Question 1

1(a) **[2]**	Replace x with $1-x$, $\left.\begin{array}{l} \mathrm{f}(x)-k\mathrm{f}(1-x)=x \\ \mathrm{f}(1-x)-k\mathrm{f}(x)=1-x \end{array}\right\}$ solving, we get $\left(1-k^2\right)\mathrm{f}(x)=k(1-x)+x$ Hence $\mathrm{f}(x)=\dfrac{k+(1-k)x}{1-k^2}$.	
(b) **[2]**	From $\mathrm{f}(x)-\mathrm{f}(1-x)=\mathrm{g}(x)$ ----(1) Replacing x with $1-x$ we get $\mathrm{f}(1-x)-\mathrm{f}(x)=\mathrm{g}(1-x)$---(2) (1)+(2) gives $\mathrm{f}(x)-\mathrm{f}(1-x)+\mathrm{f}(1-x)-\mathrm{f}(x)=0=\mathrm{g}(x)+\mathrm{g}(1-x)$. Hence $\mathrm{g}(x)+\mathrm{g}(1-x)=0$ is a necessary condition for there to be a solution.	
(c) **[2]**	Let $\mathrm{g}(x)=a_k x^k+a_{k-1}x^{k-1}+\cdots+a_1x+a_0$ with $a_k\neq 0$ Then $\mathrm{g}(1-x)=a_k(1-x)^k+a_{k-1}(1-x)^{k-1}+\cdots+a_1(1-x)+a_0$ Hence, the coefficient of x^k in $\mathrm{g}(x)+\mathrm{g}(1-x)$ is $a_k+a_k(-1)^k$. From **(b)**, since there is a solution f to $(*)$, $a_k+a_k(-1)^k=0$, which implies that k must be odd, else we have $a_k=0$, a contradiction.	
(d) **[3]**	Let $\mathrm{g}(x)=mx+c$ and so $\mathrm{g}(1-x)=m(1-x)+c$ We have $mx+m(1-x)+2c=0$ $m+2c=0$ Let $m=2, c=-1$, so $\mathrm{g}(x)=2x-1$ is a possible function. Consider $\mathrm{f}(x)-\mathrm{f}(1-x)=2x-1$. We will show that there is a solution f to the equation above. In fact, it suffices to let $\mathrm{f}(x)=x$.	
(e) **[2]**	<u>Method 1:</u> $\mathrm{g}(x)=(2x-1)^3=8x^3-12x^2+6x-1$ Let $\mathrm{f}(x)=ax^3+bx^2+cx+d$, $\mathrm{f}(1-x)=a(1-x)^3+b(1-x)^2+c(1-x)+d$ Since $\mathrm{f}(x)-\mathrm{f}(1-x)=\mathrm{g}(x)$, Compare coefficient of x^3, $a=4$ Compare coefficient of x, $b+c=-3$ Compare constant term, $b+c=-3$ Let $b=0, c=-3$, a solution is $\mathrm{f}(x)=4x^3-3x$.	

	<u>Method 2:</u> Since $g(x)+g(1-x)=0$ and $f(x)-f(1-x)=g(x)$. $\quad(*)$ If $f(x)=\dfrac{g(x)}{2}$, $f(x)-f(1-x)=\dfrac{g(x)}{2}-\dfrac{g(1-x)}{2}$ $=\dfrac{1}{2}\left(g(x)-g(1-x)\right)$ $=\dfrac{1}{2}\left(g(x)-\left[-g(x)\right]\right)=g(x)$, satisfying $(*)$. Hence $f(x)=\dfrac{g(x)}{2}=\dfrac{(2x-1)^3}{2}$ is a solution. <u>Method 3:</u> Since $g(x)+g(1-x)=0$ and $f(x)-f(1-x)=g(x)$. $\quad(*)$ If $f(x)=xg(x)$, $f(x)-f(1-x)=xg(x)-\left(1-x\right)g(1-x)$ $=xg(x)-\left(1-x\right)\left[-g(x)\right]$ $=xg(x)-xg(x)+g(x)=g(x)$, satisfying $(*)$. Hence $f(x)=xg(x)=x(2x-1)^3$ is a solution.	

