H3 Mathematics Problem Solving

Lecture 4: Reading Mathematics as Problem Solving

Dr. Ho Weng Kin

The Final Lecture

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Central Theme

Reading mathematics is itself a form of problem solving.

Book link: Chapter 6: Reading Mathematical Texts; Chapter 7: Conclusion

Why This Matters for H3 Mathematics

In H3 Mathematics, you are often asked to read:

- unfamiliar definitions;
- recursively defined objects;
- long mathematical passages;
- symbolic notation introduced on the spot;
- proof structures that unfold over several parts.

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- long mathematical passages;
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- proof structures that unfold over several parts.

Key Message

The problem is not only to calculate.
The problem is to understand what is being built.

A Provocative Thought

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Key Insight

You are not merely reading symbols.
You are reconstructing thought.

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Novel	**Mathematics**
Read continuously	Read recursively
Skip details sometimes	Every symbol matters
Emotional flow	Logical flow
Fast reading possible	Slow reading essential
Passive absorption	Active reconstruction

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Key Message

Understanding mathematics requires active engagement.

Book link: Chapter 6, Section 1

Today: Two Reading Demonstrations

We shall practise reading two dense mathematical passages.

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   Reading symbolic notation, detecting patterns, and proving a conjecture.

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We shall practise reading two dense mathematical passages.

1. **The Calkin–Wilf sequence**
   Reading a recursive construction and its proof structure.
2. **Generalisations of the product rule**
   Reading symbolic notation, detecting patterns, and proving a conjecture.

Aim

Not merely to solve the questions, but to learn how to read them.

Reading Demonstration I

Representation of Positive Rational Numbers
in the Calkin–Wilf Sequence

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Representation of Positive Rational Numbers
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Reading Task

Before solving, we must first understand what is being constructed.

What is the Passage About?

The passage begins with the idea of a **countable set**:

Countability

A set is countable if its elements can be listed so that every element occurs at least once.

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Pause and ask:

Why is this background being given?

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Examples:

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Reading Move

Pause and ask:

Why is this background being given?

It prepares us to think about listing all positive rational numbers.

The Aim of the Construction

The passage says that Calkin and Wilf gave a method for generating a sequence containing:

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1. Every term is a positive rational number.
2. Every term is in lowest terms.
3. Every positive rational number appears exactly once.

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Three Claims to Track

1. Every term is a positive rational number.
2. Every term is in lowest terms.
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Good reading means keeping these claims in mind.

Reading the Recursive Rule

At the top of the diagram is

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Each parent

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has two children:

$$\text{left child} = \frac{a}{a+b}, \qquad \text{right child} = \frac{a+b}{b}.$$

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Active Reading Questions

- What changes from parent to child?
- What is preserved?
- Why might $a + b$ be important?

The First Rows

$$\frac{1}{1}$$

$$\frac{1}{2}, \quad \frac{2}{1}$$

$$\frac{1}{3}, \quad \frac{3}{2}, \quad \frac{2}{3}, \quad \frac{3}{1}$$

$$\frac{1}{4}, \quad \frac{4}{3}, \quad \frac{3}{5}, \quad \frac{5}{2}, \quad \frac{2}{5}, \quad \frac{5}{3}, \quad \frac{3}{4}, \quad \frac{4}{1}$$

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Reading row by row gives the Calkin–Wilf sequence.

Reading Move

Do not rush past the examples.
Examples reveal the rule in action.

Question 1(a): Reading Backwards

Given that

$$\frac{4}{13} \quad \text{and} \quad \frac{17}{3}$$

appear in the diagram, write down their respective parents.

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The construction gives children from parents.
But the question asks us to reverse the construction.

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Reading Move

The construction gives children from parents.
But the question asks us to reverse the construction.

So we ask:

Was this fraction a left child or a right child?

Finding Parents

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$$\frac{a}{b}$$

is a left child, then it has form

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So the numerator is smaller than the denominator.

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Control Question

Compare numerator and denominator.

Question 1(a): Solution

For
$$\frac{4}{13},$$
we have $4 < 13$, so it is a left child:
$$\frac{4}{13} = \frac{4}{4+9}.$$
Hence its parent is
$$\boxed{\frac{4}{9}}.$$

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For
$$\frac{17}{3},$$
we have $17 > 3$, so it is a right child:
$$\frac{17}{3} = \frac{14+3}{3}.$$
Hence its parent is

Question 1(b): Reading Paths

The example says:

$$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{1}.$$

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What is Happening?

We repeatedly move from a fraction to its parent.

Question 1(b): Reading Paths

The example says:

$$\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{1}.$$

What is Happening?

We repeatedly move from a fraction to its parent.

Reading Move

A path is not merely a list.
It records repeated reversal of the construction.

