact value of l. [4]

2. **PJC/I/6**



The diagram shows a rectangle $ABCD$ inscribed in a semi-circle with fixed radius r cm. Two vertices of the rectangle lie on the arc of the semi-circle. If $AB = x$ cm, show that the perimeter P of the rectangle $ABCD$ is $2x + 4\sqrt{r^2 - x^2}$. [2]

Given that as x varies, the maximum value of P occurs when $AB : BC = 1 : k$, find k. [4]

3. **YJC/I/Q5**

(a) (i) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\left(\dfrac{2x}{1+x^2}\right)\right) = \dfrac{-2}{1+x^2}$ for $-1 \le x \le 1$.

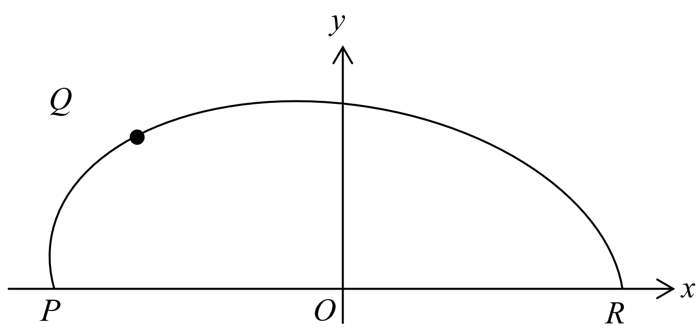
Explain, in your working, how you use the condition $-1 \le x \le 1$ to conclude your answer. [3]

(ii) Hence, determine the constants A and B satisfying the equation

$$\cos^{-1}\left(\frac{2x}{1+x^2}\right) + A\tan^{-1} x = B \text{, for } -1 \le x \le 1.$$ [3]

(b) A right cone with base area A has a fixed height of 10 cm. Given that the base area is increasing at a rate of 2 cm^2s^{-1}, calculate the rate of change of the volume of the cone. [2]

4. **DHS/2013/I/11**



The point $Q(x, y)$ lies on the curve $4x^2 + xy + y^2 = 36$, where $y \ge 0$, as shown in the diagram above. The curve cuts the x-axis at the points P and R.

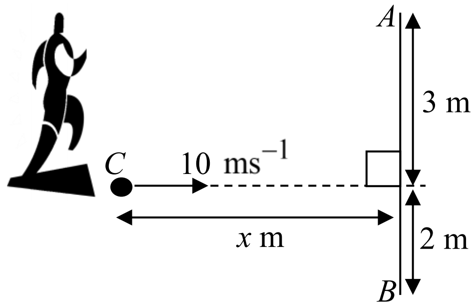
(i) Show that A, the area of triangle PQR, is given by $A = 3y$. [3]

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [2]

(iii) Hence find the value of x for which A has a stationary value. Using the second derivative test, determine the nature of this stationary value. [6]

(iv) If x increases at a constant rate of 8 units/s, find the rate of change of A when $x = 0$. [3]

5. **PJC/2010/I/Q4**



An athlete at point C is running towards the finishing line AB, at a constant speed of 10 ms^{-1} in a direction perpendicular to AB, as shown in the diagram above.

(i) Given that $\measuredangle ACB$ is θ and x is the distance of the athlete from AB, show that

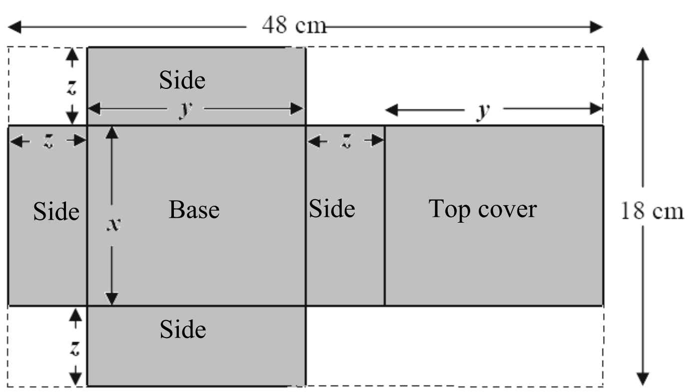
$$\theta = \tan^{-1}\frac{3}{x} + \tan^{-1}\frac{2}{x}.$$ [2]

(ii) Find the exact rate of change of θ when the athlete is 10 m from the finishing line. [3]

6. **ACJC/2011/I/7**

An isosceles triangle has fixed base of length b cm. The other 2 equal sides of the triangle are each decreasing at the constant rate of 3 cm per second. How fast is the area changing when the triangle is equilateral? Leave your answer in terms of b.