Question 2

2(a) **[2]**	$z^3-1=(z-1)(z^2+z+1)=0$ Since $z\neq 1$, $z^2+z+1=0$ (shown). Solving, $z=\dfrac{-1\pm\sqrt{-3}}{2}=-\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}\mathrm{i}$.	
(b) **[2]**	MC is obtained by rotating MB by $\dfrac{\pi}{2}$ rad in anticlockwise direction, followed by a scaling with factor $\sqrt{3}$. Hence, $c-m=\mathrm{i}(b-m)\sqrt{3}$. Thus $k=(\sqrt{3})\mathrm{i}$.	
(c) **[2]**	$c-\dfrac{a+b}{2}=\mathrm{i}\left(b-\dfrac{a+b}{2}\right)\sqrt{3}$ $c=\mathrm{i}\left(\dfrac{b-a}{2}\right)\sqrt{3}+\dfrac{a+b}{2}$ $=b+\mathrm{i}\left(\dfrac{b-a}{2}\right)\sqrt{3}+\dfrac{a}{2}-\dfrac{b}{2}$ $=b+(b-a)\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\mathrm{i}\right)$ $=b+\omega(b-a)$ (shown)	
(d) **[2]**	From **(c)**, $a\omega-(1+\omega)b+c=0$. Since $1+\omega+\omega^2=0$ we have $a\omega+b\omega^2+c=0$. Multiplying by ω^2 we get $a\omega^3+b\omega^4+c\omega^2=0$ and now using the fact $\omega^3=1$ we have the desired result $a+b\omega+c\omega^2=0$.	
(e) **[4]**	We first note that if $a+b\omega+c\omega^2=0$ then reversing the calculations in (d), we get the result in (c) and (b) which means that ABC is an equilateral triangle. Since XQP, YRQ and ZPR are equilateral, we have $x+q\omega+p\omega^2=0$, $y+r\omega+q\omega^2=0$ and $z+p\omega+r\omega^2=0$. From the result given about centroids, we have $g_1=\dfrac{x+q+p}{3}, g_2=\dfrac{y+r+q}{3}, g_3=\dfrac{z+p+r}{3}$. Hence	

	$g_1+g_2\omega+g_3\omega^2$ $=\frac{x+q+p}{3}+\left(\frac{y+r+q}{3}\right)\omega+\left(\frac{z+p+r}{3}\right)\omega^2$ $=\frac{1}{3}\left[\left(x+q\omega+p\omega^2\right)+\left(q+y\omega+r\omega^2\right)+\left(p+r\omega+z\omega^2\right)\right]$ $=\frac{1}{3}\left[\left(x+q\omega+p\omega^2\right)+\left(q\omega^2+y\omega^3+r\omega^4\right)+\left(p\omega+r\omega^2+z\omega^3\right)\right]$ $=\frac{1}{3}\left[\left(x+q\omega+p\omega^2\right)+\left(q\omega^2+y+r\omega\right)+\left(p\omega+r\omega^2+z\right)\right]$ $=0$ This thus implies that triangle $G_1G_2G_3$ is equilateral. This result is also known as Napoleon's Theorem.	

