EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2021

General Certificate of Education Advanced Level

Higher 2

Section A: Pure Mathematics [40 marks]

1 **Do not use a calculator in answering this question.**

The function f is defined by $\mathrm{f}(z)=4z^3-12z^2+13z-10$. Given that $\mathrm{f}\left(\frac{1}{2}+\mathrm{i}\right)=0$, find all the roots of $\mathrm{f}(z)$. [4]

2 It is given that $z=\cos\theta+\mathrm{i}\sin\theta$, where $0<\theta<\frac{\pi}{2}$.

(i) Show that $\mathrm{e}^{\mathrm{i}\left(\theta-\frac{\pi}{2}\right)}=\sin\theta-\mathrm{i}\cos\theta$. [1]

(ii) Hence, or otherwise, show that $\arg\left(1-z^2\right)=\theta-\frac{\pi}{2}$ and find the modulus of $1-z^2$. [3]

(iii) Hence, represent the complex number $1-z^2$ on an Argand diagram. [2]

(iv) Given that $\dfrac{z^*}{z^3\left(1-z^2\right)}$ is real, find the possible values of θ. [3]

3 **(a)** The function f is given by $\mathrm{f}:x\mapsto\cos\left(\frac{1}{2}x+\frac{1}{6}\pi\right),\quad x\in\mathbb{R},\ 0\le x\le k.$

(i) State the largest exact value of k for which the function f^{-1} exists. [1]

For the rest of the question, the domain of f is $x\in\mathbb{R},\ 0\le x\le\frac{4}{3}\pi.$

(ii) Write down the equation of the line in which the graph of $y=\mathrm{f}(x)$ must be reflected in order to obtain the graph of $y=\mathrm{f}^{-1}(x)$. Hence, sketch on the same diagram, the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, indicating the exact coordinates of the endpoints of both graphs. [3]

(iii) State the value(s) of x for which $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$. [1]

(b) The functions g and h are defined by

$$\begin{aligned}&\mathrm{h}:x\mapsto 2x^2+3, && x\in\mathbb{R},\ x\le 0,\\ &\mathrm{hg}:x\mapsto 2x+3-2a, && x>a,\ \text{where } a\in\mathbb{R}^+\end{aligned}$$

Find $\mathrm{g}(x)$ and state the domain of g. [3]

4 The curve C has equation $y=3-\dfrac{10}{x^2-2x+4}$.

(i) Without using a calculator, determine the exact coordinates of the stationary point of C. [2]

(ii) Sketch C, stating clearly the equations of any asymptotes. [2]

The region R is bounded by C, the line $3y+16x=15$ and the y-axis.

(iii) Find the exact area of the region R. [3]

(iv) Find the volume generated when R is rotated through 2π radians about the y-axis. [3]

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point P is on the line AB such that $AP:PB=m:n$ where m and n are positive integers. Point C is on OP extended such that $OP:PC=1:2$.

(i) Show that $\overrightarrow{AC}=\left(\dfrac{2n-m}{m+n}\right)\mathbf{a}+\left(\dfrac{3m}{m+n}\right)\mathbf{b}$. [3]

(ii) If $|\mathbf{a}|=|\mathbf{b}|$ and the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\pi}{3}$, find the area of the triangle ABC in terms of $|\mathbf{a}|$. [4]

(iii) Find the ratio $AP:PB$ such that AC is parallel to OB. [2]

Section B: Probability and Statistics [60 marks]

6 For events A and B it is given that $\mathrm{P}(A)=\dfrac{3}{5}$, $\mathrm{P}(A\cap B)=\dfrac{7}{60}$ and $\mathrm{P}(A'\mid B)=\dfrac{13}{20}$.

(i) Show that $\mathrm{P}(B)=\dfrac{1}{3}$. [3]

For a third event C, it is given that $\mathrm{P}(C)=\dfrac{3}{10}$, and that $\mathrm{P}(B\cap C)=\dfrac{1}{10}$.

(ii) Determine if events B and C are independent. [1]

(iii) Given also that $\mathrm{P}(A\cap B\cap C)=\dfrac{1}{15}$, find the greatest and least possible values of $\mathrm{P}(A\cap B'\cap C)$. [3]

7 John has 4 different pairs of socks in his drawer. Without looking at his drawer, he randomly draws one sock at a time from his drawer without replacement, until he obtains a matching pair. The total number of socks John draws is denoted by S.