7. **AJC/2011/I/7**

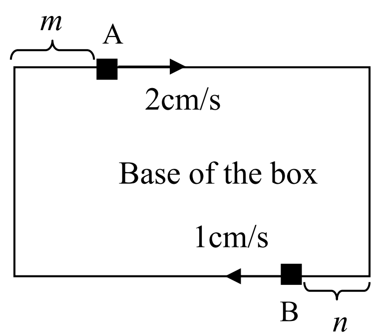


The above 48 cm by 18 cm plastic sheet is cut out to form a shape represented by the shaded region. The shape is folded into a box with width x, length y and height z as shown in the diagram.

(i) Express the volume of the box, V, in terms of y. [2]

(ii) Using differentiation, find the maximum possible volume of the box. [3]

Two small robots of negligible dimensions, A and B, are initially placed on the edge of the base of the box where $m = 0$ cm and $n = 0$ cm respectively as shown in the diagram. Robot A starts to travel along the length of the base with speed 2 cm/s. One second later, Robot B starts to travel in the opposite direction along the length of the base with speed 1 cm/s.



(iii) Given that the base of the box has length 20 cm and breadth 10 cm, find the rate of change of the distance between the two robots when $n = 4$ cm. [4]

8. **RVHS/2014/I/6**

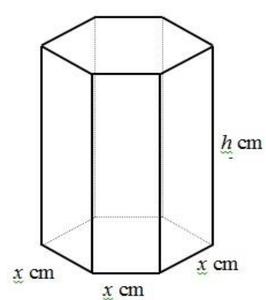
A right circular cone has a base radius of r cm and height h cm. It is given that the cone has a fixed volume of 10 cm^3 and a curved surface area of A cm^2.

(i) Show that $A^2 = \pi^2 r^4 + \dfrac{900}{r^2}$. [3]

(ii) Using differentiation, find the value of r such that the curved surface area of the cone is a minimum, giving your answer correct to 3 decimal places. [4]

9. **CJC2012/I/8**

Candy is being stored in a closed container in the form of a regular hexagonal prism, with sides x cm and height h cm (as shown in the figure below).



Given that the container has a volume of 972 cm^3,

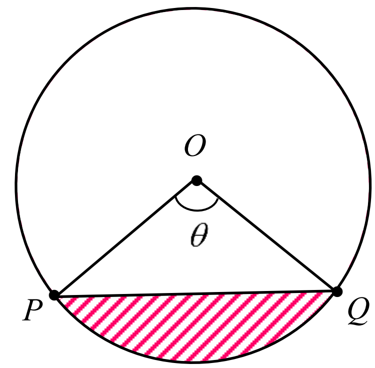
(i) show that its base area given by $\dfrac{3\sqrt{3}x^2}{2}$ cm^2. [1]

(ii) Using differentiation, find the minimum area of the material, A cm^2, that is used to make the container, leaving your answer to 2 decimal places. [6]

(iii) Given that the cost of the material for the packaging is \$ 0.05 per 100 cm^2, find the minimum cost required for the container. [1]

10. **TJC/2013/II/1**

The diagram below shows the points P and Q on the circumference of a circle with centre O, and radius $2a$ cm, where $\angle POQ = \theta$. Points P and Q are moving on the circumference so that θ is increasing at a constant rate.



