2026 J2 H2 FM

Linear Algebra (Set 2 – Solutions

1* The matrices **A** and **B** are defined as follows.

$$\mathbf{A}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\text{ with } \theta\ge 0 \text{ and } \mathbf{B}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The transformations T_1 and T_2 from $\mathbb{R}^2$ to $\mathbb{R}^2$ are defined by

$$\mathrm{T}_1:\begin{pmatrix}x\\y\end{pmatrix}\to\mathbf{A}\begin{pmatrix}x\\y\end{pmatrix}\text{ and } \mathrm{T}_2:\begin{pmatrix}x\\y\end{pmatrix}\to\mathbf{B}\begin{pmatrix}x\\y\end{pmatrix}.$$

(a) By considering $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}r\cos\alpha\\ r\sin\alpha\end{pmatrix}$ for $r>0$ and $\alpha\in[0,2\pi]$, describe the geometrical transformation T_1, explaining your answer. [2]

(b) Describe T_2 geometrically. [1]

Let matrix **M** be $\begin{pmatrix}\frac{1}{2}\sqrt{3} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\sqrt{3}\end{pmatrix}$.

(c) Without using a calculator, state the smallest positive integer value of k such that $\mathbf{M}^k$ gives the identity matrix and explain your answer. [2]

(d) Given that $\mathbf{C}=\mathbf{M}\mathbf{B}\mathbf{M}^{-1}$, find the Cartesian equations of the two lines through the origin which are invariant under the transformation given by the matrix **C**. [3]

1)	**RI_DHS_HCI_TMJC/Prelim/2023/02/Q2**	
(a)	$\mathbf{A}\begin{pmatrix}r\cos\alpha\\ r\sin\alpha\end{pmatrix}=\begin{pmatrix}r(\cos\theta\cos\alpha-\sin\theta\sin\alpha)\\ r(\cos\alpha\sin\theta+\cos\theta\sin\alpha)\end{pmatrix}=\begin{pmatrix}r\cos(\theta+\alpha)\\ r\sin(\theta+\alpha)\end{pmatrix}$ Hence T_1 gives an anticlockwise rotation of angle θ about the origin.	**M1** **A1**
(b)	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-y\end{pmatrix}$ T_2 is a reflection in the x-axis.	**B1**
(c)	**M** is obtained when we let $\theta=\frac{\pi}{6}$ in matrix **A**. Rotating a point about O for 2π radians brings it back to the original position. Since $2\pi=12\times\frac{\pi}{6}$, the smallest positive integer $k=12$.	**M1** **A1**
(d)	$\mathbf{C}=\mathbf{MBM}^{-1}=\begin{pmatrix}\frac{1}{2}\sqrt{3} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\sqrt{3}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2}\sqrt{3} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\sqrt{3}\end{pmatrix}^{-1}$, hence **C** has eigenvalues 1 and -1. Hence we have $\mathbf{Cx}_1=1\mathbf{x}_1$ and $\mathbf{Cx}_2=-1\mathbf{x}_2$ for eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$,	**M1**

	where $\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2}\sqrt{3} \\ \frac{1}{2} \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix}$.	**A1**
	So, the two invariant lines are $y = \frac{1}{\sqrt{3}}x$ and $y = -\sqrt{3}x$, which pass through the origin.	**A1**

2* **(a)** The matrix $\mathbf{A}$ has eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ with corresponding eigenvalues -1, 3 and 1 respectively. Find $\mathbf{A}$. [2]

(b) The vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are defined as:

$$\mathbf{y}_1 = \begin{pmatrix} 7 \\ -4 \\ 5 \end{pmatrix}, \mathbf{y}_2 = \begin{pmatrix} -1 \\ 8 \\ 13 \end{pmatrix}, \mathbf{y}_3 = \begin{pmatrix} 66 \\ -60 \\ 6 \end{pmatrix}.$$

Given that $\mathbf{Q}$ is a 3×3 matrix, determine whether or not the vectors $\mathbf{Qy}_1$, $\mathbf{Qy}_2$, $\mathbf{Qy}_3$ are linearly independent, justifying your conclusion. [2]

2	**ACJC\2017\Prelim\P1\Q3a & JPJC Prelim 9649/2022/01/Q10aii) and b)**	
(a)	$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 7 & 0 & -8 \\ -4 & 1 & 8 \\ 4 & 0 & -5 \end{pmatrix}$	M1 A1
(b)	Since $\begin{vmatrix} 7 & -1 & 66 \\ -4 & 8 & -60 \\ 5 & 13 & 6 \end{vmatrix} = 0 \Rightarrow \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is linearly dependent $\exists \lambda_1, \lambda_2$ and $\lambda_3 \in \mathbb{R}$ and not all zero such that $\lambda_1\mathbf{y}_1 + \lambda_2\mathbf{y}_2 + \lambda_3\mathbf{y}_3 = \mathbf{0}$ Then, $\mathbf{Q}(\lambda_1\mathbf{y}_1 + \lambda_2\mathbf{y}_2 + \lambda_3\mathbf{y}_3) = \mathbf{Q0} = \mathbf{0}$ $\lambda_1\mathbf{Qy}_1 + \lambda_2\mathbf{Qy}_2 + \lambda_3\mathbf{Qy}_3 = \mathbf{0}$ Since λ_1, λ_2 and $\lambda_3 \in \mathbb{R}$ and not all zero, $\{\mathbf{Qy}_1, \mathbf{Qy}_2, \mathbf{Qy}_3\}$ are linearly dependent.	 M1 A1