Question 3

(a) **[3]**	$I_{m,n}(x)=\int_0^x \frac{t^m}{(1+t^2)^{n+1}}\,\mathrm{d}t$ $=\int_0^x \frac{t^{m-1}(t)}{(1+t^2)^{n+1}}\,\mathrm{d}t$ $=\left[t^{m-1}\frac{(-1)}{2n(1+t^2)^n}\right]_0^x-\int_0^x (m-1)t^{m-2}\frac{(-1)}{2n(1+t^2)^n}\,\mathrm{d}t$ $=-\frac{x^{m-1}}{2n(1+x^2)^n}+\frac{m-1}{2n}\int_0^x \frac{t^{m-2}}{(1+t^2)^n}\,\mathrm{d}t$ $=-\frac{x^{m-1}}{2n(1+x^2)^n}+\frac{m-1}{2n}I_{m-2,n-1}(x)$	
(b) **[1]**	We need to have $m-1<2n$. Since m and n are integers, this means that $m\le 2n$.	
(c) **(i)** **[3]**	We have $\lim\limits_{x\to\infty} I_{0,1}(x)=\lim\limits_{x\to\infty}\int_0^x \frac{1}{\left(1+t^2\right)^2}\,\mathrm{d}t$ Using the substitution $t=\tan\theta$, $\int_0^x \frac{1}{\left(1+t^2\right)^2}\,\mathrm{d}t$ $=\int_0^{\tan^{-1}x}\frac{1}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\,\mathrm{d}\theta$ $=\int_0^{\tan^{-1}x}\cos^2\theta\,\mathrm{d}\theta$ $=\int_0^{\tan^{-1}x}\frac{\cos 2\theta+1}{2}\,\mathrm{d}\theta$ As $x\to\infty$, $\tan^{-1}x\to\frac{\pi}{2}$. Therefore, $\lim\limits_{x\to\infty}\int_0^x \frac{1}{\left(1+t^2\right)^2}\,\mathrm{d}t=\frac{1}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_0^{\frac{\pi}{2}}=\frac{\pi}{4}$.	
(c) **(ii)** **[4]**	The curve is symmetric about the y-axis and thus the required required exact volume is given by	

	$2\pi\int_0^\infty \frac{t^4}{(1+t^2)^4}\ \mathrm{d}t$ $= 2\pi \lim_{x\to\infty} I_{4,3}(x)$ $= 2\pi \lim_{x\to\infty} \frac{3}{6} I_{2,2}(x) \quad (\text{since } m-1<2n, 3<6)$ $= 2\pi \lim_{x\to\infty} \frac{3}{6}\frac{1}{4} I_{0,1}(x) \quad (\text{since } m-1<2n, 1<4)$ Altogether, we have $2\pi\int_0^\infty \frac{t^4}{(1+t^2)^4}\ \mathrm{d}t = 2\pi\left(\frac{1}{8}\right)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}$.	

Question 4

4 (a) (i) [2]	Consider a class of n students, from which we wish to choose a class committee of i students. On the one hand, this is clearly $\binom{n}{i}$. On the other hand, consider a particular student A. If student A is chosen, then we need to choose another i – 1 students from the remaining n – 1, of which there are $\binom{n-1}{i-1}$ ways to do so. If student A is not chosen, then we need to choose all i from the remaining n – 1, and there are $\binom{n-1}{i}$ ways to do so. By the addition principle, we have $\binom{n}{i}=\binom{n-1}{i}+\binom{n-1}{i-1}$.	
4 (a) (ii) [4]	Let P_n be the statement $\binom{n}{0}\binom{m}{k}-\binom{n}{1}\binom{m-1}{k}+\cdots+(-1)^n\binom{n}{n}\binom{m-n}{k}=\binom{m-n}{k-n}$. When $n=1$, LHS = $\binom{1}{0}\binom{m}{k}-\binom{1}{1}\binom{m-1}{k}=\binom{m}{k}-\binom{m-1}{k}=\binom{m-1}{k-1}$ = RHS. where we have used the result in (a), with m = n. Hence P_1 is true. Assume (*) holds for some positive integer n. $\binom{n+1}{0}\binom{m}{k}-\binom{n+1}{1}\binom{m-1}{k}+\cdots+(-1)^{n+1}\binom{n+1}{n+1}\binom{m-(n+1)}{k}$ $=\binom{n}{0}\binom{m}{k}-\left[\binom{n}{0}+\binom{n}{1}\right]\binom{m-1}{k}+\left[\binom{n}{1}+\binom{n}{2}\right]\binom{m-2}{k}+\cdots$ $+(-1)^n\left[\binom{n}{n-1}+\binom{n}{n}\right]\binom{m-n}{k}+(-1)^{n+1}\binom{n}{n}\binom{m-n-1}{k}$ $=\binom{n}{0}\binom{m}{k}-\binom{n}{1}\binom{m-1}{k}+\cdots+(-1)^n\binom{n}{n}\binom{m-n}{k}$ $-\binom{n}{0}\binom{m-1}{k}+\binom{n}{1}\binom{m-2}{k}+\cdots+(-1)^{n+1}\binom{n}{n}\binom{m-n-1}{k}$	