Find the acute angle θ at the instant when the rate of change of the area of the shaded segment is $\frac{a}{2}$ times the rate of change of the length of the minor arc PQ. [5]

11. **ACJC/2014/I/2**

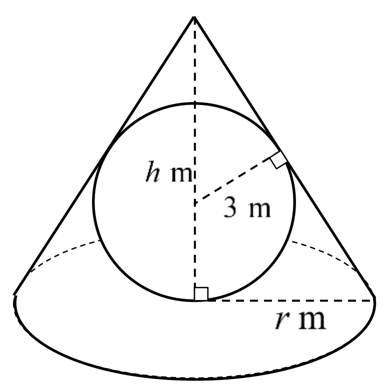
P is a variable point on the circumference of a circle with diameter AB, and Q is the point on AB such that $AQ = AP$. Given that angle $PAQ = \theta$ radians and $AB = \ell$, show that the area S of triangle PAQ is given by $S = \frac{1}{2}\ell^2\left(\sin\theta - \sin^3\theta\right)$. [3]

Use differentiation to find, in surd form and in terms of ℓ, the maximum value of S, proving that it is a maximum. [4]

12. **IJC/2017/I/11**

[It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the volume and surface area of a sphere of radius r are $\frac{4}{3}\pi r^3$ and $4\pi r^2$ respectively.]

In a distant Northern kingdom of Drivenbell, Elsanna builds a spherical snowball with radius 3 m. The snowball is inscribed in a right conical container of base radius r m and height h m. The container is specially designed to allow the snowball to remain intact with fixed radius 3 m (see diagram).



(i) By considering the slant height of the cone, show that $r=\dfrac{3h}{\sqrt{\left(h^2-6h\right)}}$. [3]

(ii) Use differentiation to find the values of h and r that give a minimum volume for the container. Find the value of the minimum volume. [6]

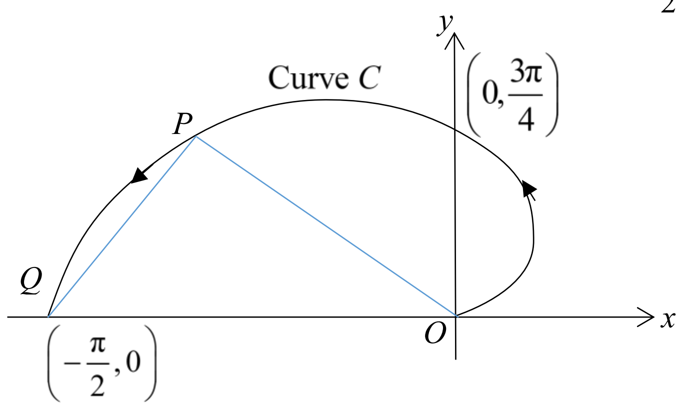
The snowball is being removed from the container and it starts to melt under room temperature.

(iii) Assuming that the snowball remains spherical as it melts, find the rate of decrease of its volume at the instant when the radius of the sphere is 2.5 m, given that the surface area is decreasing at $0.75\ \text{m}^2$ per minute at this instant. [5]

13. **SAJC/2019/I/10**

The diagram below shows a curve C with parametric equations given by

$$x=\theta\cos 2\theta,\quad y=3\theta\sin 2\theta,\qquad \text{for } 0\le\theta\le\frac{\pi}{2}.$$



The area bounded by curve C and the x-axis is a plot of land which is owned by a farmer Mr Green where he used to grow vegetables. Over the past weeks, vegetables were mysteriously missing and Mr Green decided to install an automated moving surveillance camera which moves along the boundary of the farmland in an anticlockwise direction along the curve C starting from point O and ending at point Q before moving in a clockwise direction along the curve C back to O.

At a particular instant t seconds, the camera is located at a point P with parameter θ on the curve C. You may assume that the camera is at O initially. The camera should be orientated so that the field of view should span from O to Q exactly as shown.

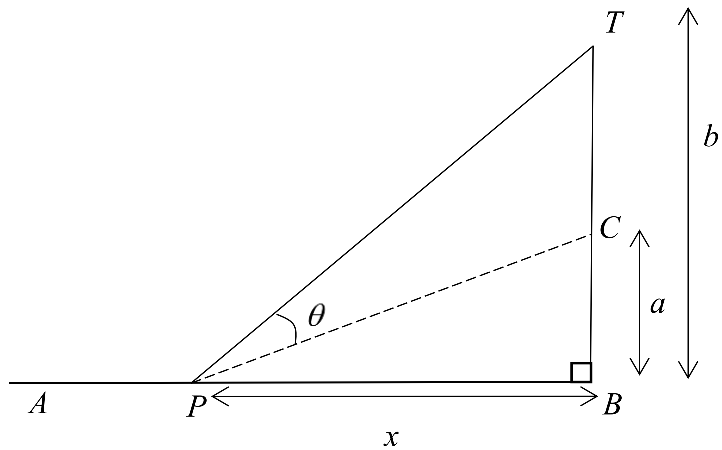
(i) Assuming that the camera is moving at a speed given by $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}=0.01$ radians/sec, find the rate of change of the area of the triangle OPQ, A when $\theta=\dfrac{\pi}{6}$. [4]