(c) **(i)** Show that the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} -4 & \frac{1}{a} & 3 \\ 2a & 1 & -a \\ -6 & \frac{2}{a} & 5 \end{pmatrix}$$

are independent of a, where a is a non-zero constant. [3]

(ii) Obtain, in terms of a, an eigenvector corresponding to each eigenvalue. [3]

(iii) Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A}^{-1} + 2\mathbf{I})^3 = \mathbf{PDP}^{-1}$. [2]

(c) (i)

Method 1: Using row operations

Recall effect of row operations on determinant (LA1 Lecture Notes last page)

> **Theorem:** Let $\mathbf{A}$ be a square matrix.
>
> **(i)** If $\mathbf{B}$ is the matrix that results when a row (or column) of $\mathbf{A}$ is multiplied by a scalar k, then $|\mathbf{B}| = k|\mathbf{A}|$
>
> **(ii)** If $\mathbf{B}$ is the matrix that results when two rows (or two column) of $\mathbf{A}$ are interchanged, then $|\mathbf{B}| = -|\mathbf{A}|$
>
> **(iii)** If $\mathbf{B}$ is the matrix that results when a multiple of one row of $\mathbf{A}$ is added to another row (or when a multiple of one column is added to another column), then $|\mathbf{B}| = |\mathbf{A}|$

$$|\mathbf{A}-\lambda\mathbf{I}| = 0$$

$$\begin{vmatrix} -4-\lambda & \frac{1}{a} & 3 \\ 2a & 1-\lambda & -a \\ -6 & \frac{2}{a} & 5-\lambda \end{vmatrix} = 0 \xrightarrow{-2R_1+R_3\to R_3} \begin{vmatrix} -4-\lambda & \frac{1}{a} & 3 \\ 2a & 1-\lambda & -a \\ 2+2\lambda & 0 & -1-\lambda \end{vmatrix} = 0$$ M1

$$\xrightarrow{2C_3+C_1\to C_1} \begin{vmatrix} 2-\lambda & \frac{1}{a} & 3 \\ 0 & 1-\lambda & -a \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$ M1

$$\Rightarrow (2-\lambda)(1-\lambda)(-1-\lambda) = 0$$

$$\Rightarrow \lambda = -1, 1, 2$$ A1

The eigenvalues of **A** are -1, 1, 2 which are independent of a.

Method 2: Direct Computation

$$|\mathbf{A}-\lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -4-\lambda & \frac{1}{a} & 3 \\ 2a & 1-\lambda & -a \\ -6 & \frac{2}{a} & 5-\lambda \end{vmatrix} = 0$$ M1

$$(-4-\lambda)\left[(1-\lambda)(5-\lambda)+2\right] - 2a\left[\left(\frac{1}{a}\right)(5-\lambda) - \frac{6}{a}\right] - 6\left[-1-3(1-\lambda)\right] = 0$$

$$(-4-\lambda)(7-6\lambda+\lambda^2) - 2a\left(-\frac{1}{a} - \frac{1}{a}\lambda\right) - 6(-4+3\lambda) = 0$$ M1

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$$(\lambda-1)(\lambda+1)(\lambda-2) = 0$$

$\therefore$ The eigenvalues of **A** are $-1, 1$ and 2 which are independent of a. A1

(ii)

For $\lambda = -1$, an eigenvector is $-\begin{pmatrix} -3 \\ \frac{1}{a} \\ 3 \end{pmatrix} \times \begin{pmatrix} 2a \\ 2 \\ -a \end{pmatrix} = -\begin{pmatrix} -7 \\ 3a \\ -8 \end{pmatrix} = \begin{pmatrix} 7 \\ -3a \\ 8 \end{pmatrix}$. B1

For $\lambda = 1$, an eigenvector is $-\frac{a}{2}\begin{pmatrix} -5 \\ \frac{1}{a} \\ 3 \end{pmatrix} \times \begin{pmatrix} -6 \\ \frac{2}{a} \\ 4 \end{pmatrix} = -\frac{a}{2}\begin{pmatrix} -\frac{2}{a} \\ 2 \\ -\frac{4}{a} \end{pmatrix} = \begin{pmatrix} 1 \\ -a \\ 2 \end{pmatrix}$. B1