Path for $\frac{4}{13}$

$$\frac{4}{13}$$

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$$\frac{4}{13-4} = \frac{4}{9}.$$

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Then:

$$\frac{4}{9} \to \frac{4}{5} \to \frac{4}{1} \to \frac{3}{1} \to \frac{2}{1} \to \frac{1}{1}.$$

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Thus the path is:

$$\boxed{\frac{4}{13}, \frac{4}{9}, \frac{4}{5}, \frac{4}{1}, \frac{3}{1}, \frac{2}{1}, \frac{1}{1}}$$

Path for $\frac{17}{3}$

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Then:

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Question 1(c): What Must Be Proved?

Explain why the Calkin–Wilf sequence:

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Reading Move

Separate the claim into two parts:

1. positivity;
2. lowest terms.

Question 1(c): Positivity

Suppose

$$a, b > 0.$$

Then

$$a + b > 0.$$

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are positive rational numbers.

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So both children

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are positive rational numbers.
Since the root

$$\frac{1}{1}$$

is positive, every descendant is positive.

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Key Invariant

Being in lowest terms is preserved from parent to child.

Question 1(c): Reading the Proof Structure

The proof has the following architecture:

1. Start from $\frac{1}{1}$, which is positive and in lowest terms.
2. Show that each child of a positive fraction is positive.
3. Show that each child of a fraction in lowest terms is also in lowest terms.
4. Conclude that every entry in the diagram has both properties.

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Reading Lesson

Many proofs depend on identifying what is preserved.

Question 1(d): The Big Claim

Complete the proof that the Calkin–Wilf sequence contains every positive rational number, each expressed in lowest terms.

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We must show that every positive rational number appears.

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How do we show it appears in the diagram?

The Strategy

Instead of trying to build

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Key Strategy

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If it has a path back to $\frac{1}{1}$, then it must appear somewhere in the diagram.

Reading Lesson

Sometimes the proof direction is opposite to the construction direction.

The Backward Algorithm

Start with

$$u_1 = \frac{p}{q}.$$

If

$$u_n = \frac{a}{b},$$

then define:

$$u_{n+1} = \begin{cases} \dfrac{a}{b-a}, & a < b, \\[2ex] \dfrac{a-b}{b}, & a > b. \end{cases}$$

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Reading Move

This is the parent-finding rule written as an algorithm.

Why Does the Algorithm Terminate?

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If $a > b$, then

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Key Control Idea

We need a quantity that must decrease and cannot decrease forever.

Why Does It End at $\frac{1}{1}$?

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Hence every positive rational number in lowest terms appears in the diagram.

Question 1(e): Are There Repetitions?

Determine whether any positive rational number is repeated in the Calkin–Wilf sequence.

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Use what has already been proved.

We know:

- every entry is in lowest terms;
- every fraction has a unique path back to $\frac{1}{1}$.

Question 1(e): Are There Repetitions?

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Reading Move

Use what has already been proved.

We know:

- every entry is in lowest terms;
- every fraction has a unique path back to $\frac{1}{1}$.

Therefore no positive rational number is repeated.

What Did We Learn from Reading This Passage?

The Calkin–Wilf question required us to read:

- a recursive construction;
- a diagram;
- a backwards path;
- an invariant;
- a termination argument;
- a uniqueness argument.

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- a recursive construction;
- a diagram;
- a backwards path;
- an invariant;
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- a uniqueness argument.

Big Lesson

Reading mathematics means tracking the structure of an argument.

Reading Demonstration II

Investigating the Generalisations
of the Product Rule

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Reading Task

This passage introduces notation, builds examples, then asks us to conjecture and prove.

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Reading Move

Before solving, translate the notation into familiar language.

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2. Derivative of the product of many functions:

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Reading Warning

Do not mix these two generalisations.
They are related, but different.

Reading $D^2(fg)$

From

$$D(fg) = fDg + gDf,$$

differentiate again:

$$D^2(fg) = D(fDg) + D(gDf).$$

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$$D(fg) = fDg + gDf,$$

differentiate again:

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Using the product rule:

$$D(fDg) = fD^2g + (Df)(Dg),$$
$$D(gDf) = gD^2f + (Dg)(Df).$$

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$$D(fDg) = fD^2g + (Df)(Dg),$$

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Therefore

$$D^2(fg) = fD^2g + 2DfDg + gD^2f.$$

What Should We Notice?

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Reading Move

When coefficients appear, ask:

Where have I seen this pattern before?

What Should We Notice?

$$D^2(fg) = fD^2g + 2DfDg + gD^2f.$$

This resembles:

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Reading Move

When coefficients appear, ask:

Where have I seen this pattern before?

The coefficient 2 is not accidental.

