RAFFLES INSTITUTION

2023 YEAR 5 TERM 3 COMMON TEST (MODIFIED)

MATHEMATICS **9758**

2 hours

Answer **all** the questions.

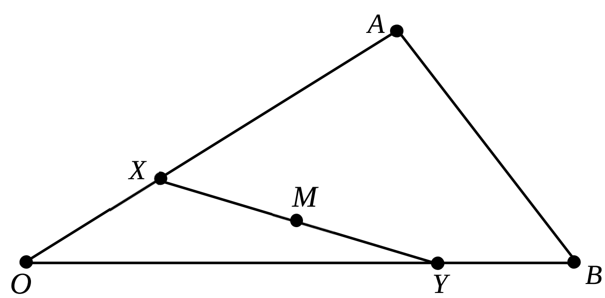
1 A cubic curve has two stationary points at $(2,\ 0)$ and $\left(-\dfrac{4}{3},\ \dfrac{500}{27}\right)$. Find the equation of the curve. [4]

2 Given that the complex numbers w and z satisfy the equations

$$w^* + 2z = \mathrm{i} \qquad \text{and} \qquad w + (2-\mathrm{i})z = 2 + 6\mathrm{i}$$

find w in the form $a + b\mathrm{i}$, where a and b are real. [4]

3



With reference to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. It is given that the point X is on OA such that $OX : XA = 1 : 2$ and the point Y is on OB such that $OY : YB = 3 : 1$. M is the mid-point of the line segment XY.

(a) Find the vector $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

Point N lies on the line AB such that O, M and N are collinear.

(b) Find the ratio $AN : NB$. [4]

4 A plane p is parallel to the line L with equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, $t \in \mathbb{R}$ and passes through the points $A(5, -4, 1)$ and $B(-3, 4, 2)$.

(a) Find a cartesian equation of p. [3]

(b) Find the exact distance between L and p. [3]

5 **Do not use a calculator in answering this question.**

By completing the square, or otherwise, show that $4x^2 - 10x + 13$ is always positive. Hence solve the inequality $\dfrac{9x^2 - 15x + 3}{x^2 - x - 2} < 5$. [5]

Hence solve the inequality $\dfrac{9x^2 - 15|x| + 3}{2 + |x| - x^2} \leq -5$. [3]

6 **(a)** Use differentiation to find the x-coordinate of the stationary point of the curve

$$y = x^2 - p^2 \ln\left(\frac{x}{q}\right),$$

where p and q are positive and negative constants respectively, and determine the nature of the stationary point. [4]

(b) **(i)** The tangent to the curve $y = \ln x$ at the point $A(a,\ \ln a)$ passes through the origin. Find the value of a. [2]

(ii) The normal to the curve $y = \ln x$ at the point $B(b,\ \ln b)$ passes through the origin. Show that B lies on the curve $y = kx^2$, where k is a constant to be determined. [2]

7 **(a)** The parametric equations of a curve are

$$x = \tan^n \theta,\ \ y = \sec^n \theta,$$

where n is an integer greater than 2 and $0 < \theta < \frac{\pi}{2}$.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, expressing your answer as a single trigonometric function in terms of n and θ. [3]

(b) Given that $2y - \tan^{-1} y - x = 0$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$, expressing your answer in terms of y, simplifying your answer.

Hence show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{ry\left(1+y^2\right)}{\left(1+2y^2\right)^s},$$

where r and s are integers to be determined. [5]

8 Functions f and g are defined by

$$\mathrm{f} : x \mapsto \left|2 - \mathrm{e}^x\right|, \quad \text{for } x \in \mathbb{R},\ x \geq -5,$$

$$\mathrm{g} : x \mapsto x^2 - 2x - 2, \quad \text{for } x \in \mathbb{R},\ x < -1.$$

(a) Sketch the graph of $y = \mathrm{f}(x)$. [2]