(i) Show that $\mathrm{P}(S=3)=\dfrac{2}{7}$. [1]

(ii) Explain why $P(S=k)=0$ for $k=6,7,8$. [1]

(iii) Determine the probability distribution of *S*. [3]

(iv) Find E(*S*) and Var(*S*). [3]

Every morning, John draws socks using the above procedure. Every evening, John returns all the drawn socks to his drawer so that, the following morning, he will again have four pairs of socks in his drawer to draw from. Assume that the number of socks drawn on any given day is independent of the number of socks drawn on any other day. Let *T* denote the total number of socks John draws in a given span of 3 days.

(v) Explain why $P(T=9)>\left(\frac{2}{7}\right)^3$. [1]

(vi) Find Var(*T*). [1]

8 Mei Li has 3 types of gemstones comprising 6 Emeralds, 4 Sapphires and 3 Labradorites. All the gemstones are of different sizes.

(i) Find the number of ways Mei Li can arrange all the gemstones in a circle, where the biggest of each type of gemstone are together. [2]

(ii) Next, Mei Li selects 9 gemstones with 3 of each type to arrange in a circle. Find the number of ways to do this, such that the gemstones alternate in the following order, clockwise: Emerald, Sapphire, Labradorite. [3]

(iii) Finally, Mei Li decides to randomly arange all 13 gemstones in a row. Find the number of ways to do this, such that no two Sapphires are next to each other and the Labradorites are all together. [3]

Mei Li selects 7 gemstones to be showcased at a conference.

(iv) Find the number of ways to select the gemstones such that at least one of each type is selected. [3]

9 A company produces light bulbs with a brightness rating of 470 lumens. It is known that the brightness of these light bulbs is normally distributed with a standard deviation of 10 lumens. A production manager wishes to test whether the mean brightness is less than 470 lumens. He collects a random sample of 40 light bulbs and finds that their mean brightness is 466.8 lumens.

(i) Explain what is meant by the term "random sample" in the context of this question. [1]

(ii) Carry out a test at the 5% level of significance for the production manager, clearly stating the *p*-value of this test. Explain what *p*-value means in the context of this question. [5]

(iii) Suppose that the production manager had tested if the mean brightness differed from 470 lumens at the 5% level of significance. Without performing another hypothesis test, determine whether your conclusion in **part (ii)** would be affected. [2]

The company decides to produce a new line of 'lowlight' bulbs with a brightness rating of 350 lumens. A random sample of 50 'lowlight' bulbs is taken and the brightness level, y lumens, are summarised as follows.

$$n=50 \qquad \sum(y-350)=-125 \qquad \sum(y-347.5)^2=4704$$

(iv) Calculate unbiased estimates of the population mean and variance for the brightness of the 'lowlight' bulbs. [2]

(v) Given instead that the standard deviation of the brightness of the 'lowlight' bulbs is known to be σ, the company found insufficient evidence from this sample to conclude that the mean brightness was less than 350 lumens at the 1% level of significance. Determine the range of values of σ. [2]

10 In this question, you should state the parameters of any distributions that you use.

A stationery factory manufactures pens for sale. The diameters (in mm) of the pens have distribution $\mathrm{N}(10, 0.003)$. The pens are packed in sets of 24 into a box. Within the box, the pens are laid out side by side. The widths of the boxes (in mm) are normally distributed with mean 240.5 mm and variance 0.02 mm^2.

(i) Find the probability that a random sample of 24 pens would fit into a randomly selected box. [3]

A pen is considered "defective" if its diameter is not within 0.1mm of the population mean.

(ii) Find the probability that a randomly selected pen is defective. [2]

Pens are produced and inspected in batches. For each batch, a sample of 12 pens is randomly selected and checked.

- If there are fewer than 2 defective pens in this sample of 12, the batch passes the inspection.
- If there are exactly 2 defective pens in this sample, a second sample of 12 pens is randomly selected and checked. If there are no defective pens in the second sample, the batch passes the inspection.
- Otherwise, the batch does not pass the inspection.

(iii) Show that the probability of a batch of pens passing the inspection is 0.871, correct to 3 significant figures. [3]

(iv) Find the probability that not more than 3 defective pens were found in a batch of pens during the inspection process, given that the batch of pens did not pass the inspection. [3]

The factory also manufactures and packs pencils and crayons in boxes of 12 each. During a promotion period, special pencils and special crayons are produced and randomly included in some boxes. The number of special pencils in a box of pencils has distribution $\mathrm{B}(12, 0.07)$, while the number of special crayons in a box of crayons has distribution $\mathrm{B}(12, 0.06)$.

(v) 40 boxes of pencils and 40 boxes of crayons are randomly selected. Find the probability that the mean number of special pencils per box of pencils is more than the mean number of special crayons per box of crayons. [4]