Question 2(a)(i): Find $D^3(fg)$

Differentiate:

$$D^2(fg) = fD^2g + 2DfDg + gD^2f.$$

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Then

$$D^3(fg) = D(fD^2g) + D(2DfDg) + D(gD^2f).$$

Compute term by term:

$$D(fD^2g) = fD^3g + DfD^2g,$$

$$D(2DfDg) = 2D^2fDg + 2DfD^2g,$$

$$D(gD^2f) = DgD^2f + gD^3f.$$

Question 2(a)(i): Solution

Collect like terms:

$$D^3(fg) = fD^3g + 3DfD^2g + 3D^2fDg + gD^3f.$$

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appears.

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Pattern

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appears.

This suggests binomial coefficients.

Question 2(a)(ii): Formulating the Conjecture

From:

$$D(fg) = fDg + gDf,$$

$$D^2(fg) = fD^2g + 2DfDg + gD^2f,$$

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We conjecture:

$$\boxed{D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} (D^i f)(D^{n-i} g)}$$

where $D^0 f = f$.

Reading the Conjecture

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Reading Lesson

A formula often tells a story.

Question 2(a)(iii): Proving the Conjecture

We prove:

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} (D^i f)(D^{n-i} g)$$

by induction on n.

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Reading a Proof by Induction

Always identify:

- the statement $P(n)$;
- the base case;
- the inductive hypothesis;
- where the inductive hypothesis is used.

Base Case

For $n = 1$:

$$\sum_{i=0}^{1} \binom{1}{i} (D^i f)(D^{1-i} g)$$

$$= \binom{1}{0} (D^0 f)(D^1 g) + \binom{1}{1} (D^1 f)(D^0 g)$$

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This is exactly the product rule:

$$D(fg) = f Dg + g Df.$$

Inductive Step

Assume true for $n = k$:

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Reading Move

This is where the inductive hypothesis is used.

Applying the Product Rule Inside the Sum

$$D^{k+1}(fg) = \sum_{i=0}^{k} \binom{k}{i} \left[(D^{i+1}f)(D^{k-i}g) + (D^{i}f)(D^{k+1-i}g) \right].$$

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So we have two sums:

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Reading Challenge

The proof now becomes an exercise in re-indexing.

Where the Binomial Coefficients Come From

After re-indexing and collecting like terms, the middle coefficient is

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Hence

$$D^{k+1}(fg) = \sum_{i=0}^{k+1} \binom{k+1}{i} (D^i f)(D^{k+1-i} g).$$

The induction is complete.

What Did We Learn from This Passage?

The product rule question required us to read:

- newly introduced notation;
- a familiar rule in unfamiliar form;
- repeated examples;
- an emerging pattern;
- a conjecture;
- an induction proof;
- a re-indexing step.

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Big Lesson

Symbolic reading means tracking structure through notation.

From Reading to Proof

Reading mathematics is not separate from proving.

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To prove well, we must first read well.

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Reading revealed:

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Reading revealed:

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In the product rule problem

Reading revealed:

- binomial pattern;
- derivative distribution;
- induction structure.

Reading Through Schoenfeld's Lens

Reading mathematics requires:

1. **Cognitive resources**
   What do I already know?
2. **Heuristics**
   What reading strategy should I use?
3. **Control**
   Do I actually understand this line?
4. **Beliefs**
   Do I believe difficult text can be understood slowly?

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Key Message

Reading is problem solving in slow motion.

Book link: Chapter 5; Chapter 6

Control While Reading

Good readers constantly monitor themselves.

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They ask:

- Do I understand the definition?
- Can I explain this line in words?
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Metacognition

Reading mathematics requires thinking about one's own understanding.

Common Reading Mistakes

- Reading symbols without meaning.
- Skipping definitions.
- Ignoring examples.
- Treating formulas as decorations.
- Failing to ask what the proof is trying to do.
- Giving up when understanding is not immediate.

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Correction

Slow down. Annotate. Test examples. Reconstruct the argument.

Confusion is Part of Understanding

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Even strong mathematicians:

- reread dense passages;
- work through examples;
- draw diagrams;
- rewrite proofs in their own words.

Looking Back at the Course

Across these lectures, we encountered:

- Pólya's stages;
- mathematical resources;
- heuristics;
- looking back and expanding;
- Schoenfeld's framework;
- reading mathematics as active reconstruction.

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Unified Perspective

Problem solving is not merely about obtaining answers.
It is about learning to think mathematically.

The Continuing Journey

Mathematics is not learned by memorising finished ideas.

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Mathematics is not learned by memorising finished ideas.

It is learned by slowly entering
another person's way of thinking.

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another person's way of thinking.

To read mathematics well
is to think mathematically.

Book link: Chapter 7: Conclusion

Final Encouragement

Read slowly.

Final Encouragement

Read slowly.

Think deeply.

Final Encouragement

Read slowly.

Think deeply.

Persist patiently.

Final Encouragement

Read slowly.

Think deeply.

Persist patiently.

And keep solving problems.

References

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