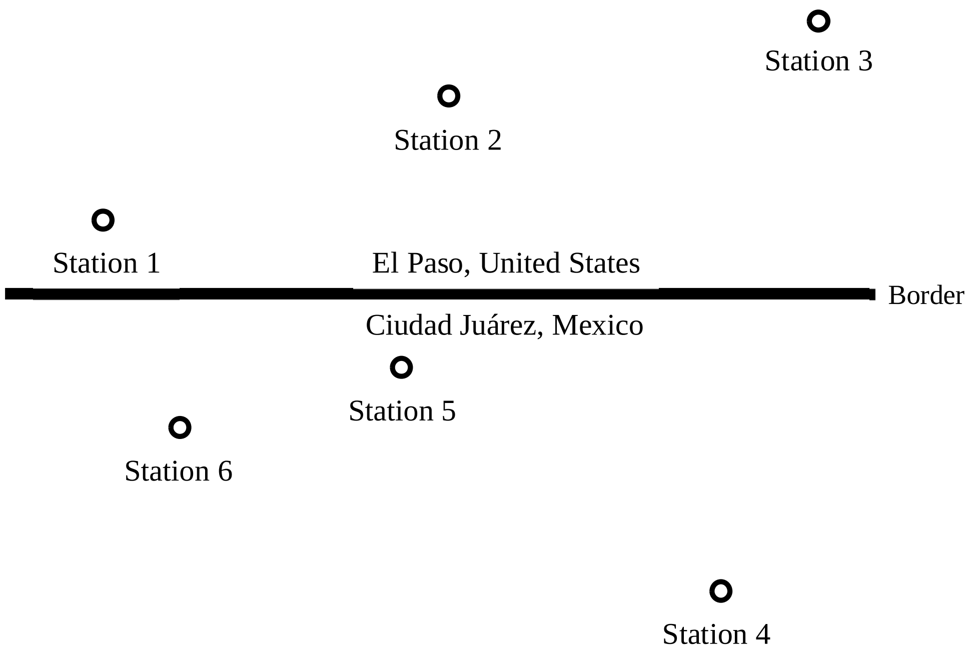
(b) Does f have an inverse? Justify your answer. [1]

(c) Find $\mathrm{g}^{-1}(x)$ and state its domain. [3]

(d) Explain why the composite function fg exists. Find an expression for $\mathrm{fg}(x)$ and write down the domain of fg. [3]

(e) State the range of fg. [1]

9 "Border Tuner" is a large-scale, participatory art installation designed to interconnect the cities of El Paso in Texas, United States (US) and Ciudad Juárez in Chihuahua, Mexico. Powerful "arms" of lights make "bridges of light" that open live sound channels for communication across the US-Mexico border. There are six interactive stations for the participants, three placed in El Paso and three in Ciudad Juárez as shown in the schematic diagram below.



Each of the interactive "Border Tuner" stations features a microphone, a speaker and a large wheel or dial. As a participant turns the dial, an "arm" of light that follows the movement of the dial is created. When two such "arms" of light meet in the sky and intersect, a bidirectional channel of sound opens automatically between the people at the two remote stations, allowing for communication between the people at these two stations. (Source: https://www.bordertuner.net/concept)

Referring the point at Station 1 as the origin O and the horizontal ground as the xy-plane, the coordinates of Stations 2, 4 and 5 are (132, 36, 2), (274, −119, 0) and (120, −32, 6) respectively.

Alice, Sofia and Juan are at Stations 2, 4 and 5 respectively. Alice's "arm" of light passes through a point with coordinates (142, 26, 22) while Sofia's "arm" of light is in the direction of $\begin{pmatrix} -7 \\ k \\ 8 \end{pmatrix}$, where k is a constant.

(a) Given that Alice's "arm" of light is represented by a line l, find a vector equation for l. [2]

(b) Find the acute angle that Alice's "arm" of light makes with the horizontal ground. [2]

(c) Find the value of k so that a channel of communication opens between Alice and Sofia. [4]

(d) Juan directs his "arm" of light such that it is perpendicular to Alice's "arm" of light and he is also able to communicate with Alice. Find a cartesian equation of the line representing Juan's "arm" of light. [5]