	$=\binom{m-n}{k-n}-\binom{m-n-1}{k-n}=\binom{m-n-1}{k-n-1}$. Since P_1 is true, $P_n \Rightarrow P_{n+1}$, by mathematical induction, (*) is true for all positive integers *n*.	
(b) **[4]**	Context: Amongst *m* players, of which *n* are star players, choose a team containing *k* players and all star players must be included within these *k* players. Clearly, since all *n* star players are chosen for the team, we need to choose remaining $k-n$ players from the $m-n$ players, and there are $\binom{m-n}{k-n}$ ways, which is the RHS of (*). For LHS of (*), let the star players be numbered 1 to *n* and let A_i denote the sets of teams of *k* players without star player *i*. Hence, we need $\left\|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}\right\|$ so that every star player is included. Without considering whether we include star players, we have a total of $\binom{m}{k}$ ways choosing *k* players from *m* players. For the ways without any one of the star players, we have $\sum_{i=1}^{n}\|A_i\|=\binom{n}{1}\binom{m-1}{k}$ and for the ways without any two of the star players, we have $\sum_{i\neq j}\|A_i\cap A_i\|=\binom{n}{2}\binom{m-2}{k}$. Similarly for the rest until we have for the ways without any of the star players, which is $\|A_1\cap A_2\cap\cdots\cap A_n\|=\binom{n}{n}\binom{m-n}{k}$. By principle of inclusion and exclusion, we have $\left\|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}\right\|$ $=\|S\|-\sum\|A_i\|+\sum\|A_i\cap A_j\|-\ldots+(-1)^n\|A_1\cap A_2\cap\cdots\cap A_n\|$ $=\binom{n}{0}\binom{m}{k}-\binom{n}{1}\binom{m-1}{k}+\cdots+(-1)^n\binom{n}{n}\binom{m-n}{k}$.	
(c) **(i)** **[1]**	$(y-1)^n=\sum_{r=0}^{n}(-1)^r\binom{n}{r}y^{n-r}$	
(c) **(ii)** **[3]**	$x^n=\left((1+x)-1\right)^n=\sum_{r=0}^{n}(-1)^r\binom{n}{r}(1+x)^{n-r}$ On multiplying both sides with $(1+x)^{m-n}$, we have	

$$x^n(1+x)^{m-n} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}(1+x)^{n-r+m-n} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}(1+x)^{m-r}$$

-- (**)

Considering the coefficient of x^k, on the LHS of (**), we just need the coefficient of x^{k-n} multiplied with x^n to give the term with x^k, and this is clearly $\binom{m-n}{k-n}$ in the binomial expansion of $(1+x)^{m-n}$.

On the RHS of (**), we need the x^k term in each $(1+x)^{m-r}$ as *r* steps from 0 to *n*.

Hence it is $\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{m-r}{k}$ which is the LHS of (*).