(ii) Using differentiation, find the value of θ that would maximize A and explain why A is a maximum for that value of θ. Hence find this value of A and the coordinates of the point P corresponding to the location of the camera at that instant. [5]

(iii) For the image to be 'balanced', triangle OPQ is isosceles. Find the coordinates of the location where the camera should be. [3]

14. **ASRJC/2021/1/Q10**

In the diagram below, A and B are two fixed points on horizontal ground and an observer is standing at point P which is x metres away from the base of a building BC of height a m. A transmission tower CT is fixed at the top of the building and the total height of the building and the tower is given as b metres. You may assume that B, C and T lies on a straight line.



It is given that the angle TPC is θ radians.

(i) Show that $\tan\theta = \dfrac{(b-a)x}{x^2+ab}$. [2]

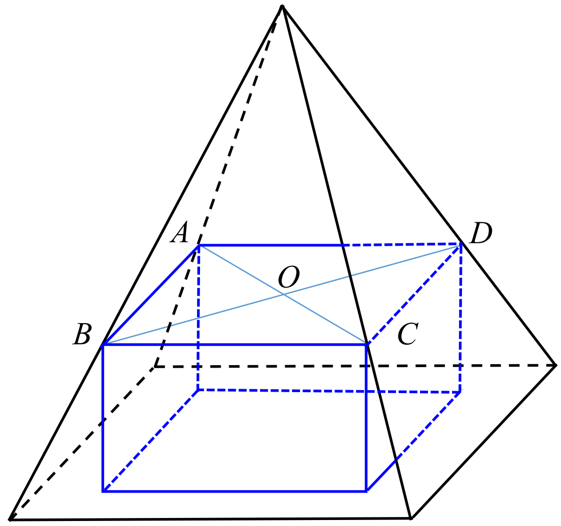
(ii) Find, in terms of a and b, the value of x such that θ is a maximum. Find also the corresponding value of $\tan\theta$. [4]

For the remaining parts of the question, take $a = 50$ and $b = 70$.

(iii) Find the range of values of $\tan\theta$ as x varies between 30 and 80. [2]

(iv) The observer at P starts to walk towards B with a steady speed of 3 ms^{-1}. Find the rate of change of θ when he is 10 m from B. [4]

15. **ASRJC/2022/2/Q4**



The product engineer of a factory crafted the design of a rectangular box, using a right pyramid, that is shown on the diagram above (not drawn to scale). The rectangular box is contained in a right pyramid with a rectangular base such that the upper four corners of the box *A*, *B*, *C* and *D* touch the slant faces of the pyramid, and the bottom four corners lie on the base of the pyramid. *O* is the point of intersection of the two diagonals, *AC* and *BD*.

The height of the pyramid is $3\sqrt{2}$ units, the length of the diagonal of its rectangular base is $12\sqrt{2}$ units, the height of the box is *b* units, where $b < 3\sqrt{2}$, and the angle *AOB* is θ radians. It is given that the box is made of material with negligible thickness.

(i) By finding the length of *OA* in terms of *b*, show that the volume *V* of the rectangular box is given by $V = 8b\left(3\sqrt{2}-b\right)^2 \sin\theta$. [3]

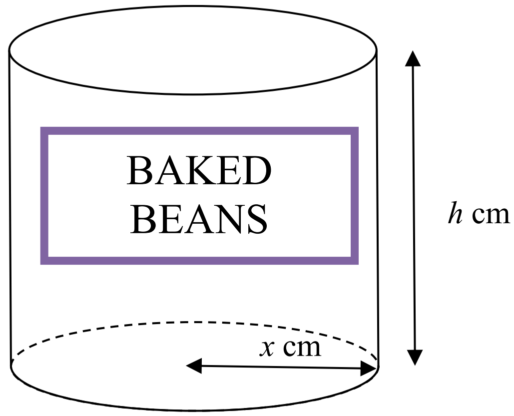
For the rest of the question, it is given that $\theta = \dfrac{\pi}{3}$.