B1

	For $\lambda = 2$, an eigenvector is $\frac{1}{2}\begin{pmatrix}-6\\ \frac{1}{a}\\ 3\end{pmatrix}\times\begin{pmatrix}2a\\ -1\\ -a\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\ 0\\ 4\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 2\end{pmatrix}$.	
(iii)	Recall that if $\mathbf{A}$ has eigenvalue λ and eigenvector $\mathbf{x}$ $\mathbf{A}^{-1}$ has eigenvalue $\frac{1}{\lambda}$ and eigenvector $\mathbf{x}$ $\mathbf{I}$ has eigenvalue 1 and eigenvector $\mathbf{x}$ (any non-zero vector is an eigenvector) Let $\mathbf{P} = \begin{pmatrix}7 & 1 & 1\\ -3a & -a & 0\\ 8 & 2 & 2\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}\left(\frac{1}{-1}+2\right)^3 & 0 & 0\\ 0 & \left(\frac{1}{1}+2\right)^3 & 0\\ 0 & 0 & \left(\frac{1}{2}+2\right)^3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 27 & 0\\ 0 & 0 & \frac{125}{8}\end{pmatrix}$ Then $\left(\mathbf{A}^{-1}+2\mathbf{I}\right)^3 = \mathbf{PDP}^{-1}$	B1 for $\mathbf{P}$ B1 for $\mathbf{D}$.

3 Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = \begin{pmatrix}2x+y\\ x-y+z\end{pmatrix} \text{ for all } \begin{pmatrix}x\\ y\\ z\end{pmatrix} \in \mathbb{R}^3.$$

(a) Find a basis for the range of T and explain why the dimension of the kernel of T is 1. [3]

(b) Find a basis $\mathbf{v}_1$ for the kernel of T, where $\mathbf{v}_1$ is in the form $\begin{pmatrix}x_1\\ y_1\\ 6\end{pmatrix}$ and $x_1,\ y_1 \in \mathbb{R}$ are constants to be determined. Hence, find a general solution $T\left(\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = \begin{pmatrix}2\\ 9\end{pmatrix}$. [3]

(c) Let $\mathbf{v}_2 = \begin{pmatrix}a\\ -a+3b^2\\ 3-2a\end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix}a^2\\ -a^2-9\\ -2a^2-9\end{pmatrix}$. Determine the conditions $a,\ b \in \mathbb{R}$ such that $\text{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\} = \mathbb{R}^3$, explaining your answers clearly. [4]

4	**TJC JC2 Prelim 9649/2019/02/Q2**	
(a)	Observe that for $\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2x+y\\x-y+z\end{pmatrix}$ $\mathbf{A}=\begin{pmatrix}2&1&0\\1&-1&1\end{pmatrix}\to\begin{pmatrix}1&0&\frac{1}{3}\\0&1&-\frac{2}{3}\end{pmatrix}$ Hence, $\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}$ is a basis for the range of T. By the Rank and Nullity theorem, since $\dim\left(\mathbb{R}^3\right)=3$ and $\operatorname{rank}(A)=2$, $\operatorname{nullity}(A)=3-2=1$.	 **M1** **A1** **B1**
(b)	Solving $\begin{pmatrix}2&1&0\\1&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}1&0&\frac{1}{3}\\0&1&-\frac{2}{3}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $-3x=z$, $\frac{3}{2}y=z$ Hence, $\left\{\begin{pmatrix}-2\\4\\6\end{pmatrix}\right\}$ is a basis for $\ker(T)$. Observe that a particular solution is $\begin{pmatrix}1\\0\\8\end{pmatrix}$. Hence, the general solution is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\0\\8\end{pmatrix}+\lambda\begin{pmatrix}-2\\4\\6\end{pmatrix}$, $\lambda\in\mathbb{R}$.	 **M1** **A1** **B1**
(c)	For $\operatorname{span}\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}=\mathbb{R}^3$, the set $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ must be linearly independent since 3 linearly independent vectors will span $\mathbb{R}^3$. Hence, the determinant of the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ cannot be 0. Let $\mathbf{X}=\begin{pmatrix}-2&a&a^2\\4&-a+3b^2&-a^2-9\\6&3-2a&-2a^2-9\end{pmatrix}\to\begin{pmatrix}-2&a&a^2\\0&a+3b^2&a^2-9\\0&a+3&a^2-9\end{pmatrix}$	**M1** **M1**

	$\det(\mathbf{X}) = -2\left[\left(3b^2+a\right)\left(a^2-9\right)-\left(a+3\right)\left(a^2-9\right)\right]$ $= -2\left(a^2-9\right)\left[3b^2+a-a-3\right]$ $= -6\left(a^2-9\right)\left(b^2-1\right)$ Since $\det(\mathbf{X}) \neq 0$, $a \neq \pm 3$, $b \neq \pm 1$.	**A1, A1**