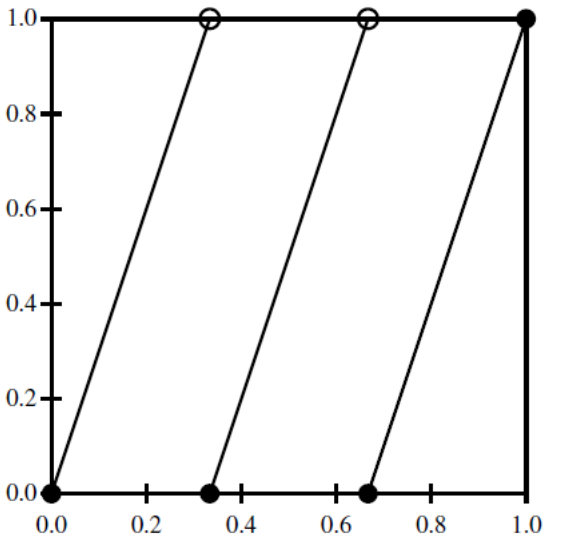
Question 5

(a) **(i)** **[3]**	By the AM-GM inequality, we have $\dfrac{x_1+\ldots+x_n}{x_{n+1}}+x_{n+1}\left(\dfrac{1}{x_1}+\ldots+\dfrac{1}{x_n}\right)$ $\geq 2\sqrt{\left(\dfrac{x_1+\ldots+x_n}{x_{n+1}}\right)x_{n+1}\left(\dfrac{1}{x_1}+\ldots+\dfrac{1}{x_n}\right)}$ $=2\sqrt{(x_1+\ldots+x_n)\left(\dfrac{1}{x_1}+\ldots+\dfrac{1}{x_n}\right)}$ $=2\sqrt{y_n}$ Hence $y_{n+1}=\left[(x_1+\ldots+x_n)+x_{n+1}\right]\left[\left(\dfrac{1}{x_1}+\ldots+\dfrac{1}{x_n}\right)+\dfrac{1}{x_{n+1}}\right]$ $=y_n+(x_1+\ldots+x_n)\dfrac{1}{x_{n+1}}+x_{n+1}\left(\dfrac{1}{x_1}+\ldots+\dfrac{1}{x_n}\right)+1$ $\geq y_n+2\sqrt{y_n}+1$	
(a) **(ii)** **[2]**	To show that no two terms are identical, we will show that the sequence is strictly increasing. To show that the sequence is strictly increasing, we need to show that the consecutive terms in the sequence of $\sqrt{y_n}$ differs by at least 1, i.e. $\sqrt{y_{n+1}}-\sqrt{y_n}\geq 1$ $\Leftrightarrow y_{n+1}\geq y_n+2\sqrt{y_n}+1.$	
(b) **(i)** **[2]**	We have $\dfrac{1}{1+a_{i-1}}-\dfrac{1}{1+a_i}$ $=\dfrac{a_i-a_{i-1}}{(1+a_{i-1})(1+a_i)}$ $>\dfrac{a_i-a_{i-1}}{(1+a_i)(1+a_i)}$ since $a_{i-1}<a_i\Rightarrow\dfrac{1}{(1+a_{i-1})}>\dfrac{1}{(1+a_i)}$ $=\dfrac{a_i-a_{i-1}}{(1+a_i)^2}$	