(ii) Find the exact value of *b* which maximises *V*. Hence find the cost of manufacturing one such box if the material used to make the box cost \$0.03 per unit^2. [6]

When the height of the box is at half the height of the pyramid, it is reducing at a rate of 2 units per second.

(iii) Determine whether the volume of the box is expanding or shrinking and find the rate at which this is happening. [3]

16. **DHS Prelim 9758/2024/01/Q9**



A food company produces cans of baked beans. Each can is in the form of a closed right cylinder with a base radius of x cm and a height of h cm (see diagram) and its capacity is V cm^3, where V is a fixed constant. The cans are made of steel metal sheets with negligible thickness. The cost to make the curved surface of the can is 1 cent per cm^2 and the cost to make the top and bottom surfaces is k cents per cm^2. Let C cents be the production cost of a can. For economic reasons, the value of C is minimised by varying the value of x.

(a) Express h in terms of π, x and V. [1]

(b) Using differentiation, show that $\dfrac{x}{h}=\dfrac{1}{2k}$ when C is a minimum. [6]

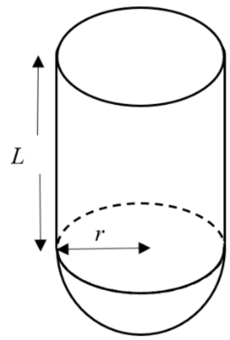
(c) **(i)** By using the relation in part **(a)**, show that, as h varies with time t, $\dfrac{\mathrm{d}h}{\mathrm{d}t}=-\dfrac{2h}{x}\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)$. [2]

(ii) Due to inflation, k increases at a constant rate of 0.1 units per month. Use the relation in part **(b)** or otherwise, to find the rate of change of x at the instant when $k = 2$ and $x = 1$. [3]

17. **DHS Prelim 9758/2025/P1/Q11**

[The surface area and volume of a sphere are given by $4\pi r^2$ and $\frac{4}{3}\pi r^3$ respectively.]

An aquarist who has a fixed budget of $\$k$ wants to make a goldfish sanctuary. His design of the tank consists of a circular cover, a cylinder of length L cm and a hemispherical base of radius r cm neatly fitted together. The costs of the material per cm^2 used for the cover, the cylindrical surface and hemispherical base are \$20, \$10 and \$30 respectively. Assume that the tank is made of material with negligible thickness.

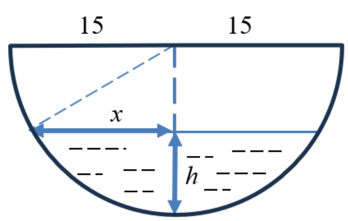


(a) Show that the volume of the tank is V cm^3, where $V = \frac{k}{20}r - \frac{10}{3}\pi r^3$. [3]

(b) As r varies, find the cost of the material used to make the hemispherical base in terms of k when V is at its maximum value. You need to show that V is a maximum. [3]

The aquarist has fixed the radius of the cylinder to be 15 cm. Initially, there is some water in the hemispherical part of the tank. However, due to a defect in the tank, water is leaking at a constant rate of 20 cm^3 per minute.

The depth of the water at the hemispherical bottom is denoted by h cm, and the radius of the water surface is x cm. The volume of water in the hemispherical bottom is given by W cm^3, where $W = \frac{\pi h\left(3x^2 + h^2\right)}{6}$.



(c) By formulating a relationship between x and h, find, when $h = 3$, the rate of change of

(i) the depth of water, [3]

(ii) the radius of the water surface. [3]

18. **SAJC Prelim 9758/2025/P1/Q2**

A closed cylinder is designed to have a fixed external surface area of p cm^2 such that its volume is maximum. Find the radius of the cylinder in terms of p. [5]

19. **TJC Prelim 9758/2025/P2/Q4**

[The arc length L of a sector of radius a and angle θ is given by $L = a\theta$;

The volume V of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$]

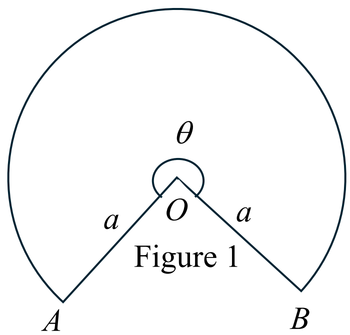


Figure 1

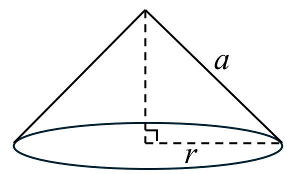


Figure 2

A metal sheet is shaped as a circular sector with angle θ and fixed radius a, as shown in Figure 1. This sector is then formed into a right circular cone with slant height a by joining the two radii OA and OB together as shown in Figure 2.