(b) (ii) [4]	As suggested, applying the Cauchy-Schwarz inequality, we have $$\begin{pmatrix} \frac{1}{\sqrt{a_1}} \\ \frac{1}{\sqrt{a_2-a_1}} \\ \vdots \\ \frac{1}{\sqrt{a_n-a_{n-1}}} \end{pmatrix} \bullet \begin{pmatrix} \frac{\sqrt{a_1}}{(1+a_1)} \\ \frac{\sqrt{a_2-a_1}}{(1+a_2)} \\ \vdots \\ \frac{\sqrt{a_n-a_{n-1}}}{(1+a_n)} \end{pmatrix} \le \left\| \begin{pmatrix} \frac{1}{\sqrt{a_1}} \\ \frac{1}{\sqrt{a_2-a_1}} \\ \vdots \\ \frac{1}{\sqrt{a_n-a_{n-1}}} \end{pmatrix} \right\| \left\| \begin{pmatrix} \frac{\sqrt{a_1}}{(1+a_1)} \\ \frac{\sqrt{a_2-a_1}}{(1+a_2)} \\ \vdots \\ \frac{\sqrt{a_n-a_{n-1}}}{(1+a_n)} \end{pmatrix} \right\|$$ $$\Leftrightarrow \left(\frac{1}{1+a_1}+\frac{1}{1+a_2}+\dots+\frac{1}{1+a_n}\right)^2$$ $$\le \left(\frac{1}{a_1}+\frac{1}{a_2-a_1}+\dots+\frac{1}{a_n-a_{n-1}}\right)$$ $$\times\left(\frac{a_1}{(1+a_1)^2}+\frac{a_2-a_1}{(1+a_2)^2}+\dots+\frac{a_n-a_{n-1}}{(1+a_n)^2}\right)$$ We now show that $\frac{a_1}{(1+a_1)^2}+\frac{a_2-a_1}{(1+a_2)^2}+\dots+\frac{a_n-a_{n-1}}{(1+a_n)^2}<1$, which will lead us to the desired result. Using (i), we know that $$\frac{a_1}{(1+a_1)^2}+\frac{a_2-a_1}{(1+a_2)^2}+\dots+\frac{a_n-a_{n-1}}{(1+a_n)^2}$$ $$<\frac{a_1}{(1+a_1)^2}+\left(\frac{1}{1+a_1}-\frac{1}{1+a_2}\right)+\left(\frac{1}{1+a_2}-\frac{1}{1+a_3}\right)+$$ $$\dots+\left(\frac{1}{1+a_{n-1}}-\frac{1}{1+a_n}\right)$$ $$=\frac{a_1}{(1+a_1)^2}+\frac{1}{1+a_1}-\frac{1}{1+a_n}$$ $$<\frac{a_1}{1+a_1}+\frac{1}{1+a_1}-\frac{1}{1+a_n} \quad \left(\text{since } \frac{1}{(1+a_1)^2}<\frac{1}{1+a_1} \text{ as } a_1>0\right)$$ $$=1-\frac{1}{1+a_n}<1$$	
(iii) [3]	We let $a_1=2^{-n}, a_2=2^{-n+1},\dots,a_n=2^{-1},a_{n+1}=1,a_{n+2}=2,\dots,a_{2n+1}=2^n$. Then the inequality becomes	

	$\left(\frac{1}{1+2^{-n}}+...+\frac{1}{1+1}+...+\frac{1}{1+2^{n}}\right)^2$ $\leq\left(\frac{1}{2^{-n}}+\frac{1}{2^{-n+1}-2^{-n}}+...+\frac{1}{2^{n}-2^{n-1}}\right)$ We note that $\frac{1}{1+2^{-n}}+\frac{1}{1+2^{n}}=\frac{2^n}{1+2^{n}}+\frac{1}{1+2^{n}}=1$ and thus the inequality simplifies to $\left(n+\frac{1}{2}\right)^2<\frac{1}{2^{-n}}+\frac{1}{2^{-n}}+\frac{1}{2^{-n+1}}+...+\frac{1}{2^{n-1}}$ $=\frac{1}{2^{-n}}+\frac{2^n(1-2^{-2n})}{1-2^{-1}}$ $=2^n+2^{n+1}(1-2^{-2n})$ $=3\left(2^n\right)-2^{-n+1}.$	