(a) If θ is measured in radians, state the base radius of the cone, r, in term of a and θ. [1]

(b) Show that the volume of the cone, V, is given by

$$V^2 = \frac{a^6}{576\pi^4}\theta^4\left(4\pi^2 - \theta^2\right).$$ [3]

(c) Using differentiation, find the exact value of θ that will maximise V. [4]

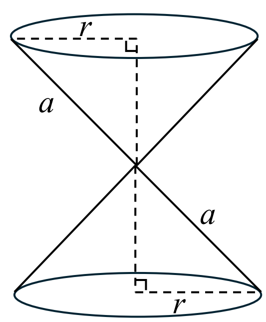


Figure 3

Two identical cones, as described in part **(c)**, are joined together tip-to-tip to form a new object, as shown in Figure 3. The upper cone is initially filled with water, which leaks out into the lower cone at a rate of $\frac{\pi}{15}$ cm^3 s^{-1}. At time t seconds, the height of the water in the upper cone is y.

(d) Show that the volume of water in the cone, W cm^3, is given by $W = \frac{2\pi y^3}{3}$. [2]

(e) Calculate the rate of change of y when $y = 2$. [2]

Answers

1	(i) $t=-1,\ x=12$ (ii) m = 15.369 (3 d.p.) $\sqrt{41}(5-\pi)$
2	$k=4$
3	(a)(ii) $A=2$ and $B=\dfrac{\pi}{2}$ (b) $\dfrac{20}{3}$
4	(ii) $\dfrac{dy}{dx}=-\dfrac{8x+y}{x+2y}$ (iii) A is maximum at $x=-\sqrt{\dfrac{3}{5}}$; decreases at a rate of 12 units^2/s
5	(ii) $\dfrac{d\theta}{dt}=\dfrac{1325}{2834}$ rads^{-1}
6	$-\sqrt{3}b\ \text{cm}^2/\text{s}$
7	(i) $V=-2y^3+78y^2-720y$; (ii) Max $V=800$; (iii) $-\dfrac{9}{\sqrt{34}}$
8	(ii) 1.890
9	(ii) 561.18 units^2, (iii) \$0.28
10	$\theta=\dfrac{\pi}{3}$
11	max $S=\dfrac{1}{2}\ell^2\left(\dfrac{1}{\sqrt{3}}-\dfrac{1}{3\sqrt{3}}\right)=\dfrac{1}{2}\ell^2\left(\dfrac{2}{3}\right)\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{9}\ell^2$
12	(ii) $V=72\pi(\approx 226.195)$ (iii) 0.9375
13	(i) 0.0327, (ii) 2.14 units^2, $(-0.449, 2.73)$, (iii) $P(-0.785, 2.55)$
14	(ii) $x=\sqrt{ab}$, $\tan\theta=\dfrac{(b-a)\sqrt{ab}}{2ab}$ (iii) $\dfrac{3}{22}<\tan\theta\le\dfrac{1}{\sqrt{35}}$ (iv) decreasing at a rate of $\dfrac{51}{3250}$ rad/s.
15	(ii) $b=\sqrt{2}$, \$4.64 (iii) $36\sqrt{3}$ units^3/s
16	a) $h=\dfrac{V}{\pi x^2}$ cii) The rate of change of x is $-\dfrac{1}{60}$ units per month
17	(b) 3k/10 (c)(i) -0.0786 (c)(ii) -0.105
18	$r=\left(\dfrac{p}{6\pi}\right)^{\frac{1}{2}}$
19	(a) $r=\dfrac{a}{2\pi}\theta$ (c) $\theta=2\pi\sqrt{\dfrac{2}{3}}$ (e) $-\dfrac{1}{120}$ cm s^{-1}