Question 6

6(a) (i) [2]	 	
(a) (ii) [1]	The graph of T_n will have n linear segments from in the interval $[0,1]$, each of them with gradient n and which increases from 0 to 1. In particular, each of the segments intersects the line $y = x$ exactly once. So there are n fixed points for T_n in $[0,1]$.	
(a) (iii) [2]	**Method 1** We note that the fixed points are the intersections of the line $y = x$ with the line segments $y = nx - k$, where k = 0, 1, …, n – 1. Hence the fixed points occur at $x = nx - k \Leftrightarrow x = \dfrac{k}{n-1}$. **Method 2** We will show that the fixed points have the form $x = \dfrac{k}{n-1}$, where $k = 0,1,2,...,n-1$. In fact, $T_n\left(\dfrac{k}{n-1}\right) = \left\{\dfrac{kn}{n-1}\right\} = \left\{\dfrac{kn}{n-1} - k\right\} = \left\{\dfrac{k}{n-1}\right\}$. Therefore for $0 \le k < n-1$, we have shown that $x = \dfrac{k}{n-1}$ are fixed points since $\dfrac{k}{n-1} < 1$ implies that $\left\{\dfrac{k}{n-1}\right\} = \dfrac{k}{n-1}$. When k = n – 1, $T_n\left(\dfrac{k}{n-1}\right) = T_n(1) = 1$. Thus it is also a fixed point and we are done.	

(b) (i) [1]	Since $T_2(x)=\{2x\}$ if $x\neq 1$, we notice that $T_2\left(\frac{1}{3}\right)=\frac{2}{3}, T_2\left(\frac{2}{3}\right)=\frac{1}{3}.$ So $\left\{\frac{1}{3},\frac{2}{3}\right\}$ is an orbit of size 2.	
(b) (ii) [2]	We need to find x_1,x_2,x_3 so that $T_3(x_1)=x_2, T_3(x_2)=x_3, T_3(x_3)=x_1.$ Since $T_3(x)=\{3x\}$ if $x\neq 1$, we notice that $T_3\left(\frac{1}{13}\right)=\frac{3}{13}, T_3\left(\frac{3}{13}\right)=\frac{9}{13}, T_3\left(\frac{9}{13}\right)=\frac{1}{13}.$ So $\left\{\frac{1}{13},\frac{3}{13},\frac{9}{13}\right\}$ is an orbit of size 3.	
(c) [2]	We have $T_b(x)=\{bx\}=bx-\lfloor bx\rfloor$. $T_a(T_b(x))=\{a(bx+m)\}=\{abx+am\}=\{abx\}=T_{ab}(x).$	
(d) [3]	Suppose not. Then $m=qk+r$, where $0<r<q$. Since k is the minimal period, $\underbrace{T_a(T_a\ldots(T_a(x)))}_{k\text{ times}}=x$, and, we also have $\underbrace{T_a(T_a\ldots(T_a(x)))}_{m\text{ times}}=x$. But splitting this composition up, we can see that $\underbrace{T_a(\ldots(T_a}_{r\text{ times}}\underbrace{(\underbrace{T_a(\ldots(T_a}_{k\text{ times}}\underbrace{(T_a\ldots(T_a(x))))))}_{k\text{ times}}}_{q\text{ copies}})=x$ $\Rightarrow \underbrace{T_a(T_a\ldots(T_a(x)))}_{r\text{ times}}=x$ which implies that r is the minimal period, contradicting the minimality of k. Hence we must have $r=0$ and k dividing m.	
(e) (i) [2]	The p-periodic points of T_a are the fixed points of T_a iterated p times. From **(c)**, we know that T_a iterated p times is T_{a^p}. We know from **(b)** that the function has a^p fixed points. Of these, a of them are the fixed points of T_a. Since p is prime, the rest of the fixed points have minimal period p under T_a (from **(d)**). This means that there are a^p-a points with minimal period p.	

(e) **(ii)** **[3]**	From **(d)**, there are $a^3 - a$ points with minimal period 3. Similarly, there are $a^5 - a$ points with minimal period 5. There are also a^{15} 15-periodic points and a of them have minimal period 1 (the fixed points). Therefore, the number of points with minimal period 15 is $a^{15} - \left(a^3 - a\right) - \left(a^5 - a\right) - a$. As before 15 must divide this number, (since each point lies in an orbit of size 15) and therefore $a^{15} - a^3 - a^5 + a \equiv 0 \text{ (modulo 15)}$ as desired.	